TIDAL CHARACTERISTICS FROM HARMONIC CONSTANTS

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The method of specifying the characteristics of the tide by means of harmonic constants has so many advantages that its use on the part of the mariner and hydrographer should become more widespread. Not only does this method afford the most concise method of specifying the various features of the tide at any place, but at the same time permits a more illuminating understanding of the complex manifestations of the tide than is possible in any other way.

The harmonic constants result from the harmonic analysis of the tide observed at any place. This analysis is a mathematical process for resolving the observed tide at any place into a number of simple constituent tides. For the purposes of the tidal specialist it is by far the most satisfactory process he can employ, both in the analysis and prediction of tides. Indeed, for the specialist in tides it is an indispensable process.

It must be admitted, however, that the harmonic analysis is a highly specialized mathematical process, and that it necessitates a very considerable amount of time-consuming numerical computations. The mariner and hydrographer therefore question whether the time and effort required to master the technique of the process is not out of all proportion to its possible use for their purpose.

Without taking sides on this question, it may be observed that, as in the case of other highly specialized processes, we may use the *results* of the harmonic analysis without the necessity of mastering the technique of the processes involved. In other words, the formulae and computations of the harmonic analysis may be left to the tidal mathematician. The results, however, can be used to great advantage by the mariner, who needs merely acquire an understanding of the principles involved. Fortunately such an understanding of the principles underlying the harmonic analysis is not at all difficult.

UNDERLYING PRINCIPLES.

As is well known, the rise and fall of the tide is due to the action of the tide-producing forces of sun and moon acting on the rotating earth, the variations in the tide at any place being brought about by the changing positions of moon and sun relative to the earth. It is clear, however, that we may regard these relative movements of earth, sun and moon as arising from the movement of sun and moon around a fixed earth. The advantage of so conceiving the matter is that by this means we may regard both sun and moon as revolving about the earth, and any argument applied to the movement of the one then becomes applicable to the other.

The conception underlying the harmonic analysis of the tide may be stated as follows. The sun and moon, *as tide producing bodies,* may be conceived as replaced by a number of simple hypothetical tide-producing bodies which, with respect to the earth, have simple movements. Instead of moving in elliptical orbits inclined to the plane of the earth's equator, these hypothetical tide producing bodies move in the plane of the equator in circular orbits with the earth as the center. Each of these tide-producing bodies is assumed to give rise to a simple tide the high water of which at any place always occurs at the same time after its upper meridian passage and the low water the same time after its lower meridian passage. The great advantage of this conception is that it permits the tide at any place to be conceived as the combination of a number of simple tides related to tide-producing bodies having simple movements.

Each of the simple tides into which the tide of nature is resolved by the harmonic analysis is called a component tide or simply a component. And for convenience the same term is used to designate the hypothetical tide-producing body which gives rise to the simple tide. Each of the components has been given a distinctive name to distinguish it from the others as, for example, principal lunar, principal solar, etc.; but as a rule they are designated by simple symbols as M_2 or S_2. The significance of these symbols will become clear later.

The *cosine* curve represents accurately the tide curve of a simple component tide of which Figure 1 is a graphic representation. Two characteristics specify it completely, namely, the amplitude and the epoch. The amplitude of a *cosine* curve is its maximum ordinate, and hence the amplitude of a component tide is the semi-range of the tide represented in the illustration by BE or FC. The epoch is the time, in angular measure, elapsing between the meridian passage of a hypothetical tide-producing body and the high water of its tide. In the figure $ABCD$ represents a complete period of a simple component tide, A being the instant of meridian passage of its hypothetical tide-producing body. The height of this tide at any instant is measured vertically up or down from the line GH, while the time is measured along this line from G toward H. Hence the distance GE, expressed as an angle, which is $(GE \div GH)\ 360°$, is the epoch of the component. The amplitudes and epochs of the component tides at any point are collectively called the harmonic constants of the tide at that point.

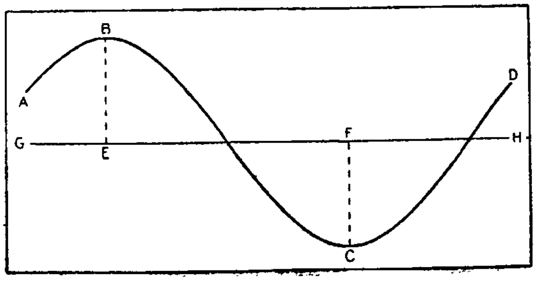

FIG. 1

The period of revolution of each of the hypothetical tidal bodies, and hence the period of each of the corresponding simple constituent tides, is computed from astronomical data and depends only on the relative movements of sun, moon, and earth. These periods being independent of local conditions are the same for all places on the earth; what remains to be determined for the tide at any place are the epochs and amplitudes of the various components, since these vary from place to place in accordance with the type, range and time of tide. It is this disentangling of the amplitudes and epochs of the constituent tides that the harmonic analysis does.

PRINCIPAL COMPONENTS.

Theoretically, the number of component tides is very large; but most of them are of such small magnitude that for practical purposes they may be disregarded. Even in the harmonic prediction of the tide with the aid of tide-predicting machines, it is rarely necessary to take into account more than 20 component tides. For deriving the characteristics of the tide for the use of the mariner or hydrographer the five principal components are all that are required.

On the average there are two high and two low waters in a tidal day of 24 hours 50 minutes. This means that the average tidal cycle has a period of 12 hours and 25 minutes. It is obvious therefore that the most prominent component of the tide must be one which every 12 hours 25 minutes or 12.42 hours gives a complete cycle of one high and one low water. This component with a period of 12.42 hours, or angular speed of $28^\circ.98$ per hour, is called the principal lunar semi-diurnal component, but is generally represented by the symbol M_2, the M indicating that it is a component derived from the moon's motion, and the subscript 2 indicating that it is a semi-diurnal component, that is, its period is half a day.

The daily rotation of the earth gives an apparent movement of the sun about the earth with a period averaging exactly 24 hours. From a consideration of the tide-producing forces, or by analogy with the moon's movement, it follows that the principal solar component should have a period of exactly 12 hours, or an angular speed of 30° per hour. This is known as the principal solar semi-diurnal component, but is preferably represented symbolically by S_2, the S denoting its relation to the apparent movement of the sun, and the subscript, as before, indicating it is a semi-diurnal component.

The moon does not move in a circular orbit, but in an elliptical one. This means that its distance from the earth is not constant, and hence its tide producing force varies, being less than the average at apogee and greater at perigee, the period from one perigee to another being on the average 27.55 days. To M_2 with its simple circular motion we must therefore add another component such that at perigee its high water will correspond with the M_2 high water, and at apogee its low water will correspond with the M_2 high water. In other words, the component which is to take account of the moon's perigean movement must, in a period of 13.78 days (one-half of 257.5), lose 180° on M_2 or at the rate of $180^\circ \div 13.78 = 13^\circ.06$ per day. Its hourly speed therefore is $28^\circ.98 - (13^\circ.06 \div 24) = 28^\circ.44$, and its period $360 \div 28.44 = 12.66$ hours. This component is known as the larger lunar elliptic semi-diurnal, and is represented by the symbol N_2.

Still another prominent feature of the movement of the moon must be taken into consideration, namely, the fact that the plane of its orbit is inclined to the plane of the equator. This means that the declination of the moon is constantly changing. For certain reasons it is of advantage to take account of the declinational effect by the use of two components rather than one, and these are denoted by the symbols K_1 and O_1. The speeds of these are determined by the following considerations: The average period from one maximum declination to another is a half tropic month, or 13.66 days. The speeds of these two components should, therefore, be such that when the moon is at its maximum declination they shall both have their maximum amplitudes and when the moon is on the equator they shall neutralize each other; that is, in a period of 13.66 days K_1 shall gain on O_1 one full revolution. The difference in their hourly speeds, therefore, is $360^\circ \div (24 \times 13.66) = 1^\circ.10$. The mean of the speeds of these two components must be that of the apparent diurnal movement of the moon about the earth, or $360^\circ \div 24.84 = 14^\circ.49$. The speeds are therefore derived from the equations $(K_1 + O_1) \div 2 = 14^\circ.49$ and $K_1 - O_1 = 1^\circ.10$, from which $K_1 = 15^\circ.04$ and $O_1 = 13^\circ.94$.

The periods of K_1 and O_1, which may be derived by dividing 360° by their respective hourly speeds, are 23.93 hours and 25.82 hours. In other words, the period of each of these two components is approximately one day, and for this reason the subscripts of their symbols are 1; K_1 is called the luni-solar diurnal and O_1 the lunar diurnal.

The five components, whose periods and hourly speeds were derived above, are the principal components. It will be convenient to have their symbols, speeds and periods summarized in tabular form. The speed is given per mean solar hour and the period likewise is expressed in mean solar hours of every-day use.

PRINCIPAL HARMONIC COMPONENTS OF THE TIDE.

Name of Component.	*Symbol*	*Speed per hour*	*Period*
		Degrees	Hours
Principal lunar semi-diurnal	M_2	28.98	12.24
Principal solar semi-diurnal	S_2	30.00	12.00
Larger lunar elliptic semi-diurnal	N_2	28.44	12.66
Luni-solar diurnal	K_1	15.04	23.93
Lunar diurnal	O_1	13.94	25.82

For any particular component the amplitude is designated by the symbol of the component while the epoch is designated by the symbol with a degree mark added. Thus in New York Harbor, at the mouth of the Hudson, the principal lunar semi-diurnal component has an amplitude of 2.11 feet and an epoch of 233°. Symbolically this would be expressed, $M_2 = 2.11$ feet, $M_2^0 = 233°$. In Tables of harmonic constants the column listing the amplitudes is frequently headed by the letter H, while the column listing the epochs is headed by K (the Greek letter *kappa*).

With these preliminary considerations relative to the harmonic constants we may now proceed to a consideration of their use in specifying various characteristics of the tide. The more important of these relate to type, time, range and age.

TYPE OF TIDE.

The rise and fall of tide manifests itself in a great variety of forms. All these however may be grouped under three types, namely, semi-daily, daily and mixed. The semi-daily type is characterized by two high waters and two low waters in a day, with morning and afternoon tides resembling each other in all respects. The daily type is one in which but one high water and one low water occur in a day. The mixed type is characterized by two high waters and two low waters in a day, but with morning and afternoon tides differing considerably.

Typical examples of each of the three types of tide are shown in Figure 2, which represents the rise and fall of the tide for the first two days of September, 1929, at three different places. The upper diagram represents the rise and fall of the tide in New York Harbor, where the tide is of the semi-daily type. Two high and two low waters occur during a day, morning and afternoon tides resembling each other in all respects. The middle diagram represents the tide at Pensacola, Florida, on the Gulf of Mexico. Here the tide is of the daily type, but one high and one low water occuring in a day.

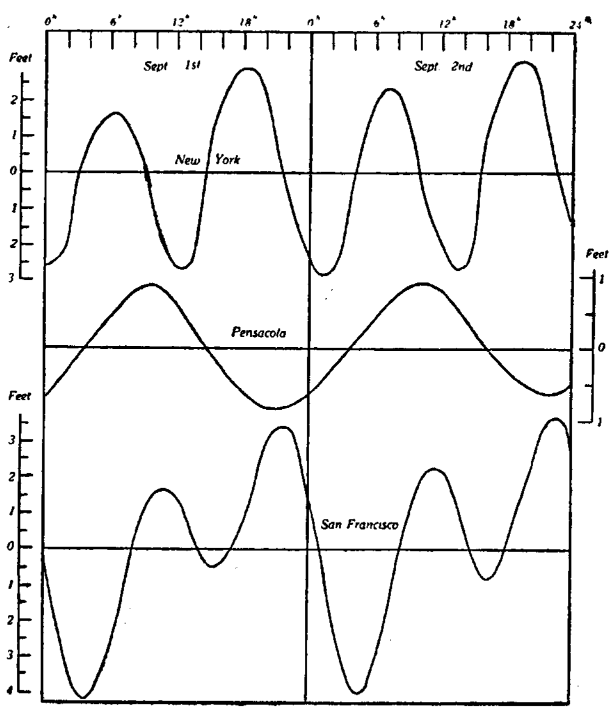

FIG. 2.

The lower diagram represents the tide at San Francisco, California, where the tide is of the mixed type. Two high waters and two low waters occur during a day, but morning and afternoon tides are different. Morning high waters are more than a foot lower than the afternoon high waters, while the morning low waters are three feet lower than the afternoon low waters.

The five principle constants permit ready determination of the type of tide. Without going into the details of the matter it may be stated that for semi-daily tides the amplitudes of $(K_1 + O_1) \div (M_2 + S_2)$ are less than 0.25; for the mixed type of tide this ratio is between 0.25 and 1.25; while for the daily type this ratio is greater than 1.25.

To exemplify the determination of the type of tide from the harmonic constants we may use the three places for which Figure 2 illustrates the tide. From a Table of harmonic constants, as for example, International Hydrographic Bureau Special Publication 12*a*, we find the following values, in centimetres, of the amplitudes of the four harmonic constants concerned:

Place.	K_1	O_1	M_2	S_2
	cm.	cm.	cm.	cm.
New York..........................	10	5	64	13
Pensacola...........................	13	12	2	1
San Francisco......................	37	23	54	12

From the above values we find $(K_1 + O_1) \div (M_2 + S_2)$ to be 0.19 for New York; 8.33 for Pensacola; 0.91 for San Francisco. In accordance with the ratios for the different types, New York has the semi-daily type, Pensacola the daily, and San Francisco the mixed type of tide. The type of tide is thus readily derived through the harmonic constants.

THE MIXED TYPE OF TIDE.

The mixed type of tide merits more detailed consideration since this type comprises a great variety of different forms. These different forms may, however, be grouped into three large classes, namely those which feature the difference between the morning and afternoon tides chiefly in the low waters, those which feature it chiefly in the high waters and those which feature it in approximately equal degree in both the high and the low waters. Typical examples of each of these different forms of the mixed type of tide are illustrated in Figure 3.

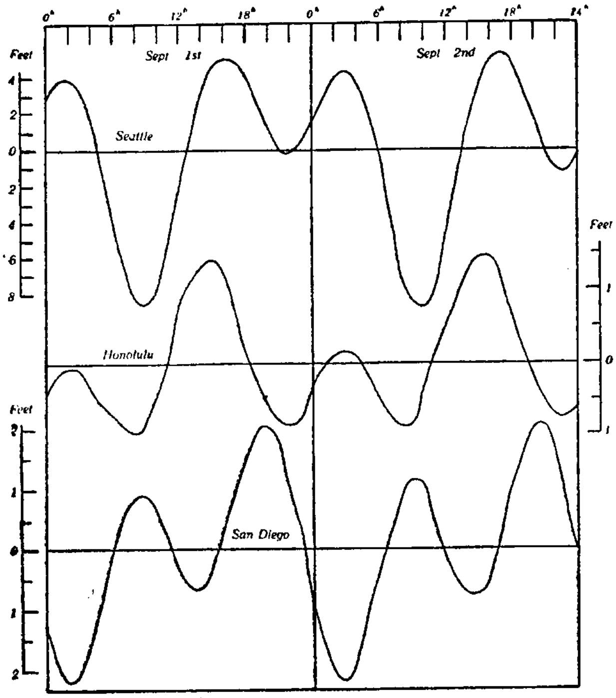

FIG. 3.

In Figure 3 are reproduced the tide curves for the first two days of September 1929, at Seattle, Washington, Honolulu, Hawaii, and San Diego, California. The upper curve, that for Seattle, illustrates the form of mixed tide in which the difference between morning and afternoon tides is exhibited chiefly by the low waters. It will be noted that the difference between the two high waters of a day is generally small, while the difference between the two low waters frequently amounts to as much as eight feet.

The middle figure, which represents the tide at Honolulu shows that here the condition is the exact opposite from that at Seattle; for here the two low waters of a day do not differ much, while the two high waters differ considerably. The lower diagram represents the tide at San Diego where, it is seen, the difference between morning and afternoon tides is featured in approximately equal degrees by both the high waters and the low waters.

It can be shown easily that the different forms of the mixed type of tide result from different phase relations between the daily and semi-daily constituents of the tide. If the difference between M_2° on the one hand and $(K_1^\circ + O_1^\circ)$ on the other is less than 60°, the difference between morning and afternoon tides will be principally in the high waters; if this difference is from 60° to 120° both high waters and low waters will exhibit differences; and if this difference is from 120° to 180° morning and afternoon tides will exhibit differences principally in the low waters.

The applicability of the above rule may be tested in connection with the tides at the three places illustrated in Figure 3. The harmonic constants, both amplitudes and epochs, for each of these three places are as follows:

Place.	K_1		O_1		M_2		S_2	
	Cm.	Deg.	Cm.	Deg.	Cm.	Deg.	Cm.	Deg.
Seattle	82	156	46	133	106	128	26	154
Honolulu	15	70	8	59	16	106	5	102
San Diego	33	95	21	79	52	276	21	274

From the above values it is found that the difference between M_2° and $(K_1^\circ + O_1^\circ)$ is 161° for Seattle; 23° for Honolulu; and 101° for San Diego. In accordance with the preceding paragraph, therefore, the tide at Seattle should show difference principally in the low waters; at Honolulu this difference should be principally in the high waters; while at San Siego the difference should be exhibited in approximately equal degree by both high and low waters. And Figure 3 proves that these are the conditions that actually obtain at each of these places.

It should be noted that in the application of the above rule the difference between M_2° and $(K_1^\circ + O_1^\circ)$ is taken without reference to the sign of the result. It should be noted, too, that when this difference is greater than 180° it is to be subtracted from 360°.

TIME OF TIDE.

The times of high and low water are generally specified in terms of the lunitidal intervals, that is, with reference to the time of the moon's local meridian passage. The high water lunitidal interval is the time elapsing between the moon's local meridian passage (upper or lower) and the occurrence of high water, while the low water interval is the time elapsing to the occurrence of low water.

Since M_2 is the principal lunar component it follows that the epoch of M_2 gives the approximate lag between the moon's meridian passage and the time of high water, that is, the high water lunitidal interval. The epoch is given in degrees, and to convert it into hours it is only necessary to divide it by the hourly angular speed of M_2 which is 28°.98. For practical purposes it is clear that instead of 28°.98 we may use 29°. The low water lunitidal interval may then obviously be derived by adding to or subtracting from the high water interval one-quater of a tidal day, that is 6.2 hours.

It is important to note that the lunitidal intervals derived from M_2 alone are only approximate, for other of the constituent tides enter in to modify the time of tide. But for tides of the semi-daily type the intervals derived from M_2 are fairly close approximations. For tides of the mixed type the approximations are not so close, while for the daily type of tide, it is clear, the intervals must be derived from the daily components K_1 and O_1, since in this type of tide M_2 is not the principal component. While the lunitidal intervals for diurnal tides can be derived from the harmonic constants, it is an involved matter; and since diurnal tides are almost without exception of small range, this matter is of but little importance to the mariner.

To exemplify the closeness with which the lunitidal intervals may be derived from the harmonic constants for the semi-daily and mixed types of tide we may take New York, Liverpool and Brest as representative semi-daily tides, and San Francisco, Hongkong and Bombay as representative mixed tides. In the table following there are given the lunitidal intervals as derived directly by dividing the epoch of M_2 by 29°, and as determined from long series of tide observations by means of the regular computations on the high and low waters.

STATION.	HIGH WATER INTERVAL.		LOW WATER INTERVAL.	
	From M_2	*From Obs.*	*From* M_2	*From Obs.*
	hours.	hours.	hours.	hours.
New York	8.0	8.1	1.8	2.1
Liverpool	11.1	10.9	4.9	5.3
Brest	3.4	3.6	9.6	10.0
San Francisco	11.4	11.7	5.2	5.1
Hongkong	9.2	9.4	3.0	2.9
Bombay	11.4	11.4	5.2	5.1

On comparing the values of the lunitidal intervals in the above tables it is seen that, as a rule, the high and low water intervals derived by the approximate method from M_2 do not differ by more than two or three tenths of an hour from the values determined from long series of high and low water observations. The greatest difference in the above Table is 0.4 hour.

RANGE OF TIDE.

Formulae can be derived foı determining with precision the various ranges of the tide from harmonic constants. Such formulae, however, are quite involved. For determining the characteristics of the tide for the purpose of the hydrographer and mariner, approximate values are sufficient and such values may be readily determined by means of simple formulae.

Since M_2 is the principle component of the tide in the demi-daily and mixed types, it follows, that the amplitude of this component is approximately equal to the half range of the tide. It is found, however, as might be expected from general considerations, that the amplitude of M_2 is, as a rule, about ten per cent less than the half range of the tide. Hence the mean range of the tide may be derived by multiplying M_2 by 2.2.

Spring ranges of the tide occur about the times of new and full moon, that is when the lunar and solar components conspire. Hence the mean value of the spring range is given approximately by $2.2\ (M_2 + S_2)$. Neap ranges occur about the times of the moon's first and third quarters when the lunar and solar components oppose each other. Hence the mean value of the neap range is given approximately by $2.2\ (M_2 - S_2)$. Similar considerations prove that approximate values of the perigean and apogean ranges are given by $2.2\ (M_2 + N_2)$ and $2.2\ (M_2 - N_2)$ respectively.

In tides of the daily type the semi-diurnal components are small and hence the ranges are derived from the daily components. In tides of this type the greatest rise and fall occurs, not at the times of spring tides or perigean tides but at the times of tropic tides, that is, at the times when the moon is near her fortnightly maximum north and south declination. The mean range of daily tides we may take as being given by 1.5 $(K_1 + O_1)$, while the tropic range we may take as 2.0 $(K_1 + O_1)$.

As examples for determining the ranges of the tide from harmonic constants we may take Boston as representative of the semi-daily tide, San Francisco of the mixed and Pensacola of the daily tides. The following table gives the results from the harmonic constants in the first column under each station, while the second column gives the ranges as derived from tide observations directly:

RANGE FROM HARMONIC CONSTANTS AND OBSERVATIONS.

	Boston		San Francisco		Pensacola	
	Harm. Const.	*Obs. Tides.*	*Harm. Const.*	*Obs. Tides.*	*Harm. Const.*	*Obs. Tides.*
	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.
Mean range	9.6	9.4	3.9	3.9	1.2	1.1
Spring range	11.2	10.9	4.8	4.6		
Neap range	8.1	7.8	3.0	3.1		
Perigean range	11.8	11.2	4.7	4.6		
Apogean range	7.4	8.0	3.1	3.3		
Tropic range					1.6	1.7

A comparison of the ranges derived from the harmonic constants with those derived from the observed high and low waters shows that the simple formulae used give very good approximations. But it is to be emphasized that these simple formulae are intended to give approximations only.

AGE OF TIDE.

Between any astronomical condition or cause and the resulting maximum effect upon the tide, there is usually a lag known as the age of the tide. Thus spring tides come not on the days of new or full moon but a day or two later. A similar lag is found to exist between the moon's first or third quarter and the occurrence of neap tides.

There are various ages of the tide, each designated by its astronomic condition. Of these, however, three ages are of practical importance, namely, the phase age, the parallax age and the diurnal age. The phase age is the interval elapsing between new or full moon and the occurrence of spring tides; it is also the interval between the moon's first and third quarters and neap tides. The parallax age is the interval between the moon's perigee or apogee and the occurrence of perigean or apogean tides. The diurnal age is the interval between the moon's maximum semi-monthly declination and the occurrence of tropic tides.

The existence of various ages of the tide finds reflection in the epochs of the harmonic constants. If, for example, the phase age is zero at any place, it means that the epochs of M_2 and S_2 have the same value. The difference in the epochs thus permits the different ages to be readily computed. For practical purposes we have the following formulae, the age in each case being given in hours: phase age = 0.98 $(S_2^\circ - M_2^\circ)$; parallax age = 1.84 $(M_2^\circ - N_2^\circ)$; diurnal age = 0.91 $(K_1^\circ - O_1^\circ)$.

Various other features of the tide may be derived through the harmonic constants. Those discussed above are the more important ones. The ease with which these may be derived from the harmonic constants should commend the latter to the favorable notice of the mariner and hydrographer.