SEMI-RIGOROUS ADJUSTMENT OF TRIANGULATION

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The pamphlet on the "Adjustment of Triangulation" issued to H. M. Surveying Ships in September 1929 describes methods of reducing simple geometrical figures to geometrical consistency, simple methods of satisfying the "equations of condition" being devised, which result in attaining the desired geometrical consistency, but which do not attempt to consider the actual probability of the distribution of the errors.

Rigorous adjustment by the method of least squares is a laborious process, and is not necessary in Hydrographic Surveys, but methods which may be termed "Semi-rigorous", in which the uneliminated errors are reduced to such small quantities that their effect is almost negligible, can be employed with little extra labour than is involved in adjusting by "rule of thumb" methods such as are described in the pamphlet referred to.

The "equations of condition" which have to be fulfilled in order to balance a figure are of two kinds :- (1) Angle equations, and (2) Side equations.

Rigorous adjustment necessitates the simultaneous solution of all equations of condition, but in the semi-rigorous method, the angle and side equations are rigorously solved independently.

In the semi-rigorous method, the first step is the rigorous adjustment of the angle equations to obtain the corrections necessary to "close" the figure. The corrections having been applied to the observed angles, the side equation is then adjusted rigorously, resulting in further adjustments to the angles, which, whilst satisfying the side equation, result in upsetting the "closing" of the figure.

The whole process has then to be repeated until the remaining errors become so small as to be inappreciable — it is, in fact, a method of successive approximation.

For the purpose of Hydrographical Surveys, a sufficient degree of accuracy can usually be obtained by combining the rigorous adjustment of angle equations with a non-rigorous adjustment of the side equation, *i. e.*, the angle equations are first adjusted rigorously, and the side equation is satisfied by the method given in the pamphlet of September 1929.

If this procedure results in an over large correction for the side equation, the rigorous adjustment of the side equation should be applied, after which the angles would have to be re-adjusted rigorously before a further non-rigorous adjustment of the side equation is applied to obtain geometrical consistency.

The calculation of the corrections for the angle equations from the rigorous formulae involves a certain amount of extra work in the majority of cases, but the extra work is not out of proportion to the gain in accuracy.

The rigorous adjustment of the side equation is little more laborious than the non-rigorous, the only additional data required being the squares of the differences for one second in the *log sines* (Dablas) which can be obtained with sufficient accuracy by slide rule.

The semi-rigorous method of adjusting simple figures will now be described.

QUADRILATERAL.

In the case of the quadrilateral, the rigorous adjustment of the angle equations gives precisely the same result as the method used in the pamphlet of September 1929, and the latter can therefore still be used.

The only difference between the rigorous adjustment of the side equation and the method previously employed is that whereas, in the latter case, a permanent correction to all angles is found by dividing the difference of sums of *log sines* by the sums of the "dablas", in the former case the correction to each angle is found by multiplying the "dabla" for each angle by a constant factor consisting of the difference of *log sines* divided by the sum of the squares of the dablas.

It will be obvious, therefore, that the whole adjustment can be made on a very similar form to that given in the pamphlet previously referred to.

The following example of the rigorous adjustment of the side equation will make the procedure clear. The figures are the same as in the example of the Quadrilateral adjustment in the 1929 pamphlet, so that a comparison between the rigorous and non-rigorous methods is furnished.

	Adjusted Angles.	*Log Sines.*	∇	∇²	*Angle Corrn.*	*Angles (final).*	*Log Sines (final).*
1	67.28.15	9.9655237	8.7	76	+ 2".3	67.28.17.3	9.9655258
3	27.13.45	9.6604392	40.9	1675	+ 10".6	27.13.55.6	9.6604826
5	54.01.15	9.9080723	15.3	234	+ 4".0	54.01.19.0	9.9080784
7	32.03.45	9.7249669	33.6	1129	+ 8".7	32.03.53.7	9.7249962
	*S*1 =	39.2590021					0830
2	52.50.15	9.9014177	16.0	256	— 4".2	52.50.10.8	9.9014110
4	32.27.45	9.7297700	33.1	1095	— 8".6	32.27.36.4	9.7297415
6	66.17.15	9.9616939	9.2	85	— 2".4	66.17.12.6	9.9616917
8	27.37.45	9.6662812	40.2	1617	— 10".4	27.37.34.6	9.6662394
	*S*2 =	39.2591628		6167			0836
	*S*1 =	39.2590021					
		D = 1607					

$$\text{Factor} = \frac{1607}{6167} = .26$$

The symbol ∇ indicates "Dabla".

The squares of the "dablas" and the angle corrections can conveniently be calculated by the slide rule, and similarly the corrections to the *log sines*, which are found by multiplying the "dabla" of each angle by its correction.

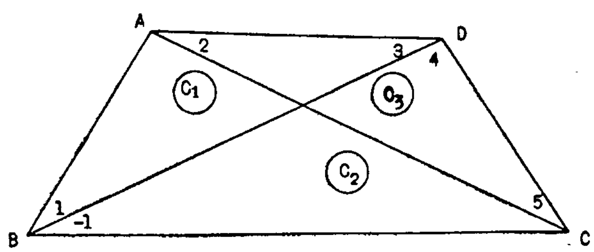
This example forms a good illustration of the differences in the corrections found by the rigorous method compared with those found by the non-rigorous method.

The final angles given are correct as regards the side equation, but *not* as regards the angle equations. The angles must therefore be again corrected, after which it will be found that the correction necessary to satisfy the side equation and obtain geometrical consistency by the non-rigorous method will be very small — in this case only 0".2.

QUADRILATERAL WITH ONE TRIANGLE ALREADY ADJUSTED.

When one triangle of a quadrilateral is already adjusted, the adjustment of the figure is carried out as follows :-

Let *ABCD* be a quadrilateral of which the triangle *ABC* has already been adjusted.



Number the angles as shewn in the diagram, and let the corrections to these angles be x_1, x_2, etc...

Let the corrections to the closure of the triangles *ABD*, *DBC* and *ADC* be C_1, C_2 and C_3.

Then $C_1 + C_2 = C_3$, in virtue of the fact that the triangle ABC is already adjusted and fixed.

Moreover, since this is so, the correction to $\angle ABD$ must be equal and opposite to that of $\angle DBC$, and the correction to $\angle DAC$ must be equal to that of $\angle BAD$, both being x_2. Similarly x_5 corrects both $\angle DCA$ and $\angle DCB$.

The rigorous solution of the angle equations gives us the following expressions for the corrections :-

$$x_1 = \frac{1}{4}(C_1 - C_2)$$

$$x_2 = x_3 = \frac{1}{4} C_1 + \frac{1}{8} C_3$$

$$x_4 = x_5 = \frac{1}{4} C_2 + \frac{1}{8} C_3$$

So that all that is necessary to adjust the angles is the solution of 3 simple equations in which the expressions C_1, C_2 and C_3 are known.

The side equation governing the three unfixed sides, AD, BD and DC is

$$\frac{DA}{DB} = \frac{DB}{DC} = \frac{DC}{DA} = 1 \qquad \text{or} \qquad \frac{\sin DBA}{\sin DAB} = \frac{\sin DCB}{\sin DBC} = \frac{\sin DAC}{\sin DCA} = 1$$

and the adjustment is made by making the sum of the *log sines* of the upper angles equal to the sun of the *log sines* of the lower angles in this expression.

Now there are only three corrections involved in this adjustment, as the corrections to $\angle DBA$ and $\angle DBC$ are equal and opposite, the corrections to $\angle DCA$ and $\angle DCB$ are equal, and the corrections to $\angle DAC$ and $\angle DAB$ also equal.

The same corrections therefore appear on both sides of the equation.

The corrections to the angles are found by multiplying the algebraical sum of the "dablas" of angles with similar corrections by a factor determined by dividing the difference of the sum of the *log sines* by the sum of the squares of the algebraical sums of the "dablas", which are essentially negative on the left hand side of the equation.

The working will be clear from the following example, which gives the rigorous adjustment of both angle and side equations.

It is to be noted that a non-rigorous adjustment of the side equation can be combined with the rigorous adjustment of the angle equations if the corrections by the former are small.

QUADRILATERAL WITH ONE TRIANGLE ALREADY ADJUSTED.

Example.

ANGLE ADJUSTMENT.

Triangle	*Observed Angles*	*Corr^n^*	*Adjusted angles*
	° ′ ″		
A	66.17.15		
B	86.28.45		Already adjusted.
C	27.14.00		
	180.00.00		
*A*2	98.21.30	— 30″	98.21.00
*B*1	54.01.00	+ 15″	54.01.15
*D*3	27.38.15	— 30″	27.37.45
	180.00.45		180.00.00

*A*2	32.04.15	— 30"	32.03.45
*C*5	52.51.00	— 45"	52.50.15
*D*3+4	95.07.15	— 75"	95.06.00
	180.02.30		180.00.00

B-1	32.27.45	— 15"	32.27.30
*C*5	80.05.00	— 45"	80.04.15
*D*4	67.29.00	— 45"	67.28.15
	180.01.45		180.00.00

$C_1 = -45''$ $C_2 = -105''$ $C_3 = -150''$.

$$x_1 = \frac{1}{4}(C_1 - C_2) = +15''.$$

$$x_3 = x_2 = \frac{1}{4}C_1 + \frac{1}{8}C_3 = -11''.25 - 18''.75 = -30''.$$

$$x_5 = x_4 = \frac{1}{4}C_2 + \frac{1}{8}C_3 = -26''.25 - 18''.75 = -45''.$$

Side Adjustment.

Nº	Angle		Corrn (*)	Final
1	*DBA*	54°.01'.15"	+3".1	18".1
5	*DCB*	80 .04 .15	—0 .8	14 .2
2	*DAC*	32 .03 .45	+2 .4	47 .4

Nº	Angle		Corrn	Final
2	*DAB*	28°.21'.00"	+2".4	02".4
—1	*DBC*	32 .27 .30	—3 .1	26 .9
5	*DCA*	52 .50 .51	—0 .8	14 .2

Nº	∇	Log Sine.	Corrn (**)	Corrn Log Sine.
1	15	9080723	+ 47	0770
5	3.5	9934458	— 3	4455
2	33.5	7249669	+ 80	9749
		6264850		4974

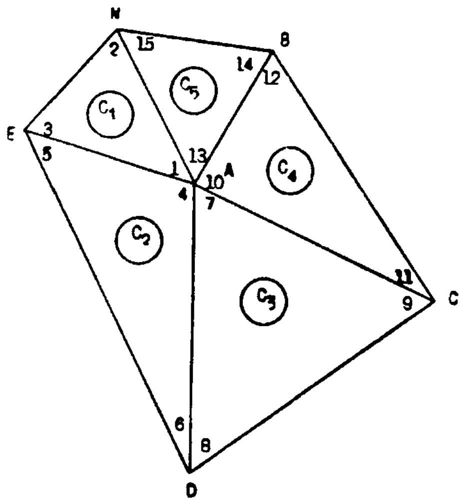
Nº	∇	Log Sine.	Corrn	Corrn Log Sine.
2	— 3	9953717	— 7	3710
—1	33	7297204	— 104	7100
5	16	9014177	— 13	4164
		6265098		4974
		4850		
		D = +248		

Angle	Σ∇	Σ∇²	Corrn
1	15 —(—33) = + 48	2304	+ 3".1
2	33.5 —(— 3) = + 36.5	1332	+ 2".4
5	3.5 — 16 = — 12.5	156	— 0".8
		3792	

$$\text{Factor} = + \frac{248}{3792} = + .0654$$

(*) *The correction to an angle* = Σ ∇ × Factor.
(**) *The correction to* log sine *of an angle = Angle Corrn.* × ∇
The final angles obtained are correct as far as the side equation is concerned, but not *as regards the angle equations. The angles should therefore be again corrected, after which the side equation can be adjusted again by non-rigorous methods which will not disturb the closure of the figure, and will result in geometrical consistency.*

CENTRAL-POINT POLYGON.



Let *BCDEN* be a polygon with central station *A*.

The first step in the rigorous treatment of this figure is to make the angles at $A = 360°$, dividing the error equally between the component angles.

Then, if C_1, C_2, etc. be the corrections to the closures of the various triangles, and C = the correction to the closure of the exterior angles of the polygon,

$$C_1 + C_2 + C_3 + C_4 + C_5 = C$$

Now, if we call the angles at the centre "centre angles" and the other angles "base angles",

the rigorous solution of the angle equations gives the following expressions for the corrections to the angles:

Corrn. to centre angle $= \frac{1}{3}$ Triangle Correction minus $\frac{1}{3n}$ Figure Correction,

Corrn. to base angle $= \frac{1}{3}$ Triangle Correction plus $\frac{1}{6n}$ Figure Correction,

where Figure Correction is C, or the sum of Triangular Corrections and n = the number of triangles in the figure.

In this particular case, therefore, if x_1, x_2, x_3 are the corrections to the angles 1, 2 and 3, in the triangle *AEN*,

$$x_1 = \frac{1}{3} C_1 - \frac{1}{15} C \qquad x_2 = x_3 = \frac{1}{3} C_1 + \frac{1}{30} C$$

It will therefore be apparent that the rigorous adjustment of the angle equations requires the solution of 2 simple equations for each triangle included in the figure.

The rigorous adjustment of the side equation is precisely similar to the quadrilateral, that is to say :- the corrections are found by multiplying the "dabla" of each angle by a factor consisting of the difference of the *log sines* divided by the sum of the squares of the "dablas". Otherwise the adjustment is exactly similar to the non-rigorous adjustment previously described in the pamphlet of September, 1929. The rigorous adjustment of the side equation will again result in upsetting the closure of the angles, so that the angles must be again adjusted rigorously before applying a final non-rigorous adjustment of the side correction to obtain exact geometrical consistency.

The following example will make the procedure clear :-

CENTRAL-POINT POLYGON.

Example.

ANGLE ADJUSTMENT.

Angle	*Observed Angles.*	*Central Angles.*	1st Corrn	*Angles*-1st *Adjustment*	2nd Corrn	*Angles*-2nd *Adjustment*
1	35.05.00	35.05.00	+ 18"	35.05.18	+ 36"	35.05.54
2	71.47.00			71.47.00	+ 33"	71.47.33
3	73.06.00			73.06.00	+ 33"	73.06.33
	179.58.00			179.58.18		180.00.00
4	124.44.00	124.44.00	+ 18"	124.44.18	+ 26"	124.44.44
5	35.42.00			35.42.00	+ 23"	35.42.23
6	19.32.30			19.32.30	+ 23"	19.32.53
	179.58.30			179.58.48		180.00.00
7	72.44.00	72.44.00	+ 18"	72.44.18	— 34"	72.43.44
8	48.49.00			48.49.00	— 37"	48.48.23
9	58.28.30			58.28.30	— 37"	58.27.53
	180.01.30			180.01.48		180.00.00
10	57.39.00	57.39.00	+ 18"	57.39.18	— 24"	57.38.54
11	36.00.00			36.00.00	— 27"	36.59.33
12	86.22.00			86.22.00	— 27"	86.21.33
	180.01.00			180.01.18		180.00.00
13	69.46.30	69.46.30	+ 18"	69.46.48	— 4"	69.46.44
14	59.24.30			59.24.30	— 7"	59.24.23
15	50.49.00			50.49.00	— 7"	50.48.53
	180.00.00	359.58.30		180.00.18		180.00.00

$C_1 = +102 \quad C_2 = +72 \quad C_3 = -108 \quad C_4 = -78 \quad C_5 = -18 \quad C = -30$

$x_1 = \frac{1}{3} C_1 - \frac{1}{15} C = 34 + 2 = +36 \qquad x_2 = x_3 = \frac{1}{3} C_1 + \frac{1}{30} C = 34 - 1 = +33$

$x_4 = \frac{1}{3} C_2 - \frac{1}{15} C = 24 + 2 = +26 \qquad x_5 = x_6 = +23.$

$x_7 = \frac{1}{3} C_3 - \frac{1}{15} C = -36 + 2 = -34 \qquad x_8 = x_9 = -36 - 1 = -37.$

$x_{10} = \frac{1}{3} C_4 - \frac{1}{15} C = -26 + 2 = -24 \qquad x_{11} = x_{12} = -27.$

$x_{13} = \frac{1}{3} C_5 - \frac{1}{15} C = -6 + 2 = -4 \qquad x_{14} = x_{15} = -6 - 1 = -7.$

SIDE ADJUSTMENT.

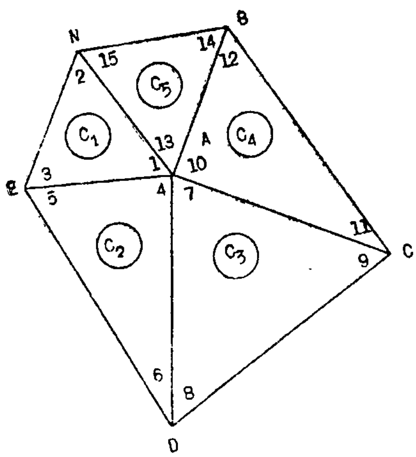
Angle	Angles 2nd *Adj.*	*Log sines*	∇	∇²	*Angle Corrn*	*Angles final*	*Log sines final*
2	71.47.33	9776921	6.9	48	+ 0".9	71.47.33.9	9776927
5	35.42.23	7661389	29.3	860	+ 3 .9	35.42.26.9	7661504
8	48.48.23	8764998	18.4	338	+ 2 .5	48.48.25.5	8765044
11	35.59.33	7691404	29.0	840	+ 3 .9	35.59.36.9	7691518
14	59.24.23	9349016	12.4	154	+ 1 .7	59.24.24.7	9349038
	*S*1 =	3243728					4031
3	73.06.33	9808484	6.4	41	— 0".9	73.06.32.1	9808479
6	19.32.53	5245225	59.3	3510	— 8 .0	19.32.45.0	5244751
9	58.27.53	9306018	12.9	166	— 1 .7	58.27.51.3	9305996
12	86.21.33	9991226	1.3	2	— 0 .2	86.21.32.8	9991226
15	50.48.53	8893616	17.2	296	— 2 .3	50.48.50.7	8893576
	*S*2 =	3244569	193.1	6255			4028
	*S*1 =	3728					
	D =	841					

$$\text{Factor} = \frac{841}{6255} = .134.$$

It is interesting to note that the correction by non-rigorous method in this case would be

$$\frac{841}{193.1} = 4''.4.$$

CENTRAL-POINT POLYGON WITH ONE OR MORE TRIANGLES ALREADY ADJUSTED.



When one or more triangles of a central-point polygon have already been adjusted, the procedure is not substantially altered.

The first step is to make the central angles equal to 360°, dividing the error equally between the angles contained in the unadjusted triangles only.

The rigorous solution of the angle equations for the unadjusted triangles is represented by the following equations:

$$\text{Corrn. to centre angle} = \frac{1}{3}\ \text{Triangular Corrn. minus}\ \frac{1}{3\,(n\text{-}m)}\ \text{Figure Corrn,}$$

$$\text{Corrn. to base angle} = \frac{1}{3}\ \text{Triangular Corrn. minus}\ \frac{1}{6\,(n\text{-}m)}\ \text{Figure Corrn,}$$

where n = number of triangles in the figure,
m = number of triangles previously adjusted.

Thus, if triangle *ABN* is already adjusted,

$$x_1 = \frac{1}{3}\,C_1 - \frac{1}{12}\,C \qquad x_2 = x_3 = \frac{1}{3}\,C_1 - \frac{1}{24}\,C$$

In other words it is only necessary to treat the figure as if it were composed of the unadjusted triangles only.

If only two triangles in a polygon have not been previously adjusted, they should be treated as a quadrilateral with one triangle previously adjusted, *i. e.* :

If triangles *ABC*, *ADC*, and *AED* are already adjusted, the figure *AENB* should be treated as a quadrilateral with the triangle *AEB* already adjusted.

The rigorous treatment of the side equation is precisely similar to the ordinary method, except that the adjusted angles must not be altered.

The factor is found by dividing the difference of sums of all the *log sines* by the sum of the squares of the "dablas" of unadjusted angles only.