ACCURATE DETERMINATION OF THE POSITION AT SEA.

by

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PREAMBLE.

It is an incontestable truth that the quality of hydrographic charts depends, mainly, on the quality of the determinations of positions.

Also, if we consider the question solely from the practical point of view, it is evident that safety of navigation depends on the accurate representation of the dangers, the banks, etc., a large number of which are located far from the shore and the positions thereof can be determined astronomically only. *This fact is so self-evident that, in enunciating it, I fear that I may be accused of banality.* Nevertheless, in spite of this obviousness, it will be found that "*astronomical position at sea*" is a subject which is almost entirely neglected in hydrographic bibliography. Therein, the most scientific questions of geodesy are dealt with ; there will even be passages on the deviation of the vertical, gravity measurements and on isostasy, but nowhere, or practically nowhere, will anything be found directly concerning our problem. At most, it may happen that a brief summary of the methods ordinarily employed in navigation will be found and often only a reference to works on astronomical navigation — *materia minor* — and if such treatise on navigation be opened we may almost always search *in vain* for the solution of the following problem: *What astronomical measurements are necessary to fix the position with the maximum of accuracy attainable by observations at sea? What degree of accuracy is it possible to attain?*

Some hydrographers of the United States Coast & Geodetic Survey have recently sought to answer these questions on the subject of the surveys made in the vicinity of the Hawaiian Islands and in connection with the hydrographic survey of Georges Bank (in the North Atlantic about 160 miles to the eastward of Nantucket). In the *Bulletin of the Association of Field Engineers* for the month of December 1930 (*see International Hydrographic Bulletin*, March 1931, p. 66), we find a series of articles devoted to this question (*Star sight positions*: G. D. COWIE and K. T. ADAMS). The question of multiple observations of stars is treated therein and reference is made to the *method of altitude bisectors.* In regard to this method, there is a truly fundamental work entitled: *Sulla teoria e pratica della nuova navigazione astronomica* by Dr. A. ALESSIO, Lieutenant (now Admiral) R. I. N., which was published as a supplement to the July-August 1908 issue of the *Rivista Marittima*, Rome.

In 1919, during the first International Hydrographic Conference (London), Commander ALESSIO tried to bring this question before the Conference, but lack of time and other reasons did not allow it to be discussed. Only a very interesting, but too brief, trace of it remains in the *Report of Proceedings* of the Conference (pages 227-229).

Brief summaries of the method of bisectors have been made in the *Admiralty Manual of Navigation* 1928, Vol. II., pages 105-107, but these are scarcely adequate.

The Author of these notes gave a description of this method in his *Elementi di Navigazione Astronomica* (1919), but in a very elementary form owing to the limitations imposed on a text-book for use in Naval Academies.

Under these circumstances, since the work of ALESSIO is out of print and as there is no translation of it, it appears opportune to set down the principal of this theory in a note in the *Hydrographic Review, for the benefit of hydrographers.*

In this note nothing new is touched upon, with the possible exception of the form of certain demonstrations and some questions of detail; its essential purpose is to direct attention to a question which, in my opinion, is of a strictly hydrographic character and which my colleague ALESSIO had already brought up some time ago.

In conclusion, I would add that the methods which are about to be described here and recommended, are not only the outcome of theoretical considerations but also and *above* all the results of long personal experience obtained in the capacity of navigating officer and hydrographic surveyor, for it is certainly not my nature (and I wish to emphasise this) to delight in abstractions.

1. *DEFINITION. - THE ALTITUDE BISECTOR.*

Let a pair of Sumner lines 11, 22, be considered (fig. 1) obtained by measurements taken at the same place from observations of stars of different azimuths.

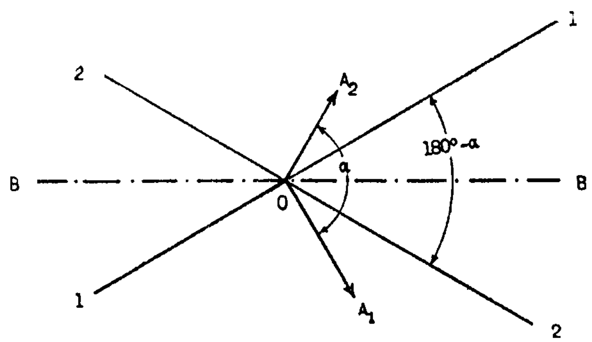


Fig. 1

Let OA_1 and OA_2 be the respective directions of the observed azimuths and let α (difference in azimuth) be the smaller of the two angles formed by the two straight lines corresponding to these directions.

It should be noted that of the two angles formed by the Sumner lines, one is equal to α and the other to $180^\circ - \alpha$. The straight line BOB which divides into two equal parts the angle the value of which is $180^\circ - \alpha$, will be referred to as the *altitude bisector* or, for shortness, as the *bisector*. No doubt can exist regarding the choice of this angle because the bisector under consideration is the one which also equally divides the angle formed by the vectors OA_1 and OA_2.

2. *ERROR OF THE BISECTOR.*

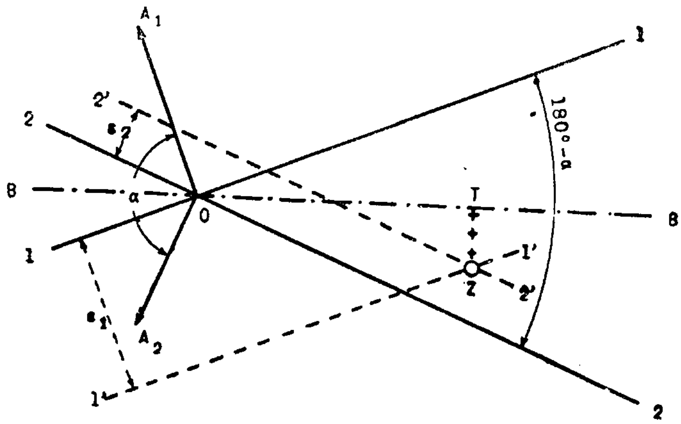


Fig 2.

Let it be assumed that the errors ε_1 and ε_2 (both as to magnitude and sign) are known by which, owing to defects in measurement, the true values of h_1 and h_2 which served to determine the two Sumner lines, are respectively affected. The *true* position of observation which it is desired to determine is not at O but at Z, the intersection of the two lines transferred to their exact positions 1'1' and 2'2' (*see* fig. 2 and 3). In fig. 2 the case is considered where both ε_1 and ε_2 are positive ; in fig. 3 where ε_1 is positive and ε_2 is negative).

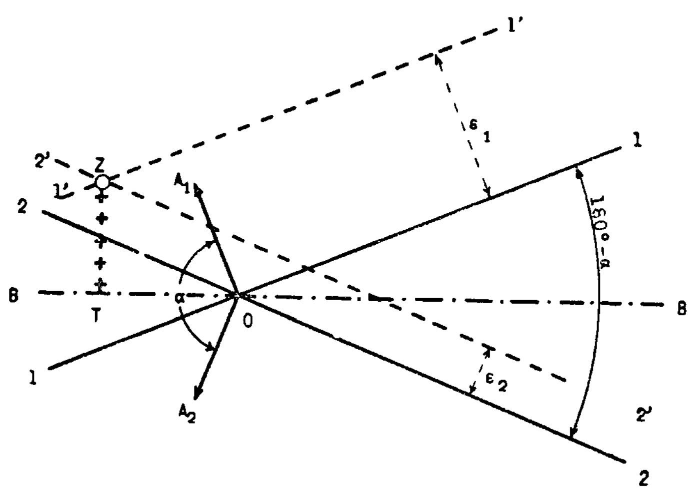


Fig. 3.

From elementary geometric considerations it is evident that the distance $ZT = X_b$ from the *true* point of observation to the bisector is given (in nautical miles) by the formula :

$$(1) \qquad X_b = \frac{1}{2} \operatorname{cosec} \frac{\alpha}{2} \left| \varepsilon_1 - \varepsilon_2 \right|$$

in which $\left| \varepsilon_1 - \varepsilon_2 \right|$ represents the absolute value of the algebraic difference $\varepsilon_1 - \varepsilon_2$ measured in minutes of arc.

In order to deduce important and practical consequences from the preceding considerations and in particular from formula (1), it is necessary to consider the conditions under which the altitudes h_1 and h_2 were measured :

(a) *by a single observer,*
(b) *with the same sextant,*
(c) *with short intervals of time between observations.*

This method of taking observations, which is after all the *normal* method, ensures that the altitudes h_1 and h_2 are *subject to the same systematic error* s due to the coexistence of three sources of error :

(*a*) the personal equation
(*b*) imperfect determination of the index correction,
(*c*) inadequate knowledge of the dip of the horizon.

Further, we must assume that all other systematic errors due to imperfections of the sextant (the value of which depends on the magnitude of the measured angle) have been accurately determined, and we will assume also that the corrections pertaining thereto have been properly applied.

The requirements governing the total duration of the group of observation, *i.e.* the necesstiy for taking the measurements during a short interval of time, allow us to make certain that the following conditions obtain :

(1) that the dip of the horizon has not undergone any change ;

(2) if the observations are taken from a vessel under way one of the lines may be transferred to the zenith of the other without introducing any *appreciable error due to the transfer.*

This being granted and neglecting the systematic error due to the anomalies of astronomical refraction (an error which is really quite negligible when the observed altitudes *are not very small*) we may take :—

$$\varepsilon_1 = s + x_1$$
$$\varepsilon_2 = s + x_2$$

in which x_1 and x_2 represent the true *accidental errors* made in the measurements ; and consequently the equation (1) becomes :

$$(2) \qquad X_b = \frac{1}{2} \operatorname{cosec} \frac{\alpha}{2} \left| x_1 - x_2 \right|$$

If we take the bisector as the *line of position of the observer*, the quantity

X_b is a measure of the error (or parallel displacement) of this line and it is therefore termed the *bisector-error.*

From formula (2) we deduce the following fundamental proposition : -- *the altitude bisector* (determined by means of a pair of altitudes observed in the manner indicated above) is a *line of position independent of the systematic error, whatever the magnitude of the latter*; the error of the bisector depends solely on the accidental errors made in the measurements.

1st NOTE : Let it be stated at once that we assume implicitly in the arguments we shall have occasion to make, that the conditions of observation, which we have described in this paragraph and which are necessary for the employment of the bisector method, have been fulfilled, particularly in the case where it is no longer a question of two altitudes but of a set of n altitudes.

2nd NOTE : We have implicitly assumed and will assume also in that wnich follows, that the dip of the horizon, though it may differ from that given in the correction tables, remains the same in all azimuths of observation. ARAGO in an article which appeared in the *Connaissance des Temps* of 1827, after analysing the observations made by E. PARRY in his earlier polar expeditions, by B. HALL in the China Sea as well as by P. GAUTTIER in the Mediterranean, concluded that *there was no reason to doubt the accuray of such assumption.* However, we find that in RAPER's treatise *Practice of Navigation,* several exceptional cases are mentioned on which different values of the dip have been observed at different points of the horizon. It will be demonstrated that a rational interpretation of the observations may reveal the existence of this exceptional anomaly. This is one, and not the least, of the advantages of the rational employment of the bisector method.

3. *MEAN BISECTOR ERROR.*

Having therefore demonstrated that the bisector error is of purely accidental character and as such falls within the realm of the calculation of probabilities, we may discuss the *accuracy* of the determinations by reasoning, not from the true error X_b — knowledge of which we assume by hypothesis only and the evaluation of which can only be considered arbitrary — but by considering the *mean error* M_b, a factor which the *theory of errors* permits us to estimate with sufficiently close approximation when based on the results furnished by the observations.

A fundamental theorem in the theory of the combinations of observations (*a*) permits us to pass directly from X_b to M_b; thus, taking m_1 and m_2 as the mean errors of the two altitudes, we have :—

$$M_b = \frac{1}{2} \operatorname{cosec} \frac{\alpha}{2} \sqrt{m_1^2 + m_2^2}$$

Let us assume (as is permissible in the case of a pair of altitudes observed in the manner indicated) that $m_1 = m_2 = m$, we obtain the formula which gives the *mean bisector error*, a factor which, in a certain sense, we may take as the measure of the degree of accuracy of this line of position.

$$(3) \qquad M_b = \frac{m}{\sqrt{2}} \operatorname{cosec} \frac{\alpha}{2}$$

The most accurate determination of the bisector obtains when $\alpha = 180^\circ$ or, in other words, when two stars are observed in opposed vertical semi-planes. In this case

$$M_b = \frac{m}{\sqrt{2}}$$

If $\alpha = 90^\circ$, then $M_b = m$, that is, by observing two altitudes with a difference in azimuth of 90°, we obtain a bisector which is precisely affected by the mean accidental error of observation. It may be stated further that if a sufficiently good determination is made with $\alpha = 60^\circ$, we shall have $M_b = m\sqrt{2}$ *Differences of azimuth of less than* 60° *should therefore be avoided.*

4. *POSITION BY FOUR ALTITUDES. MEAN ERROR OF THIS DETERMINATION.*

To those who make practical observations at sea it is well-known that the systematic error, *particularly as a result of anomalies in the dip of the horizon*, may assume very large values, and also that its *certain* elimination is a *necessary condition* for obtaining an accurate position.

From this follows the necessity of not using any lines of position for astronomical determinations of position at sea except those which are free from systematic error; in other words, the bisectors. Two bisectors determine the fix. Three altitudes are necessary and sufficient to determine two bisectors, but since it is advisable to have a *check*, one should not consider, as a *normal determination* of astronomical position at sea, any fix obtained with *less than* four altitudes. Consequently we shall give priority to determinations of this kind.

However, before commencing to treat this question, we believe it necessary to draw attention once more to a *point of capital importance.*

The theory which we put forward is valid on condition that all of the altitudes employed for the determination of the position, regardless of their number, belong to a set of homogeneous observations of equal accuracy, i.e. *obtained under the conditions described in paragraph* 2.

The application of the theory to a case where the altitudes have been observed under different conditions (i.e. *either by different observers, with different sextants, etc.*) *can only give rise to uncertainties and to deplorable errors.*

Having chosen two altitudes from amongst the four observed altitudes, one bisector may be deduced from these two altitudes. A second bisector will be obtained from the two remaining altitudes. We thus obtain two bisectors which are totally independent of each other and which, by their intersection, determine the observation point.

At what azimuths is it necessary to observe the four altitudes in order to obtain the maximum accuracy from this set of observations ?

To answer this question it is necessary to determine the *mean error of the intersection of the two bisectors.* For this purpose we shall make use of a procedure employed in geodesy (*b*) whenever the point P (fig. 4) is determined

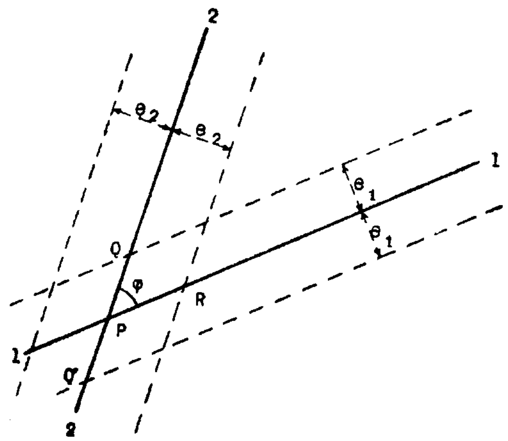


Fig. 4.

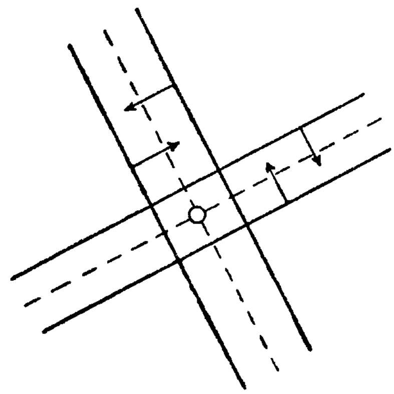


Fig. 5.

by the intersection of the two straight lines 1-1 and 2-2, the respective directions of which are correct and which cut at an angle of φ to each other, but the positions of which are uncertain owing to the fact that parallel displacements or mean errors e_1 and e_2 may occur. Let the effects produced by each of the above-mentioned displacements be considered *separately*. If one only of the straight lines is displaced — for instance the straight line 1-1 — the point P will move along the other straight line 2-2 and will fall at the point Q (or Q'). The displacement of the point is therefore measured by the vector PQ and we shall have

$$PQ = e_1 \operatorname{cosec} \varphi$$

If, in its turn, the straight line 2-2 is displaced while the position of the straight line 1-1 remains fixed, the point will move along the straight line 1-1 and reach R. The displacement will be measured by the vector PR.

$$PR = e_2 \operatorname{cosec} \varphi$$

According to the theory of combination of observations, the mean error E of the point of intersection P of the two given straight lines is assumed to be *the square root of the sum of the vectors* PQ and PR, determined above.

$$E = \sqrt{\overline{PR}^2 + \overline{PQ}^2} = \operatorname{cosec} \varphi \sqrt{e_1^2 + e_2^2}$$

In the case under consideration we have, in accordance with formula (3) :—

$$e_1 = \frac{m}{\sqrt{2}} \operatorname{cosec} \frac{\alpha_1}{2}$$

$$e_2 = \frac{m}{\sqrt{2}} \operatorname{cosec} \frac{\alpha_2}{2}$$

in which α_1 and α_2 are the differences in the azimuths relative to each of the pairs of the Sumner lines, and m is the accidental error of each altitude pertaining to the group of four altitudes under consideration.

Consequently, if E_4 be the mean error obtained by four Sumner lines, we have :—

$$(4) \qquad E_4 = \frac{m}{\sqrt{2}} \operatorname{cosec} \varphi \sqrt{\operatorname{cosec}^2 \frac{\alpha_1}{2} + \operatorname{cosec}^2 \frac{\alpha_2}{2}}$$

It follows from this formula that, to obtain the most advantageous conditions, it is necessary to combine the groups of Sumner lines in such a manner that the angle of intersection φ shall be as near 90° as possible and that the angles α_1 and α_2 shall be near 180°. The most favourable conditions obtain when the azimuths of observation are 90° apart since, by pairing the Sumner lines observed at opposite azimuths, we obtain, for a given value of m, the minimum of e_1 and of e_2, and, further, the two bisectors will intersect at right angles. (See for example figure 5). We then have $E_4 = m$.

5. *POSITION BY ANY EVEN NUMBER OF ALTITUDES.*

A truly accurate determination of position must not be limited to the observation of two pairs of stars. Normally, an even number of stars $2n$ ($n > 2$) will be observed which satisfy the requirement that they be located in vertical semi-planes approximately equidistant from each other right around the horizon. However, in practice, the pairs observed will not generally exceed three or four, since a limit is imposed not only by the distribution in azimuth of the observations but by the *magnitude* of the stars which may be observed by sextant and by the necessarily brief duration of the observations (for which favourable conditions exist at morning and evening twilights). We plot the bisectors corresponding to pairs of altitudes observed at opposite azimuths, or considered as such (*within limits of* 20° *or* 30°). It follows that each bisector will be affected by the same accidental error M_b, in such a manner that if there are *three* bisectors, the most favourable position will be that which ie located at distances proportional to the corresponding sides of the triangls formed by the bisectors. On the other hand, if there are more than three bisectors, it will be necessary, according to the theory of errors, to fix the point in such a manner that the sum of the squares of the distances to the various bisectors shall have a minimum value. We wish to emphasise, however, that all of this is *of no practical value* and at most is capable of furnishing an argument for a discussion which can be just as learned as it is useless. In every case, the triangle or the polygon formed by the bisectors is so small that it will suffice to locate the observed fix by *simply estimating* its position.

NOTE : We have read somewhere that the method of bisectors for the determination of position is applicable only to *simple cases i.e.*, when a small number of stars is observed; but that, if numerous altitudes be employed graphical plotting becomes complicated and consequently tends to confusion. We should like to suggest two simple devices by means of which the graph may be given the desired clearness in cases of apparent confusion :—

1) The choice of the pair of Sumner lines best suited for derivation of the bisectors can be accomplished by taking a separate compass rose on which are plotted the various azimuths of observation.

2) When the superposition of the various Sumner lines brings about a state of confusion which mars the simplicity of the graph, a new graph should be made applying the following rule :— The Sumner lines are displaced parallel to themselves, *all* by the same quantity k (for instance 2' or 3') and *all* in the same sense with regard to their respective azimuths, or else *all* in the opposite direction. In this manner a graph will be obtained with an *empty space* in the central part, and this *empty space* will favour the clearness of subsequent operations. This operation is equivalent to augmenting (algebraically) the systematic error s, by which each altitude is affected, by a quantity k (positive or negative). Naturally, it is necessary to take this artificial modification into account subsequently when computing the value of s.

6. *DETERMINATION OF THE POSITION BY MEANS OF THREE ALTITUDES. MEAN ERROR OF THIS DETERMINATION.*

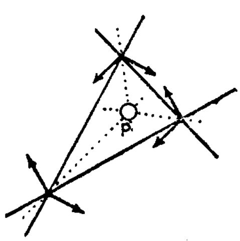


Fig. 6.

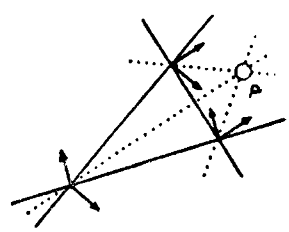


Fig. 7.

The position of the observer is determined by the meeting point of the three bisectors, amongst the six possible bisectors (internal or external) of the triangle formed by the three Sumner lines. The choice of the three bisectors is made in accordance with the rule set out in paragraph 1. Let us refer, for example, to figures 6 and 7. The determination of the mean error of the point thus obtained is expressed in a general manner as follows :—

A point P (fig. 8) is determined by the meeting point of the three bisectors of a given triangle ABC, the sides of which are well determined with respect to their direction but uncertainty prevails with regard to their positions owing to possible individual parallel displacements or mean errors, e_1, e_2, e_3. Let us consider the effects produced by each of the above-mentioned displacements *separately*.

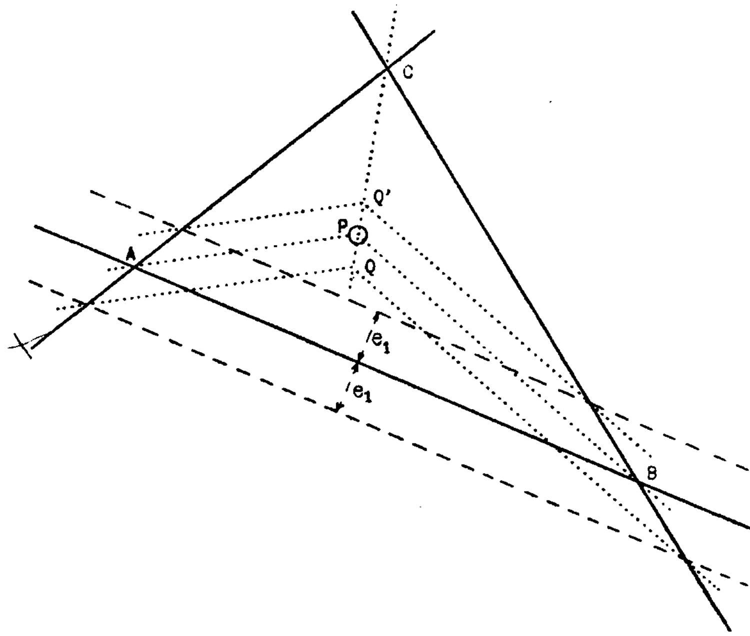


Fig. 8.

Let us displace the side AB without changing the positions of the other two sides. The side AB being displaced parallel to itself by the quantity e_1, carries with it the two bisectors of the angles A and B; the point of intersection P of these bisectors, *moving along the bisector issued from the opposite angle* C arrives at Q (or Q'). Thus the vector PQ measures the displacement of the point. It may readily be seen that we have :

$$PQ = \frac{e_1}{2} \sec \frac{A}{2} \sec \frac{B}{2}$$

in which

$$A = \widehat{BAC} \quad , \qquad B = \widehat{ABC}$$

In like manner if we displace the side BC without changing the positions of the other two sides, we obtain a displacement PR along the bisector drawn from the opposite point A and the value of this displacement will be :

$$PR = \frac{e_2}{2} \sec \frac{B}{2} \sec \frac{C}{2}$$

Finally, if we consider the displacement of the third side CA we see that the vector PS which measures the displacement corresponding to the point P along the bisector from the point B will be given by the formula :—

$$PS = \frac{e_3}{2} \sec \frac{C}{2} \sec \frac{A}{2}$$

According to the theory of the combination of observations, the mean error E of the point P is assumed to be equal to the square root of the sum of the squares of the vectors PQ, PR, and PS, thus determined :

$$E = \sqrt{\overline{PQ}^2 + \overline{PR}^2 + \overline{PS}^2}$$

$$= \frac{1}{2}\sqrt{e_1^2 \ sec^2 \frac{A}{2} \ sec^2 \frac{B}{2} + e_2^2 \ sec^2 \frac{B}{2} \ sec^2 \frac{C}{2} + e_3^2 \ sec^2 \frac{C}{2} \ sec^2 \frac{A}{2}}$$

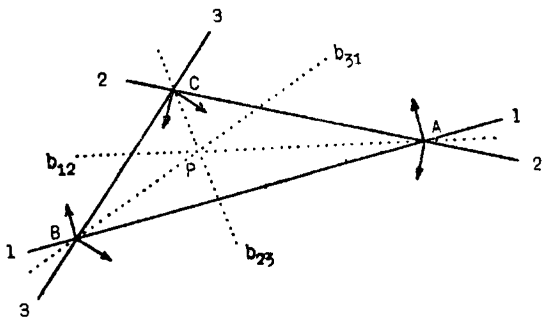


Fig. 9.

Let us apply this equation to the triangle ABC formed by the three Sumner lines 11, 22, 33 and let b_{12}, b_{31}, b_{23} be the three bisectors which we obtain by combining the three given lines in pairs (fig. 9). Instead of taking the three angles A, B, and C of which the bisectors are b_{12}, b_{31}, b_{23}, let us consider their supplements :—

$$\begin{aligned} 180^0 - A &= \alpha_{12} \\ 180^0 - B &= \alpha_{31} \\ 180^0 - C &= \alpha_{23} \end{aligned}$$

which measure the differences in the azimuths of observation of the three pairs of altitudes ; further, let the mean accidental error which affects each of the altitudes of the group be designated m, as before, $(e_1 = e_2 = e_3 = m)$ and let E_3 be the mean error of the point obtained by means of the three altitudes. We shall then have :—

$$(5) \qquad E_3 = \frac{m}{2}\sqrt{cosec^2 \frac{\alpha_{12}}{2} \ cosec^2 \frac{\alpha_{31}}{2} + cosec^2 \frac{\alpha_{31}}{2} \ cosec^2 \frac{\alpha_{23}}{2} + cosec^2 \frac{\alpha_{23}}{2} \ cosec^2 \frac{\alpha_{12}}{2}}$$

It may be shown that the minimum value of E_3, for a given value of m, occurs when

$$\alpha_{12} = \alpha_{31} = \alpha_{23} = 120^0$$

or, in other words, when the three altitudes are taken with a difference of 120° in the azimuths. In such case :—

$$E_3 = \frac{2}{\sqrt{3}} \, m = 1{,}15 \, m$$

7. *EVALUATION OF ERRORS OF OBSERVATION.*

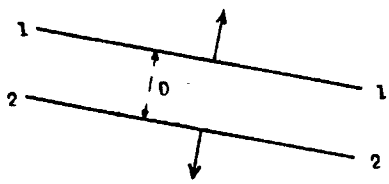


Fig. 10.

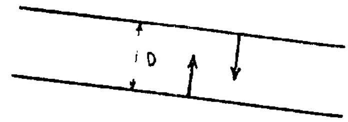


Fig. 11.

Let us consider the problem "*ab ovo*".

The distance D (fig. 10 & 11) which separates the two Sumner lines 1 and 2 *of opposite azimuths* (*which distance we assume to be positive in the case where the arrows indicate that the azimuths are directed towards the exterior of the space between the two straight lines* as in fig. 10 and negative if the arrows point inwards as in fig. 11), is manifestly equal to the sum of the errors

$$\varepsilon_1 = s + x_1 \quad , \quad \varepsilon_2 = s + x_2$$

which affect the altitudes h_1 and h_2 which were used to determine the Sumner line in question :—

$$D = 2s + x_1 + x_2$$

Consequently the distance d of a point on the bisector b from each of the straight lines is :—

$$d = s + \frac{x_1 + x_2}{2}$$

We may say then that the distance d of any point on the bisector from one or other of the Sumner lines (considered as positive in the case shown in fig. 10 and negative in the case of fig. 11) is a measure, both in the magnitude and sign, of the *systematic error* s *within the true accidental error* $\frac{x_1 + x_2}{2}$ to which corresponds the *mean error* $\sqrt{\frac{m_1^2 + m_2^2}{2}}$ and which, in the case $m_1 = m_2$, becomes $\frac{m}{\sqrt{2}}$

Extending this line of reasoning to n pairs of Sumner lines deduced from $2n$ observations of equal accuracy, we may say that each pair gives a value of the systematic error s within a mean error M given by the equation

$$(6) \qquad M = \frac{m}{\sqrt{2}}$$

in which m is the mean accidental error of observation of the altitudes of the group.

The first result of this reasoning is that the arithmetic mean N of the n values of d should be considered as *the most probable value* of the unknown s.

$$N = \frac{d_1 + d_2 + \ldots + d_n}{n}$$

The determination of the unknown m offers no greater difficulty.

The theory of errors shows us how to determine the value of M as a function of the differences v between the various values $d_1, d_2 \ldots\ldots d_n$ and their arithmetic mean N and of the number n of the values d.

In fact, if we take :—

$$\begin{aligned} v_1 &= d_1 - N \\ v_2 &= d_2 - N \\ &\ldots\ldots\ldots \\ v_n &= d_n - N \end{aligned}$$

the theory of errors gives :—

$$(7) \qquad M = \sqrt{\frac{[v\,v]}{n-1}}$$

in which formula in accordance with ordinary practice the symbol $[v\,v]$ indicates the sum of the squares of the differences v ;

$$[v\,v] = v_1^2 + v_2^2 + \ldots + v_n^2$$

From formulas (6) and (7) we nave :—

$$\frac{m}{\sqrt{2}} = \sqrt{\frac{[v\,v]}{n-1}}$$

and consequently

$$(8) \qquad m = \sqrt{\frac{2}{n-1}}\sqrt{[v\,v]}$$

Thus a formula has been obtained which permits us to determine the value of the *mean accidental error which affects the altitudes of the group under consideration.*

(9) For two pairs of Sumner lines we have therefore :—

$(n = 2)$, $m = 1.4\sqrt{[v\,v]}$

for three pairs we have :—

$(n = 3)$, $m = 1.0\sqrt{[v\,v]}$

for four pairs we have :—

$(n = 4)$, $m = 0.8\sqrt{[v\,v]}$

Let us note finally that another theorem permits us to estimate the accuracy of the unknown s, obtained, as we have stated, by taking the arithmetic mean of the values d.

It is well known that the mean error of the *arithmetic* mean (which error will be called p_s) is given by the formula :—

$$(10) \qquad p_s = \sqrt{\frac{[v\,v]}{n\,(n-1)}}$$

We may therefore take in the usual manner :—

$$s = N \pm p_s$$

and we have thus :—

$$(11) \quad \begin{cases} (n = 2) \quad , \quad s = N \pm 0.7 \sqrt{[vv]} \\ (n = 3) \quad , \quad s = N \pm 0.4 \sqrt{[vv]} \\ (n = 4) \quad , \quad s = N \pm 0.3 \sqrt{[vv]} \end{cases}$$

In practice the procedure which we have just described for the determination of m and p_s (formulas (8) and (10)) is applicable also when the two altitudes which determine each bisector have not been observed in exactly opposite azimuths. *In order to give some guide it may be stated that the condition* $\alpha = 180°$ *is sufficiently fulfilled if within* 20° *or even* 30°. But, in such case, it is necessary to take as the distance d that which separates one or the other Sumner line from the point O on the bisector which point is the closest to the position P of the observer (fig. 12).

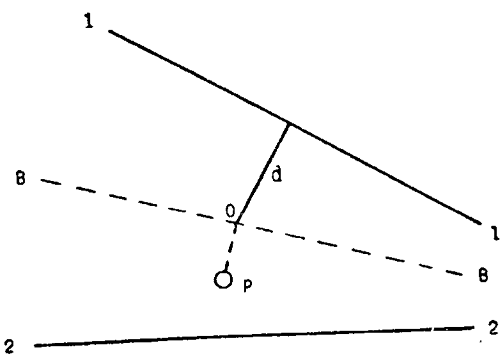


Fig. 12

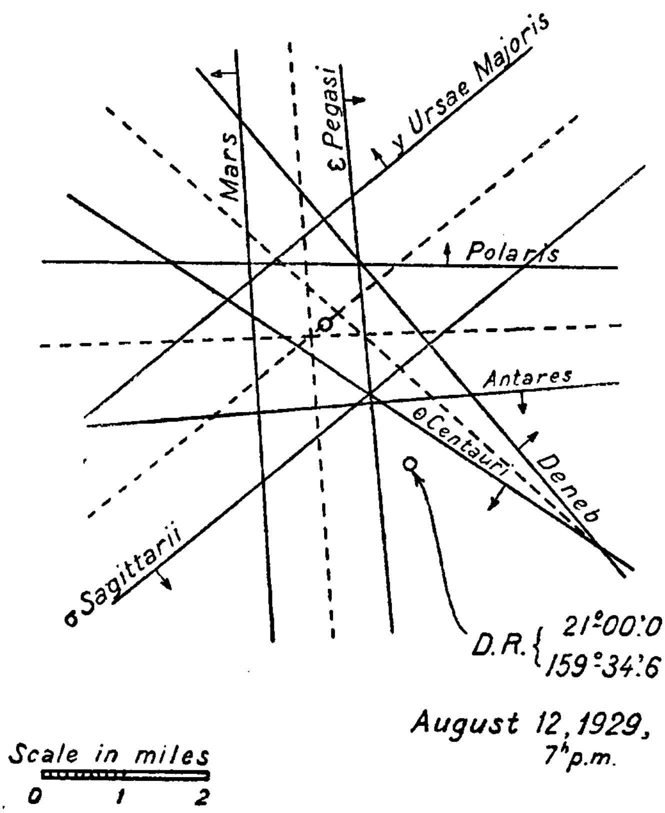
It is evident that the values of m and p_s calculated for small values of n (formulas (9) and (10)) should be taken *"cum grano salis"* in the sense that they should only be interpreted as *giving an indication of the order of magnitude of the errors.* In particular the determination of m and of p_s made only by means of two pairs of Sumner lines, remains but a rough approximation ; but it is almost always sufficient to show up possible important anomalies in the observations. In this case it is easy to see that the first formula under (9) and the first under (11) are equivalent to the following :—

$$(n = 2) \quad \begin{cases} m = \pm (d_1 - d_2) \\ p_s = \pm \dfrac{d_1 - d_2}{2} \end{cases}$$

This very simple check is *not possible, however, when the determination of the position has been made with only three Sumner lines.* In that case, indeed, one is led to attribute to the systematic error s the value of the distance from the meeting point of the bisectors to each of the three Sumner lines (which distance is the same for all three lines) and to the accidental error a value of zero. Consequently, if the errors made in the angles of dip are different at different azimuths (*see* 2nd Note of paragraph 2) and if gross errors have been made in the observations (such as collimation deduced from a false horizon, error in reading the sextant or the chronometer, etc...) the observer has no possible means of knowing that they have been made.

It is for this reason that a fix with three lines should be obtained *only when it is impossible to get others.*

Let us give an example for $n = 4$ with the data obtained from the graph No 6, page 19, cited in the *Bulletin of the Association of Field Engineers,* December 1930 (Fig. 13).



$$\left.\begin{array}{l} d_1 + 0'.6 \\ d_2 + 0'.8 \\ d_3 + 1'.0 \\ d_4 + 0'.7 \end{array}\right\} \quad , \; s = +\,0'.8 \pm 0'.1 \; ; \quad m = \pm\,0'.2$$

Fig. 13.

8. *SOME PRACTICAL CONSIDERATIONS ON THE OBSERVATION OF ALTITUDES.*

Let us add a few words on observing instruments. We consider as superfluous, and consequently improper, the use of special reflecting circles, such as the BORDA circle, the AMICI-MAGNAGHI circle, the PISTOR and MARTINS circle (Spiegel-Prismenkreis), etc.

In our opinion, *a sextant of good construction, carefully tested and well adjusted*, is always preferable. The sextant is an instrument the use of which is very familiar to seamen and therefore to hydrographic surveyors. This in itself is a prime requisite for good observations.

It would be difficult to imagine a more handy instrument and one better balanced for this purpose. May I also be allowed to express an opinion against the introduction of certain modern mechanical improvements made for the purpose of facilitating the reading of the angles or for any other *convenience*, because I believe that with these, new sources of error have been introduced.

The sole innovation which appears desirable, in the ordinary type sextant, is the adoption of a non-tarnishing graduation (See *Hydrographic Review*, Vol. VII. November 1930, p. 204), and of the prismatic telescope instead of the usual astronomical telescope. The preference accorded to the prismatic telescope is not due to the desire to obtain greater magnification (on the contrary it appears advisable, for various reasons, not to exceed the ordinary magnification of 6) but in order to obtain the greatest possible field with the maximum illumination: a quality which is highly desirable for observations at twilight and at night.

We have stated that the sextant should be well tested and in so doing we advise against taking into account certain correction tables issued by the manufacturers and other persons. The sextant should be examined, tested and adjusted personally by the hydrographic surveyor, neither more nor less than is done by the astronomer for the instruments in his own observatory. Consequently the table of instrumental errors should be calculated by the surveyor himself in accordance with his own personal observations.

It is superfluous to add that it is always best to choose one of the best sextants from amongst the good makes.

Finally we should like to be permitted to make a recommendation. In certain articles and in certain treatises, the idea is put forward of giving a certain *weight* to the observations: weights which are based solely on the *opinion* of the observer. It appears to me that that *opinion* only too often, if not always, is *purely arbitrary* and I do not hesitate to express myself as definitely opposed to this practice which I believe to be an abuse.

For the same reason, I am of the opinion that in principle one should never reject an observation on the sole pretext that the Sumner line shows a somewhat large discrepancy.

The assessment of errors made *a posteriori* in accordance with the rule discussed in paragraph 7, should be the only method employed, since it constitutes a well-considered and peremptory judgment of the quality of the determination of position, considered as a whole and not in its elements; for it is only in this manner that one may assume, *without the intervention of deplorable arbitrariness* that the position determined from a given group of observations may be considered as approximating more or less a certain quantity (*c*).

NOTES.

a) If *the true error* X, which affects the results ot an experimental measurement may be expressed by the summation :—

$$X = k_1 x_1 + k_2 x_2 + \ldots + k_n x_n$$

in which x_1, $x_2 \ldots x_n$ are *the true errors* made in n observations, of which *the mean errors* are respectively m_1, $m_2 \ldots m_n$, *the mean error* M of the result will be : –

$$M = \sqrt{k_1^2 m_1^2 + k_2^2 m_2^2 + \ldots + k_n^2 m_n^2}$$

b) *See* W. JORDAN, *Handbuch der Vermessungskunde,* 3rd edition, Part I, Chapter V., para. 93.

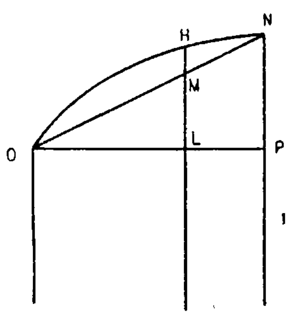
c) In order to obtain a sufficiently approximate idea of the magnitude of the *true* errors which are committed in observing a series of altitudes, we may employ a graphic process which it appears opportune to describe as the conclusion of these notes. We do not pretend to attribute any theoretical importance to this procedure, but simply wish to invite attention to its *practical value.* The procedure is based on a simple artifice which, however, to the best of our knowledge, has not been mentioned up to now in any treatise on nautical astronomy.

Let us assume that the altitudes h_0, h_1, $h_2 \ldots h_n$ of a star have been observed at the successive instants t_0, t_1, $t_2 \ldots t_n$.

Let us suppose then that, point by point (with the times t_0, $t_1 \ldots$ as abscissae and the altitudes h_0, $h_1 \ldots$ as ordinates), the arc ON (fig.) of the curve representing the altitudes as a function of the times be constructed.

Then draw the straight line joining the extremities O $(t_0\, h_0)$ and N $(t_n\, h_n)$ of this arc.

For any altitude h comprised between the extreme altitudes of the series having the figurative point H on the arc under consideration, we may take :—



$$h = h_0 + LM + MH$$

But

$$LM = \frac{PN}{OP} \times OL$$

whence

$$LM = \frac{h_n - h_0}{t_n - t_0} \quad (t - t_n)$$

in which formula t is the instant corresponding to h.

Let us designate the constant quantity $\dfrac{h_n - h_0}{t_n - t_0}$ by c and further let $MH = \alpha$; we then have :—

$$h = h_0 + c\,(t - t_0) + \alpha$$

Except in the case where the series has been observed near the culmination of the body, the term

$$c\ (t - t_0)$$

varies very rapidly as t varies. On the contrary the term α which is of the second order with respect to the quantity $(t - t_0)$, varies much more slowly and remains very small during the whole duration $(t_n - t_0)$ of the series, even though the total interval be somewhat large.

We may draw several interesting practical conclusions from these considerations. It is not ordinarily possible to construct a curve which represents the values of h as functions of the times on a sufficiently large scale – at least without making the drawing of exaggerated dimensions. *But we may, on the other hand, plot on a sufficiently large scale the arc of the curve which represents the quantity α as a function of the times, without incurring the inconvenience which we have just mentioned.* It should be noted that the curve of the α's is none other, in fact, than the curve of the values taken by the segments MH comprised between the chord OP and the curve of altitudes in Fig. and consequently its aspect is identical with that of the latter curve.

The value of α corresponding to an altitude h, is determined by the general formula :—

$$\alpha = (h - h_0) - c\ (t - t_0)$$

We therefore calculate a series $\alpha_0, \alpha_1, \alpha_2 \ldots$ of quantities connected with the observed altitudes $h_0, h_1, h_2 \ldots$ by means of the following formulas :—

$$\begin{aligned}
\alpha_0 &= (h_0 - h_0) = zero \\
\alpha_1 &= (h_1 - h_0) - c\ (t_1 - t_0) \\
\alpha_2 &= (h_2 - h_0) - c\ (t_2 - t_0) \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
\alpha_n &= (h_n - h_0) - c\ (t_n - t_0) = zero
\end{aligned}$$

Having calculated the values of α and drawing on a plane the two orthogonal axes, we then plot as abscissae the time intervals $t_0, t_1 \ldots t_n$, on a sufficiently large (but not too large) scale. In order to give an approximate guide we may state that in each case a scale may be chosen so that it be possible to read within a tenth of a minute; it will suffice, therefore, to use for instance a scale of ½ centimetre per minute. We then lay off on the corresponding ordinates the respective values of α, calculated as stated above, using a scale which permits readings to within one-tenth of a minute of arc; for instance, ½ centimetre for one minute of arc. Often, if the duration of the series is not too great, a scale of 1 c/m per minute of arc may be taken.

We then obtain on the figure as many points as there are observations, and a curve may be drawn in by hand which passes as nearly as possible through the points, thus *compensating* the result of the observations, which are naturally affected by accidental errors. It should be noted that the curve should contain no point of inflection, except at the point corresponding to the passage over the prime vertical or that which corresponds to the maximum elongation of the body under consideration. But in these special cases the curve will *practically* coincide with a straight line. In all other circumstances the curve will have a parabolic character.

Let us note also that the curve need not necessarily pass through the extreme points (α_0 t_0) and (α_n t_n) since the observations which correspond to these points, even as all others in the series, may be affected by accidental errors.

A study of the curve of α's will permit us to appreciate immediately with sufficient accuracy, the magnitude of the *true* accidental errors made during a series of observations. In fact the discrepancies between the ordinate points α_0, α_1, $\alpha_2 \ldots \alpha_n$ and the regular curve (which may be called the *mean curve*) give a measure of the above-mentioned errors.

The same diagram may also be employed with great ease and sufficient accuracy to determine the value h of the altitude at any instant comprised within the interval $t_n - t_0$.

In fact, as we have already seen, for any altitude comprised within the interval ($t_n - t_0$) we have

$$h = h_0 + c\,(t - t_0) + \alpha$$

The value of the term $c\,(t - t_0)$ is determined by making a numerical calculation, since we know the value of the constant c which has been used to determine the values of α ; on the other hand the value of the term α is deduced by reading directly from the diagram the ordinate of the mean curve corresponding to the instant t.