THE USE OF AIRCRAFT IN SURVEYING

by

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In the preceding Notes (See *Hydrographic Review* IV-2, V-1, VII-1) we have discussed various methods by which photographs taken from aircraft may be utilised without resorting to the very expensive and elaborate restitution apparatus, a great many of which have been invented and of which numerous reproductions have also been given in the *Hydrographic Review*. These methods appear to be of all the more interest to Hydrographic Offices because many of these do not possess restitution apparatus and, further, the special conditions of hydrographic surveying make their use less indispensable. It may be useful, therefore, to give greater details.

I. *THE TAKING AND PREPARATION OF PHOTOGRAPHS.*

Photographs taken from aircraft are very valuable aids, which are becoming more and more appreciated and used by Hydrographic Expeditions; they should allow the topographer's field work to be reduced to very little though supplying elements for more complete and perfect topography than would be possible without their aid (1). We do not believe, however, that the field work of the topographer should be entirely dispensed with, at least under normal conditions; the manner in which the operations should be carried out appears to us to be as follows:

The photographs are taken from an airplane which maintains as constant an altitude as possible and follows rectilinear courses, not making any exposures at the turns and maintaining the optical axis of the camera as nearly vertical as possible. In this article we assume that the optical axis does not deviate from the vertical by more than a small number of degrees, generally not more than 5° or 6°. The successive photographs should have an overlap of about 2/3. They should mostly contain a representation of a portion of the coast line.

The topographical operator, when he receives proofs of these photographs, should visit the coast and assure himself that the details of interest to mariners are clearly visible on them. It may happen that small rocks or objects may be indistinct and thus risk being overlooked in the work of restitution, in spite of their great interest from the maritime point of view. The topographical operator should make them stand out on his proof; at the same time he should mark the positions of the stations which he will make along the coast at places which can be clearly recognised on the photographs. He

(1) We are not speaking here of the utility, equally very important, of the photographs in searching for rocks and other features of submarine topography.

will determine his position by means of triangulation points and will note ths azimuth and angles of altitude of some prominent marks or of clearly visible points in the interior which he desires to fix; at least one of these pointe should be clearly recognisable on the photographs.

These stations may be much fewer than would be the case if no photographs were available; two or three on each photograph will suffice. The two most widely separated stations on the same negative give two points of known position and altitude. We shall have the fu.ther advantage that these will generally be at the same altitude, *i. e.* that of sea level, a difference of two or three metres in altitude being quite negligible in this case. The negative may not contain any triangulation points, but the operator will always have been able to determine the position and altitude of some point which is clearly marked on the photograph by means of angles to the stations made and to select this point so that its altitude will be as small as possible compared to that of the plane, less than 1/20th for example, and so that it may be sufficiently far removed from the line joining the two extreme stations on the photograph and from the circumference of the circle which passes through the stations and the principal point (centre) of the photograph (See *Hydrographic Review*, Nov. 1927, p. 92 and May 1930, p. 105).

It may therefore nearly always be assumed that we shall know the positions and altitudes of three points on each photograph without the necessity of any very special work to obtain this result, but simply by taking advantage of the indispensable examination of the coast to be surveyed which the topographer must make. The necessity, in restitution, of determining the position and altitude of three points on each photograph, which was so difficult in terrestrial surveys and which one seeks to avoid at any price, is, on the contrary, very easily and almost necessarily effected on photographs taken by planes for use in hydrographic surveys.

We should like to point out several rapid and very simple, almost entirely graphic, processes which appear to us to be suitable for use in this case for making the topographical survey of a coast.

The fact that on each negative the positions of two points at the same altitude are known simplifies matters, but this is by no means essential.

II. *RADIAL METHOD.*

Let fig. 1 represent a negative A' taken from station S; — it is assumed that we know its principal point O, *i. e.* the foot of the perpendicular dropped from S onto its plane (plate perpendicular), as well as the focal length f of the objective (1) (principal distance).

Let V' be the point of intersection of the vertical from S and the negative A'. Draw SI the bisector of the angle OSV' and consider the horizontal

(1) The determination of O and f should be made very carefully. On this subject *see Hydrographic Review*, May 1930, page 101.

In this article we have made reference continually to the negative because it is on this that the measurements are most accurate; but in practice use may be made of a diapositive or even of a print, provided certain precautions are taken.

plane A passing through I. This plane is at the same distance from the point S:

$$SV = SO = f$$

as the negative A'. It occupies the position which would have been occupied by the photographic plate if the optical axis had been vertical at the instant of exposure. In order to compare the negative A' with the photograph which would have been obtained with the plate in position A, let us turn the plane A' onto the plane A by revolving it about the horizontal line passing throug I.

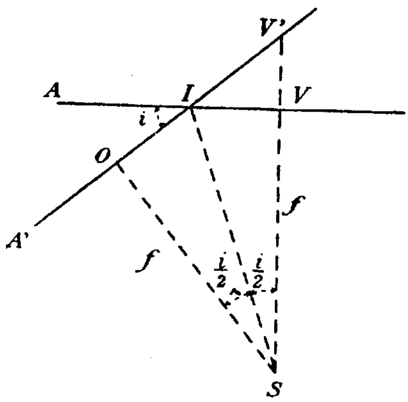

Fig. 1.

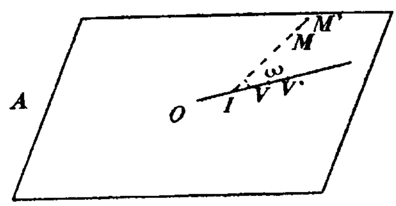

Fig. 2.

In view of the symmetry of the two planes A and A' with respect to SI, tilting the negative does not alter the angles originating at the point I; these will be superposed after the tilt, but the corresponding points will be at different distances from the point I, which is called the *isocentre* of the photograph (it is also called the focal point, orthocentre and metapole).

If i is the tilt of the plane A' with respect to the plane A, we have:

$$OI = IV = f \tan \frac{i}{2}; \quad OV' = f \tan i.$$

Consider (fig. 2) the plane A onto which the negative A' has been turned and let M' be a point on this negative; the perspective of the same point on the ground on the plane A will be formed on the line IM' at a point M such that:

$$\overline{IM} = \frac{\overline{IM'}}{1 + \frac{\overline{IM'}}{f} \cos \omega \sin i}$$

ω being the angle which the straight line IM' makes with the straight line $OIVV'$.

The displacement undergone by the point M', called the *tilt correction*, is therefore:

$$(1) \qquad \overline{MM'} = \frac{\frac{\overline{IM'}^2}{f} \cos \omega \sin i}{1 + \frac{\overline{IM'}}{f} \cos \omega \sin i}$$

which correction is readily calculated when the tilt i and the position of the isocentre have been ascertained, and, besides, this reduces to the value:

$$\text{(1 bis)} \qquad \frac{\overline{IM'}^2}{f} \cos \omega \sin i$$

when the angle i does not exceed a few degrees, as was assumed.

But the points M and M', which are the images of a point m in space, are not the images of the vertical projection m_1 of this point on the plane of reference (see fig. 3) if the point m is not within this plane but lies at a distance from it equal to its altitude h.

Let μ be the point at which the straight line Sm meets the plane of reference; the points M and M' are then the images of the point m and of the point μ.

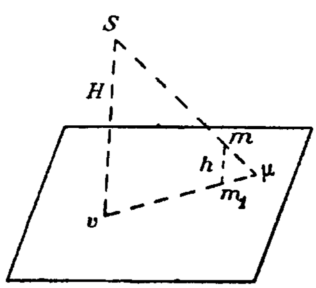

Fig. 3.

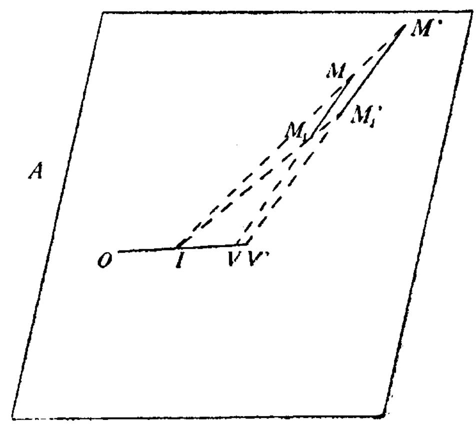

Fig. 4.

Taking H as the distance Sv of the point S from the plane of reference we have:

$$\overline{m_1\mu} = \frac{h}{H}\,\overline{v\mu}$$

The verticals, such as m_1m produce straight line images on the plane A' (fig. 1) which pass through the point V' and, on the plane A, give straight line images passing through the point V. The images M_1 and M'_1 of the point m_1 on the planes A and A' turned down (see fig. 4) will also be straight lines passing through the isocentre and then:

$$(2) \qquad \overline{M_1M} = \frac{h}{H}\,\overline{VM}$$

This displacement is called the *altitude correction*; moreover an approximate determination of the quantity $\frac{h}{H}$ is often sufficient to calculate its value when this is small.

The ascertainment of the isocentre and consequently of the points V and V' as well as the altitudes h and H, thus permits us to determine the exact position which the image of the vertical projection m_1 of the point m in space will assume on a horizontal negative exposed at the point S (1).

The same operation carried out for all the points on the negative A' will transform it into a representation which is, to a certain scale, exactly the chart desired.

A first approximation consists in neglecting the lengths OI and OV which are of the same order as the tilt (2) and therefore on the assumption that the displacements $M'M$ and MM_1 take place along the line joining the point M' with the principal point O. From this the *radial method* has been deduced (ARUNDEL Method) of which a description was given in the *Hydrographic Review* of May 1928, pages 81 to 87. It is a very simple and rapid method and generally will suffice for the topography of coasts of low elevation, provided that the airman succeeds in maintaining the optical axis of his instrument very near the vertical. Each photo therefore serves the same purpose as a station made at the point S from which the horizontal directions of all the points on the negative have been measured. The radial method requires but few control points ; but we are of the opinion that the great accuracy required in charts will not allow the number to be reduced to less than the minimum of three points for each negative, as previously laid down.

III. *METHOD OF THE ISOCENTRE.*

We may also make use of these three points to give greater accuracy to the radial method.

a) We will assume first that all three control points lie in the same horizontal plane, which we adopt as the plane of reference.

Let A', B', C' (fig. 5) be the images of these three points on a tilted negative of which the principal point is O. We compare the angles A', B', C' with the angles a_1, b_1, c_1 of the triangle of corresponding points which are on our construction sheet. The differences found are due to the tilt i of the negative, which we will now determine.

(1) It is evident that these constructions, as well as all of the others which are discussed below, cannot be done on the negative, but on a tracing. A thin sheet of celluloid on which fine lines may be drawn is very useful for this purpose (*See "Hydrographic Review"* of May 1928, pages 82 and 85).

(2) These quantities are proportional to f; with great focal lengths their values attain an appreciable magnitude.

i \ f	0.30 m.	0.50 m.
2°	OI = 5.2 mm. OV = 10.5 mm.	OI = 8.7 mm. OV = 17.5 mm.
4°	OI = 10.5 mm. OV = 21.0 mm.	OI = 17.5 mm. OV = 34.9 mm.

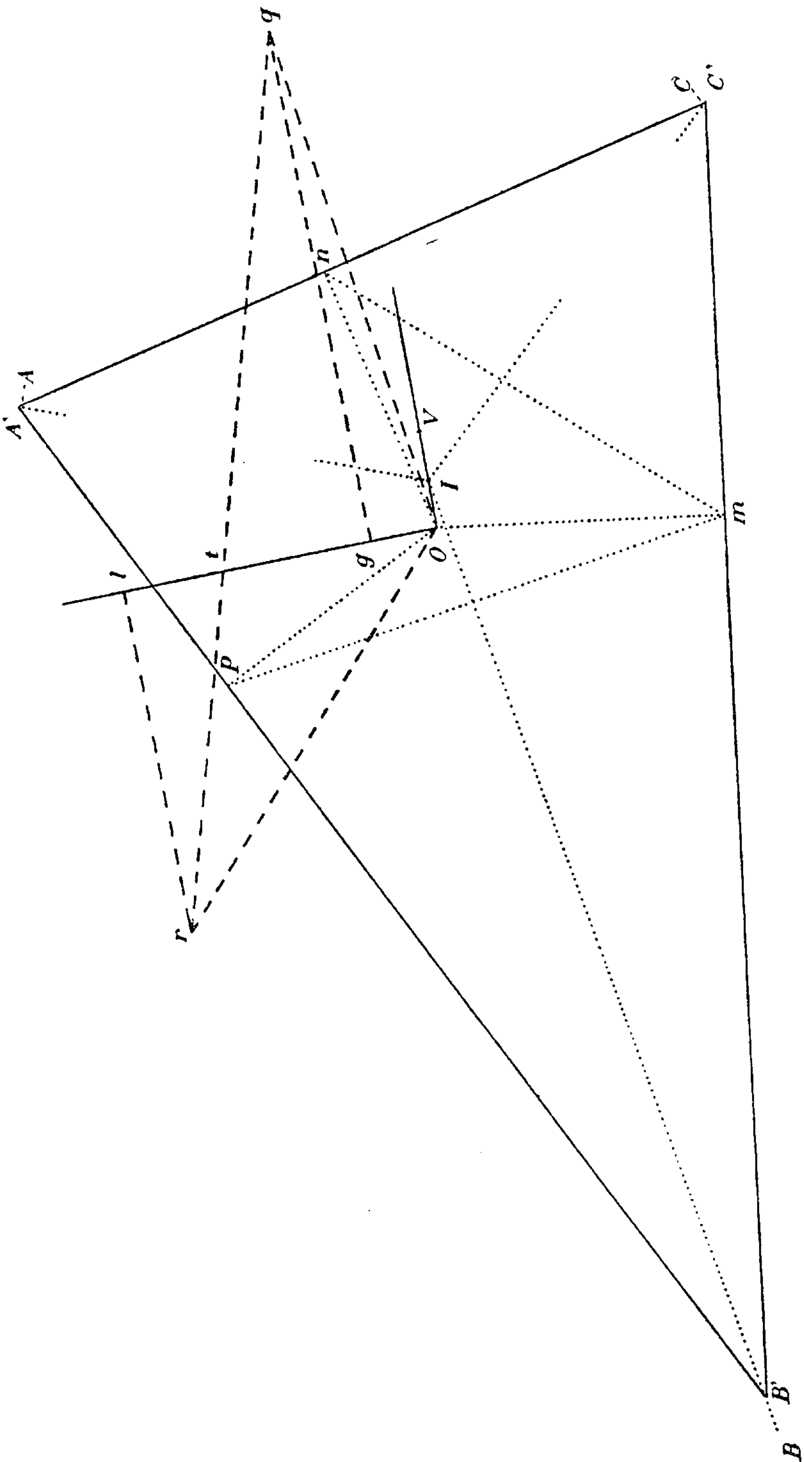

Fig. 5

From the principal point O drop the perpendiculars Om, On and Op onto the sides of the triangle, then draw Oq and Or perpendicular to mp and mn, and make:

$$Oq = mp \qquad \text{and} \qquad Or = mn$$

Let β and γ be the differences found for the angles B' and C'; divide rq into parts proportional to these differences. Let t be the point thus obtained; it will lie in the interior or the exterior of the segment rq according as β and γ are of contrary or the same signs.

$$\frac{tq}{tr} = \frac{\beta}{\gamma}$$

Draw the straight line Ot and drop the perpendiculars qg and rl onto it. The tilt sought will be given by the formula:

$$i = f\frac{\beta}{qg} = f\frac{\gamma}{rl}$$

The point I will lie on a perpendicular to Ot at a distance $f \tan \frac{i}{2}$. It will lie on the same side of the point O as are those angles of the triangle $A'B'C'$ which are smaller than those of the triangle $a_1\, b_1\, c_1$.

In figure 5 we have assumed: $B' - b_1 = 60'$; $C' - c_1 = 40'$; $f = 225$ m/m.

We find: $i = 3^\circ 15'$ and $OI = 6.4$ m/m.

As a check, apply the correction given by the formula (1 *bis*) to the points A', B', C': we shall obtain the points A, B, C and we have:

$$\widehat{C'B'A'} - \widehat{CBA} = 60' \quad \text{and} \quad \widehat{B'C'A'} - \widehat{BCA} = -40'$$

The demonstration of this method for determining the tilt of the negative will not be given here as it would be somewhat long. It may be found in the French *Annales Hydrographiques* of 1917, p. 73-80.

It will be seen that this very simple graphic construction, which requires but a few minutes and which need not be made with very great accuracy, gives the points I and V (the calculation of the differences β and γ only need be made with great care). *See* Appendix III.

By means of the formula 1 *bis* (which is easily put in the form of a table or diagram) any point M' on the negative may now be rectified to M on IM' The point M_1, which is sought, will lie in VM at a point which, generally, we are unable to determine as yet because we do not know the relation $\frac{h}{H}$ in formula (2). A second photograph will determine this point by intersection. The scale of the negative (which has been made horizontal) will be given by comparison of the lengths BC on the negative and $b_1\, c_1$ on the sketch; from this the value of H, the altitude of the airplane, can be deduced. The points I and V, as well as any given point M on the negative (made horizontal), may be tranferred to the sketch by a consideration of the scale, or, which amounts to the same thing, by means of the similar triangles with bases BC and $b_1\, c_1$. By means of the points v, which are the points V of successive photographs transferred to the sketch, we make a *nadiral triangulation* which will be much more accurate than that indicated in paragraph II. By means of this we may locate as many new points as we wish and can obtain the altitude of each (deduced from formula 2, in which the correction M_1M is now known). Those of these points which are within a part common to three photographs will provide control points on the third photograph, should they be necessary.

b) But the hypothesis made in paragraph *a*), that the control points would be in the same horizontal plane, is but rarely true. If they are not more than two or three metres out of this plane such differences may be neglected. More frequently as we have seen, the points *b* and *c* will lie on the shore line at the altitude taken as the zero and it is almost always possible to select point *a* the altitude *h* of which is less than one twentieth of *H*, the altitude of the airplane; *i. e.* not exceeding 100 to 150 metres if the airplane flies at about 3000 metres.

In this case the point A' on the negative is replaced by the point A'_1 on the line OA' and such that: (see fig. 6)

$$\overline{A'_1 A'} = \frac{h}{H}\,\overline{OA'}$$

For *H* the value given by the altimeter of the airplane is taken or else the product of *f*, (the focal distance) and the ratio of the distances between the points *b* and *c* on the ground and on the negative.

This position A'_1 is not the exact image on the tilted negative of the point a_1 on the ground (vertical projection of the point *a* on the plane of reference) but differs from it only by a quantity of the order of $i\,f\,\frac{h}{H}$ and may be neglected provided $\frac{h}{H}$ does not exceed the value stated above.

The procedure employed in paragraph (*a*) will now be applied to the triangle $A'_1\ B'\ C'$, we then obtain the tilt of the negative and the position of the point *I*. If necessary a second approximation must be made. Taking this position of the point *I* as approximate, determine a more accurate position of A'_1 by the procedure indicated in fig. 4, then a new position of point *I* and a new value for the tilt. For this second approximation the points *r* and *q* of fig. 5 may be retained; the point *t* will be displaced if the values of β and γ are changed. Thus the entire operation may be carried out very rapidly.

An identical method may be followed if the difference in altitude between *b* and *c* cannot be neglected; B' is replaced by a point B'_1 in the same way.

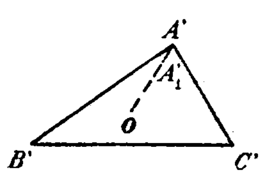

Fig. 6.

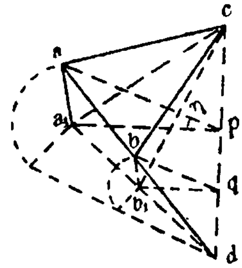

Fig. 7.

c) When the differences in altitude between the three control points exceed the limits stated above, the procedure outlined in the preceding paragraph is not applicable.

Then let *abc* be the triangle formed by the three control points in space and $a_1\, b_1\, c$ that of their projections on the plane of reference (fig. 7). The altitudes of *a* and *b* being known, a very simple graphic construction will give us the point *d* on the sketch where the straight line *ab* meets the plane of reference (by turning the quadrilateral $a\, a_1\, b_1\, b$ down). From the point a_1 drop a perpendicular $a_1 p$ on *dc*. In the triangle aa_1p, turned down, measure the inclination η of the plane *abc* to the plane of reference and the length *pa*.

If we turn the figure *dbac* about *dc*, the points *a* and *b* will fall on the prolongations of pa_1 and qb_1 at distances *pa* and *qb* and the angles of the triangle *abc* may then be measured.

Comparing these angles with those of the triangle *A' B' C'* of the negative, the tilt ε of the negative to the plane *abc* and the line of intersection of the two planes may be determined by the procedure described in paragraph *a*. If the negative be now made parallel to the plane *abc* we will obtain the triangle *A" B" C"* in which the angles will be equal to those of the triangle *abc*.

We shall also find the direction *D" C"* making with *B" C"* the angle $\widehat{bcd}$ measured on the construction sheet.

Let 1 be the first position of the negative and 2 the position parallel to the plane *abc*; it remains to bring this into a position 3, which is parallel to the horizontal plane, by turning it through an angle η about a line parallel to the intersection of position 2 and the horizontal plane.

Let us consider (fig. 8) a sphere the centre of which is the objective *S* and of radius *f*. Draw the radii SO_1, SO_2 and SO_3, which are perpendicular to the positions 1, 2 and 3 of the negative. The angle $\widehat{O_1SO_2}$ is equal to ε and the angle $\widehat{O_2SO_3}$ to η. In positions 1 and 2 the negative is perpendicular to the plane $O_1\, SO_2$; in position 3 it is tangent to the sphere at O_3. The tangent at O_3 to the circle $O_2\, O_3$ meets the bisector of the angle $\widehat{O_2SO_3}$ at the point I_{23} (1) and the radius SO_2 at a point q_3. The perpendicular at I_{23} to the plane $O_2\, SO_3$ meets the plane $O_1\, SO_2$ at the point p_{23}. The line $p_{23}\, q_3$ is therefore the intersection of the plane 3 and the plane $O_1\, SO_2$. The plane 1 cuts this same plane along the line $O_1\, I_{12}$ which meets the line $q_3\, p_{23}$ at r_{13}. The point r_{13} is, therefore, a point common to the planes 1 and 3. Another point common to these two planes will be given by the intersection u_{123} of the normal from I_{12} to the plane $O_1\, SO_2$ and the line $I_{23}\, p_{23}$. Therefore the line $u_{123}\, r_{13}$ is the intersection of the planes 1 and 3.

This straight line is determined on the negative in the following manner: The negative having been brought into the position 2 the point O_2 is marked on it such that $OI_{12} = I_{12}\, O_2$ and the direction *D"C"* which is that of the straight line $I_{23}\, p_{23}$ of fig. 8 has been laid off on it. Let a perpendicular be

(1) The indices to the letters employed here indicate the planes in which the points are located.

drawn through O_2 to this line (see fig. 9), of length $O_2\,I_{23}$ equal to $f\ tan\ \frac{\eta}{2}$ and through I_{23} draw a parallel to $D''C''$ which meets OO_2 at p_{23}. On the perpendicular at O_2 to OO_2 select a point q_3 such that:

$$\overline{O_2\,q_3} = \overline{O_2\,I_{23}}\ tan\ \eta = f\ tan\ \eta\ tan\ \frac{\eta}{2}$$

Join $q_3\,p_{23}$ and take the point of intersection r_{13} of this straight line with another drawn through I_{12} at an angle ε with OO_2. By laying off on OO_2 a length $I_{12}\,R_1$ (1) equal to $I_{12}\,r_{13}$ we obtain the image of the point r_{13} on the negative 1. Let us take further the intersection u_{123} of $I_{23}\ p_{23}$ and a perpendicular to OO_2 drawn through I_{12}; the straight line $u_{123}\ R_1$ is the image on negative 1 of the intersection of the planes 1 and 3. It is but necessary to drop a perpendicular OI from O on to this straight line and to locate thereon a point V, symmetrical to O with respect to I, to obtain the points I and V, thus permitting us to pass directly from negative 1 to the horizontal negative 3 by means of formulae (1) and (2). The inclination θ of the planes 1 and 3 with respect to each other is given by the formula:

$$tan\ \frac{\theta}{2} = \frac{OI}{f}$$

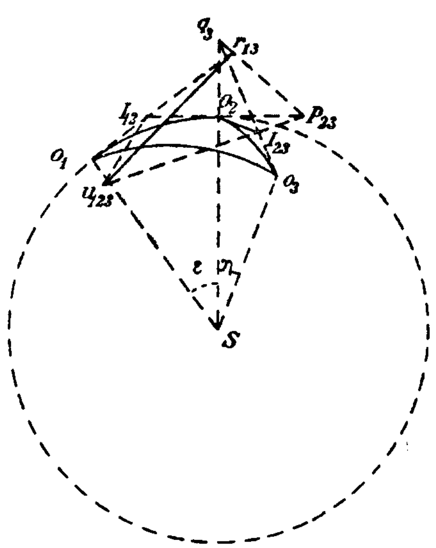

Fig. 8.

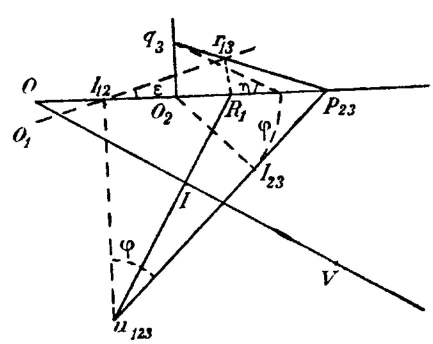

Fig. 9.

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(1) It will be more accurate to calculate the length $I_{12}\,R_1$, the expression for which is (where φ is the angle $\widehat{I_{23}\,O_2\,P_{23}}$):

$$I_{12}\,R_1 = f\ \frac{tan\ \eta}{sin\ \varepsilon}\ \frac{1 + \dfrac{tan\dfrac{\varepsilon}{2}}{tan\dfrac{\eta}{2}}\ cos\ \varphi}{1 + \dfrac{tan\ \eta}{tan\ \varepsilon}\ cos\ \varphi}\ tan\ \frac{\eta}{2}$$

Fig. 9 shows that the graphic constructions are much more rapidly executed than described and that they need not be made with very great accuracy (1). As a check, the corrections for tilt θ and for altitude should be applied to the points $A'B'C'$ and a triangle similar to the triangle $a_1\, b_1\, c$ on the plotting sheet should be obtained.

It should be noted however that the angle ε, between the negative and the plane *abc*, should be very small in order that the procedure in paragraph (*a*) can be applied, but the angles η and θ may be of any value. The altitudes of the points *a* and *b* may also have any value. This shows that the photographer above steeply sloping ground should endeavour to maintain the negative parallel to the ground rather than horizontal; but this desideratum appears to be difficult to obtain in ordinary practice.

IV. *CONJUGATE PHOTOGRAPHS.*

In the preceding paragraphs the successive photographs of the same strip have been considered as independent of each other. The determination of three control points on each of them permits us to ascertain its tilt, its altitude and its orientation. We have, therefore, been enabled to determine the position and the altitude of any point common to two of the photographs.

If, as frequently happens, three photographs have a part in common, the altitude and position of the points in the common part may be determined by means of two photographs only. These points may therefore be used as substitutes in the third photograph for the control points which may be lacking. Nevertheless, in practice, it will still be necessary to determine one control point on the ground for each photograph (except for the first two which should contain 3), because the points common to three photographs, being usually located within a narrow strip, do not provide more than two useful control points.

Consideration of conjugate photographs permits the number of control points to be still further reduced regardless of the unevenness of the ground photographed. Although of prime importance in the case of terrestrial surveys, this method does not appear to us to be so essential for the topography of coasts; we will give the principles, however.

The points common to two photographs correspond to two pencils of rays of which the homologues rest against each other and are therefore in the same plane as the line which joins the two stations, which line we shall refer to as the *base line*. Knowing the focal length and the principal points of the photographs these suffice, together with the photographs themselves, to define

(1) Instead of making this graphic construction, the angles θ and $\widehat{\overline{I_{12}}\, u_{123}\, \overline{R_1}}$ may be calculated by the formulae:

$$\tan \widehat{\overline{I_{12}}\, u_{123}\, \overline{R_1}} = \frac{\dfrac{\sin \varphi}{\cos \varepsilon}}{\cos \varphi + \dfrac{\tan \varepsilon}{\tan \eta}}$$

$$\sin \theta = \frac{\sin \varphi}{\sin \widehat{\overline{I_{12}}\, u_{123}\, \overline{R_1}}} \sin \eta$$

these two pencils, of which the positions with respect to each other (nearly to scale) depends on 5 parameters. These may be determined by consideration of five common points.

The two negatives may therefore be set and oriented with respect to each other even though no ground control point be known.

It will suffice to identify five points in the common part, which may be facilitated by the use of any type of stereoscope.

The complete solution of the problem leads to intricate calculations but may be appreciably simplified by including in the five points the two principal points and assuming certain angles to be very small. It will be easier to make the calculations in this case on the positive prints rather than on the negatives themselves.

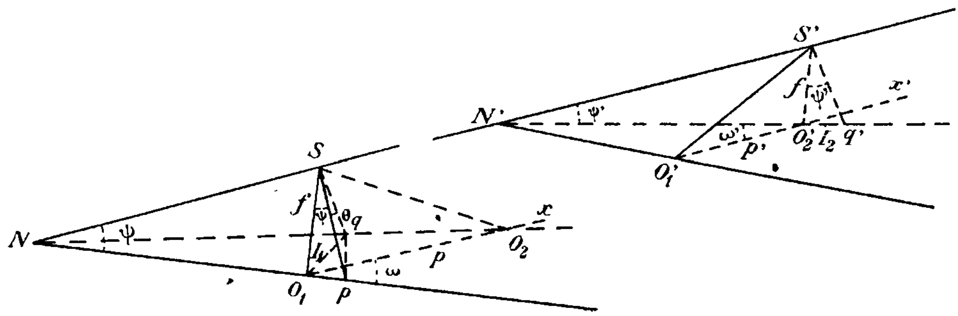

Fig. 10.

In fig. 10 let SS' be the base line which meets the planes of the photographs at the points N and N' which will be called the *nuclear points*. Let O_1 and O'_2 be the principal points of these photographs and O'_1 and O_2 the images of these points on the other photograph. Also let:

$$\theta = \text{the angle between the two planes } SS'O_1 \text{ and } SS'O'_2.$$

$$\frac{\pi}{2} - \psi = \text{the angle } NSO_1.$$

$$\frac{\pi}{2} - \psi' = \text{the angle } N'S'O'_2.$$

On the first photograph we take O_1 as the origin of the coordinates and $O_1 O_2$ as the x-axis. Let p be the length $O_1 O_2$ and ω the angle between this straight line and NO_1 produced.

On the second photograph, we take O'_2 as the origin and $O'_1 O'_2$ produced as the x'-axis; let p' be the length $O'_1 O'_2$ and ω' the angle between the x'-axis and $N'O'_2$ produced.

The angles θ, ψ and ψ' are assumed to be of very small magnitude of the 1st order; it will be seen that the angles ω and ω' are also of very small magnitude of the 1st order.

The coordinates xy, $x'y'$ of a point common to the two photographs are connected with the quantities θ, ψ and ψ' by the following formula in which the terms of the 2nd order are neglected (1):

(1) The proof of this formula and the next one will be found in: *Professional Paper* N° 6, Appendix, pages 110 to 112.

$$(3) \qquad \theta \left[y + \frac{f^2}{y}(1 - \frac{x}{p} + \frac{x'}{p'}) \right] + \psi x - \psi' x' = \frac{y - y'}{y} f$$

As may be readily seen, the quantity $\frac{y-y'}{y}$ is a very small quantity of the 1st order; and becomes nil if θ, ψ and ψ' are nil. The factor $y - y'$ is called the *want of correspondence.*

The consideration of three points common to two photographs (other than the principal points) permits us to establish three equations of this kind and from them to obtain the values of θ, ψ and ψ'. By employing more than three points we obtain a check and a greater degree of accuracy.

We might make a second approximation by calculating with these values of θ, ψ and ψ' and by adding to the second member of equation (3) the following quantity which takes the terms of the second order into account:

$$(y - y')\frac{x\psi + x'\psi'}{2y} - f\theta^2 \frac{x + x'}{p} + f\theta(\psi - \psi')\frac{y - \frac{xx'}{y}}{p} + \frac{1}{2} f (\psi^2 - \psi'^2)$$

Obtaining other solutions of the equations (3), we get definite values for the unknown quantities.

When the number of equations exceeds 3, it will be much more advantageous to avoid calculation for their solution by employing a graphic method. An example will be given later. (See Appendix I).

The angles ω and ω' are given by the equations:

$$sin\ \omega = tan\ \theta\ (\frac{f}{p} cos\ \psi + sin\ \psi\ cos\ \omega)$$

$$sin\ \omega' = tan\ \theta\ (\frac{f}{p} cos\ \psi' - sin\ \psi'\ cos\ \omega')$$

Neglecting the quantities of the 3rd order we have:

$$\omega = \theta\ (\frac{f}{p} + \psi)$$

$$\omega' = \theta\ (\frac{f}{p'} - \psi')$$

By means of the five quantities just calculated we may determine the modifications to be made to the two photographs in order that they may represent the perspectives of the same ground if the optical axes had both been perpendicular to the base line SS' and in the same plane $SS'O_2O'_2$.

The optical axis which occupied the position SO_1 in the first photograph, should be in the position Sq. In order to determine the point q, it will suffice to lay off on a straight line at an angle ω to the axis O_1x, a length O_1p equal to $f\ tan\ \psi$, then at p to erect a perpendicular to O_1p of length pq equal to $f\ \frac{tan\ \theta}{cos\ \psi}$

The angle through which the optical axis should be turned is $\widehat{O_1Sq}$ and:

$$tan\ \widehat{O_1Sq} = \frac{\sqrt{sin^2\ \psi + tan^2\ \theta}}{cos\ \psi}$$

The isocentre I_1 corresponding to this rotation will lie on O_1q at a distance from O_1 :

$$O_1I_1 = f\ tan\ \frac{\widehat{O_1Sq}}{2} = f\ cos\ \theta\ \frac{\sqrt{sin^2\ \psi + tan^2\ \theta}}{1 + cos\ \psi\ cos\ \theta}$$

Neglecting the quantities of the 3rd order, the point I_1 may be taken at the centre of O_1q.

In the new position of the photograph, q becomes the principal point (or rather a point which is distant from it by a negligible quantity of the 3rd order) ; O_2 will take up a new position on the line $I_1\ O_2$ at a point $\underline{O}_2$ given by formula (1) ; the same will occur for all points on the photograph. Thus the line $q\underline{O}_2$ becomes the new x-axis. The coordinates XY of any point common to the two photographs will again be measured with respect to this axis, with q as the origin.

The optical axis which occupied the position $S'O'_2$ in the second photograph should occupy the position $S'q'$, turning through the angle ψ'. In the direction ω' we lay off the length O'_2q' equal to $f\ tan\ \psi'$ and the isocentre I_2 half way between O'_2 and q' ; we then apply the correction obtained from formula (1) to the points of the photograph and re-measure the coordinates $X'\ Y'$, of any point common to the two photographs, with respect to the new point of origin q' and the new x-axis ($N'O'_2$ produced) at an angle ω' from the first.

As a check the same values for the coordinates Y and Y' of the same point should be obtained.

Further on (in Appendix II) we give the method of converting the coordinates x and y of a point on the photographs into the coordinates X and Y by calculation ; but the graphic measurement which has just been described will generally be sufficient.

Having thus obtained the coordinates X, Y and X', Y' of the points common to the two photographs of which the optical axes have been made parallel and perpendicular to the base line, it is easy to deduce the coordinates of the same points in space with respect to the same axes XY and an axis o the Z's passing through the point S.

Let a be a point of which the images are A and A', and let D be the distance SS'. Then :

$$X_a = D\ \frac{X_A}{X_A - X'_A} \qquad Y_a = D\ \frac{Y_A}{X_A - X'_A} \qquad Z_a = f - D\ \frac{f}{X_A - X'_A}$$

or, if the quantity $\dfrac{1}{X_A - X'_A}$ be referred to as T_A :

$$(4) \qquad X_a = D\ X_A\ T_A \qquad Y_a = D\ Y_A\ T_A \qquad Z_a = f - Df\ T_A$$

It remains to find the direction of the vertical with respect to the new directions of the optical axes which we have adopted and the distance D between the two points of exposure. These cannot be obtained without measuring some elements on the ground.

1st Case. Assume that we know the altitudes of 3 points and the distance between two of them. Let $a, b, c,$ be these points, h_a, h_b and zero their altitudes and l the distance bc (1).

These points should be in the part common to two photographs; thus their coordinates on the photographs may be measured and their coordinates in space obtained by means of the formulae (4).

Let a point e be located on the line ab so that $Z_E = Z_C$ (and consequently $T_E = T_C$).

Then:

$$\frac{\overline{ea}}{\overline{eb}} = \frac{X_e - X_a}{X_e - X_b} = \frac{Z_c - Z_a}{Z_c - Z_b} = \frac{X_E T_C - X_A T_A}{X_E T_C - X_B T_B} = \frac{T_C - T_A}{T_C - T_B} = \frac{(X_E - X_A)\ T_A}{(X_E - X_B)\ T_B} = \frac{\overline{EA}\ T_A}{\overline{EB}\ T_B}$$

The point E may then be located on the line AB on the photograph by means of the equation:

$$\frac{\overline{EA}}{\overline{EB}} = \frac{T_B}{T_A} \frac{T_C - T_A}{T_C - T_B}$$

The line EC is parallel to the line ec in space, and:

$$\frac{\overline{EC}}{\overline{ec}} = \frac{f}{f - Z_c} = \frac{1}{D\ T_C}$$

Consider the point a'', the orthogonal projection of the point a on a plane parallel to the plate and passing through $c\ e$. The coordinates of this point a'' are:

$$\begin{aligned} X_{a''} &= X_a = D X_A T_A = D X_{A''} T_C \\ Y_{a''} &= Y_a = D Y_A T_A = D Y_{A''} T_C \\ Z_{a''} &= Z_c \end{aligned}$$

whence:

$$X_{A''} = X_A \frac{T_A}{T_C} \qquad Y_{A''} = Y_A \frac{T_A}{T_C}$$

The point A'' is therefore, on the photograph, a point on the straight line joining the point A with the origin of coordinates X, Y and such that:

$$\frac{\overline{OA''}}{\overline{OA}} = \frac{T_A}{T_C}$$

If we drop a perpendicular $A''P$ from the point A'' onto CE, we shall have (see fig. 11):

$$\overline{a''p} = \overline{A''P}\ DT_C$$

and

$$Z_a = Z_c = Df\ (T_C - T_A) = \overline{A''P}\ DT_C \tan \varepsilon$$

from which:

$$\tan \varepsilon = \frac{f}{A''P} \left(1 - \frac{T_A}{T_C}\right)$$

(1) It will often be of advantage to adopt as the plane of reference a horizontal plane having two of the points abc on either side of it so that the points E and G (see further on) do not lie outside the limits of the sheet.

ε being the angle between the plane of the photograph and the plane containing the three points *abc*. With the same scale reduction we will obtain a representation of the triangle *abc* on space by locating a point A_1 on the prolongation of *PA"* at a distance:

$$\overline{A''A_1} = \overline{A''P}\ (\frac{1}{\cos\varepsilon} - 1)$$

It will suffice to make the graphic construction by means of the angle ε which has just been calculated. The point B_1 on the straight line A_1E will be such that:

$$\frac{\overline{E\,B_1}}{\overline{E\,A_1}} = \frac{\overline{eb}}{\overline{ea}} = \frac{T_C - T_B}{T_C - T_A}$$

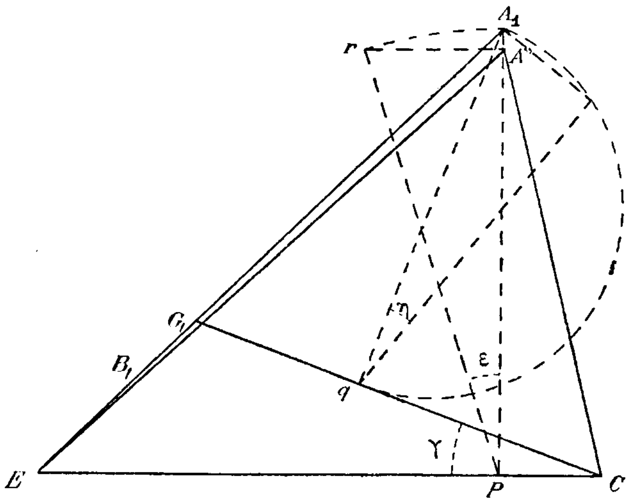

Fig. 11.

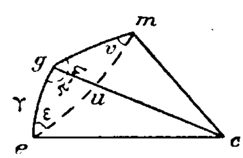

Fig. 12.

Further we take a point G_1 on EA_1 which corresponds to a point on the straight line *ab* at the same altitude (zero) as *C*. It is defined by the equation:

$$\frac{\overline{G_1\,A_1}}{\overline{G_1\,B_1}} = \frac{h_a}{h_b}$$

The perpendicular A_1q dropped onto CG_1 is the hypothenuse oi a right triangle, of which one side of the right angle, taking the reduction in scale into account, is:

$$\frac{h_a}{DT_C}$$

The angle η included between the horizontal plane and the plane *abc* is given by:

$$\sin\ \eta = \frac{h_a}{\overline{A_1q}\ D\,T_C}$$

The length A_1q may be measured on the figure. We measure also on the figure the angle $\widehat{G_1CE}$ which will be called γ. The angle η may also be measured on a graphic construction, but in any case it is necessary to calculate the value D, the distance between the two points of exposure. This will be given by the known distance l between the points b and c:

$$D = \frac{l}{\overline{B_1C}\ T_C}$$

We have therefore:

$$\sin \eta = \frac{h_a}{A_1q} \frac{\overline{B_1C}}{l}$$

If we consider the plane abc, the plane parallel to the plate passing through c, and the horizontal plane passing through c — these three planes form a pyramid with its apex at c (see figure 12); the face in the plane abc is ecg, the angle of which at the apex is γ. The angle between the plane abc and the plane of the plate is ε, and that between abc and the horizontal is η. Let cm be the third edge of the pyramid; — this is a parallel to the intersection of a horizontal plane and that of the plate. Let u be the angle $\widehat{ecm}$ and v the angle $\widehat{gme}$. These quantities are given by the equations (in which attention should be paid to the signs of the angles):

$$\tan u = \frac{\dfrac{\sin \gamma}{\cos \varepsilon}}{\cos \gamma - \dfrac{\tan \varepsilon}{\tan \eta}} \quad , \quad \sin v = \frac{\sin \gamma \sin \eta}{\sin u}$$

We then draw a line through the principal point q on the photograph, which has been partially adjusted and is the origin of the coordinates X and Y, at an angle of $90^o - u$ to CE (taking care to draw this line in the suitable direction). This line contains the vertical passing through S at an angle v to Sq; it then intersects the plane of the photograph at a point q_1 at the distance $f \tan v$ from the point q. A point q_2, the position of which is identical with respect to the axes is then plotted on the second photograph.

We are now in a position to bring the two photographs taken at S and S' into a horizontal position (see fig. 10). These have the optical axes SO_1 and $S'O'_2$. Their rectification with respect to their initial position corresponds to a rectification through the angles $O_1 Sq_1$ and $O'_2S'q_2$ and the method of the isocentre and the principal point is strictly applicable.

It should be noted that the procedure just indicated is applicable regardless of the tilt of the photograph with respect to the horizon, this is not assumed to be small, and also regardless of large differences in the altitudes of the points on the ground. It has only been necessary to assume that the optical axes make an angle of approximately 90^o with the base line and that, within a few degrees, they lie in the same plane with it.

It should be noted also that the length of a single side only of the triangle abc was known. If the three lengths had been known, as in the problem treated in paragraph III, it would still be preferable to employ this last method which has the advantage of giving, in one operation, results for the two photographs of the pair and supplying a check. The position found for the projection a_1 of the point a on the horizontal plane should, in fact, be that which was known before and the knowledge of the lengths CA_1 and $B_1 A_1$ in figure 11 should confirm the results of the construction of that figure.

2nd CASE. It sometimes happens that levelling operations result in the altitudes of numerous points being known with great accuracy though their positions or distances may not be known. We are therefore led to consider the case in which the altitudes of four points in the part common to two photographs are known though no positions or distances have been determined.

Let a, b, c and d represent these four points in space; h_a, h_b, h_c and zero their altitudes and H the unknown altitude of the point of exposure S. Let v be the angle between the vertical and the optical axis (made perpendicular to the base line as described above) and u the angle made by the plane containing these two lines with the plane $Y = 0$.

The point of exposure S being at a height $H - h_a$ above the point a, and taking into account the formulae (4), we have the equation:

$$(X_A \cos u + Y_A \sin u)\, T_A \sin v + f\, T_A \cos v = \frac{H}{D} - \frac{h_a}{D} \tag{5}$$

By establishing similar equations for each of the four points, we are able to calculate linearly the quantities $\tan u$, $\tan v$, $\frac{H}{D}$ and $\frac{1}{D}$.

But the solution of these equations may be facilitated in the following manner:

On the sides AB and AC of the photograph, let two points E and F be determined which are the image of the points e and f such that:

$$Z_e = Z_f = Z_d' \qquad \text{(see fig. 13)}$$

We have seen in the first case, examined above, that the position of the point E on the photograph is given by the equation:

$$\frac{\overline{EA}}{\overline{EB}} = \frac{T_B}{T_A}\,\frac{T_D - T_A}{T_D - T_B}$$

Similarly:

$$\frac{\overline{FA}}{\overline{FC}} = \frac{T_C}{T_A}\,\frac{T_D - T_A}{T_D - T_C}$$

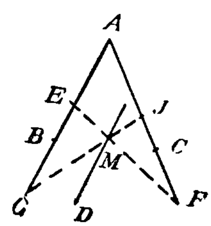

Fig. 13.

Let two points G and J of altitude zero be located on the same sides. We then have:

$$\frac{\overline{GA}}{\overline{GB}} = \frac{T_B}{T_A}\,\frac{h_a}{h_b} \quad \text{and} \quad \frac{\overline{JA}}{\overline{JC}} = \frac{T_C}{T_A}\,\frac{h_a}{h_c}$$

and:

$$T_G = \frac{h_b T_A - h_a T_B}{h_b - h_a}$$

The plane defined by the points *efd* is parallel to the plate, and that defined by the points *gjd* is parallel to the horizon.

The two straight lines *ef* and *gj*, lying in the plane of the points *abc*, meet at a point *m* which, joined to *d*, gives the direction of the intersection of the planes of the plate and of the horizon.

Thus, on the photograph, the line *MD* is perpendicular to the direction defined by the angle *u*. Hence we are able to determine this angle. Equation (5) applied to point *G* and point *D* will give the equations :

$$(X_G \cos u + Y_G \sin u)\, T_G \sin v + f\, T_G \cos v = \frac{H}{D}$$

$$(X_D \cos u + Y_D \sin u)\, T_D \sin v + f\, T_D \cos v = \frac{H}{D}$$

Seeing that $X_D \cos u + Y_D \sin u$ represents the distance between the origin and the straight line *DM*, which we call Δ_D ; and $X_G \cos u + Y_G \sin u$ the distance Δ_G from the origin to a straight line parallel to *DM* drawn through *G*, we may write :

$$\tan v = f \frac{T_G - T_D}{T_D \Delta_D - T_G \Delta_G}$$

The quantity $\frac{H}{D}$ will then be given by the formula :

$$\frac{H}{D} = T_D \ (\Delta_D \sin v + f \cos v)$$

Finally, we can calculate *D* by the following formula, choosing the point the altitude of which differs most from that of the point *d* ; take for example *a* :

$$D = \frac{h_a}{(T_D\, \Delta_D - T_A\, \Delta_A) \sin v + f\,(T_D - T_A) \cos v}$$

The value of *D* thus obtained cannot be very accurate unless the altitude h_a is nearly of the same order of magnitude.

Once the second photograph has been fixed in place, the third may also be fixed with respect to the second, and so on ; the distance between the points of exposure of the second and of the third photograph being deduced from a distance between two points identified on the three photographs in the part common to them all, without requiring any position or any altitude of control on the ground. Nor is it necessary to determine the altitude of the point of exposure or its distance from the previous exposure point. Two points on the third photograph, in the part common to the three photographs and which have already been transferred to the plotting sheet by means of photographs 1 and 2, are then rectified (it is not necessary to calculate their altitudes). These points subtend at the principal point an angle of which the containing arc is then drawn on the plotting sheet with the two points as base. The intersection of this arc and the direction in which this point lies with respect to the second photograph, fixes the principal point of the third photograph.

It is possible also to calculate the distance D' between S' and S'' by the following process in which the image of but any single point common to the three photographs (See Fig. 14) is used.

Let a be this point and D' the distance $S'S''$.

The distance $S'a$ expressed by means of the coordinates of the image of the points a on the first pair of photographs is :

$$S'a = DT_A \sqrt{X'^2_{A'} + Y'^2_{A'} + f^2}$$

Applying the indicator 1 to the quantities connected therewith, the second pair of photographs gives :

$$S'a = D'T_{1A} \sqrt{X_{1_A}^2 + Y_{1_A}^2 + f^2}$$

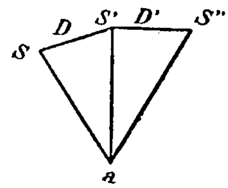

Fig. 14.

We have therefore :

$$D' = D \frac{T_A}{T_{1A}} \sqrt{1 + \frac{(X'^2_{A'} - X_{1_A}^2) + (Y'^2_{A'} - Y_{1_A}^2)}{X_{1_A}^2 + Y_{1_A}^2 + f^2}}$$

The quantities $X'_{A'}$ and X_{1_A}, $Y'_{A'}$ and Y_{1_A} are the coordinates of the same point on the second photograph, but referred to a different origin and different axes.

The use of a strip of photographs in which the first pair only is based on control marks on the ground will doubtless be fairly rare in hydrography ; but the procedure given above may still be of service if one or two of the photographs should show exceptional shortage of overlap or of points of control.

We do not pretend to have dealt with all of the special cases which may arise ; it is probable, however, that they could be solved by methods analogous to those described.

We have not discussed the correction due to the curvature of the earth ; this is entirely negligible for an isolated photograph but it must be applied if we transfer the direction of the vertical step by step over all the photographs of a strip. At each step this direction should be corrected by a quantity corresponding to the curvature of the earth.

V. *THE FOURCADE STEREOGONIOMETER.*

The *Air Survey Committee* has just published in London Publication N° 7 under the signature of Captain M. HOTINE, R. E. The use of this instrument invented by Mr. FOURCADE is described in detail by him in the *Transactions of the Royal Society of South Africa*, Vol. XIV, Part I (1926) ; Vol. XVI, Part I (1928) ; Vol. XVII, Part I (1928) ; Vol. XVIII, Part III (1929). It is investigated in Publication N° 7 with numerous examples and practical recommendations which cannot be reproduced here. A photograph of the *Fourcade Stereogoniometer* appeared in the *Hydrographic Review* of May 1930, page 104.

This apparatus, which provides a mechanical solution for the adjustment of correspondence between two photographs, introduces appreciable simplification into the problem of the use of a pair of photographs. Not only is the calculation of the quantities which we have designated by θ, ψ, ψ', ω and ω' avoided but it is not necessary to assume them to be very small. Therefore the general problem may be solved completely whatever the tilts of the photographs at the moments of exposure. The *Fourcade Stereogoniometer,* which may be further supplemented by special parts which permit automatic drawing of the chart, is thus of almost indispensable assistance in the use of strips of photographs taken above broken ground or where there are few points of control and, also, to facilitate the use of multiple objective photographic cameras.

APPENDIX I.

The group of equations (3) may be solved by the following graphic process which appears to be of particular interest when the number of equations exceeds three.

We will apply it to the example given in the "*Extension of the Arundel Method*", by Captain M. HOTINE, R. E., pages 47 and 48. In these equations account is taken of the terms of the second order. By giving to θ the value zero these become:

$$\begin{aligned} 11.9\,\psi + 82.4\,\psi' &= 3.25 \\ 2.1\,\psi + 81.9\,\psi' &= 0.65 \\ 92.2\,\psi - 4.3\,\psi' &= 5.90 \\ 73.5\,\psi + 3.9\,\psi' &= 1.89 \end{aligned}$$

Construct (fig. 15) the four straight lines represented by these equations of which ψ and ψ' are the coordinates, adopting a scale of 0.75 m/m per 0.001. Repeat the same operation giving θ a value of 0.05 (a cursory examination shows us that θ lies between 0 and 0.05). The second members of the equations then become: 2.54, 2.94, 5.34 and 5.745.

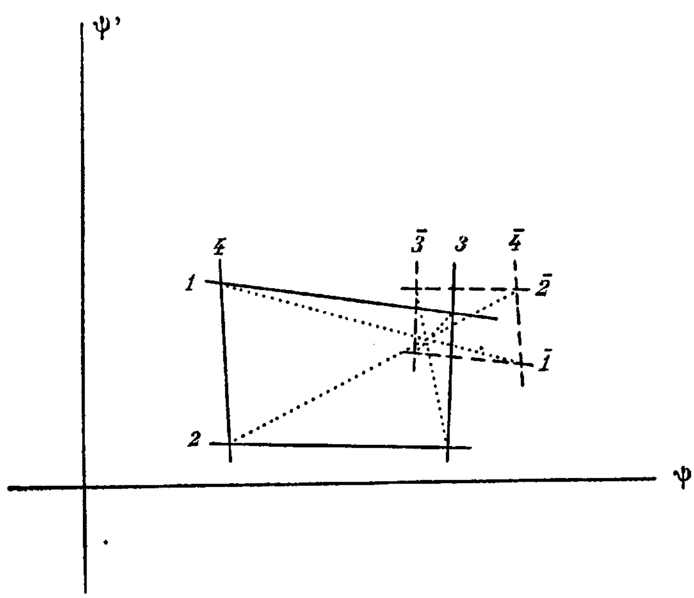

Fig. 15.

The four straight lines which these represent give us a second quadrilateral the sides of which are parallel to those of the first. By joining in pairs the corresponding corners of the two quadrilaterals we obtain four straight lines which should intersect at the same point if the second order approximation is sufficient. The method may be applied to three points or more. Four points permit a check to be obtained. It appears useless generally to exceed this number.

The coordinates of the point of the straight lines joining the corresponding corners of the polygons give us the values of ψ and ψ'

$$\psi = 0.061 = 3^\circ 30'$$
$$\psi' = 0.025 = 1^\circ 26'$$

APPENDIX II

Having measured on a pair of photographs the coordinates which have been designated by x, y and x', y', we can calculate the new coordinates X, Y and X', Y'.

The optical axis of the first photograph will be turned through the angle $\widehat{O_1Sq}$, which we call γ (See fig. 10 and 16).

And:

$$\sin\gamma = \sin\theta \sqrt{1 + \frac{\sin^2\psi}{\tan^2\theta}}$$

$$\cos\gamma = \cos\theta \cos\psi$$

$$\tan\frac{\gamma}{2} = \frac{\sin\theta}{1 + \cos\theta\cos\psi}\sqrt{1 + \frac{\sin^2\psi}{\tan^2\theta}}$$

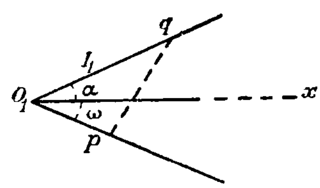

Fig. 16.

Let α be the angle $\widehat{xo_1q}$. It is given by:

$$\sin\alpha = \frac{\cos\psi\cos\omega - \frac{f}{p}\sin\psi}{\sqrt{1 + \frac{\sin^2\psi}{\tan^2\theta}}}\cos\psi$$

$$\cos\alpha = \frac{\sin\psi\cos\omega + \frac{f}{p}\cos\psi\sin^2\theta}{\sin\theta\cos\theta\sqrt{1 + \frac{\sin^2\psi}{\tan^2\theta}}}$$

Take the isocentre I_1 as the origin of coordinates and the direction I_1q as the axis of the y's. The new coordinates ξ and η are:

$$\xi = x\sin\alpha - y\cos\alpha$$

$$\eta = x\cos\alpha + y\sin\alpha - f\tan\frac{\gamma}{2}$$

If we then rectify the photograph through the angle γ; and if, retaining the direction of the axis, we transfer the origin to the new principal point, the coordinates then become:

$$\xi_1 = \frac{\xi}{1 + \frac{\eta}{f} \sin\gamma}$$

$$\eta_1 = \frac{\eta \cos\gamma - f \tan\frac{\gamma}{2}}{1 + \frac{\eta}{f} \sin\gamma}$$

The point O_2 will have taken up a new position which, joined to the last origin, gives the axis of the X's which lies at an angle β with that of the ξ's.

The angle β is given by:

$$\sin\beta = \frac{\sin\psi}{\sin\theta\sqrt{1 + \frac{\sin^2\psi}{\tan^2\theta}}} \qquad \cos\beta = \frac{\cos\psi}{\sqrt{1 + \frac{\sin^2\psi}{\tan^2\theta}}}$$

Finally we have:

$$X = \xi_1 \cos\beta + \eta_1 \sin\beta$$
$$Y = -\xi_1 \sin\beta + \eta_1 \cos\beta$$

If we neglect the terms of order greater than the first, the coordinates X and Y take the values:

$$X = x - \theta y \left(\frac{f}{p} + \frac{x}{f}\right) - \psi f\left(1 + \frac{x^2}{f^2}\right)$$

$$Y = y + \theta f \left(\frac{x}{p} - 1 - \frac{y^2}{f^2}\right) - \psi\frac{xy}{f}$$

For the second photograph, by turning the axes through the angle ω' we obtain the coordinates:

$$\xi' = x' \cos\omega' - y' \sin\omega' \qquad \eta' = x' \sin\omega' + y' \cos\omega'$$

Transferring the origin to I_2 and turning the optical axis through the angle ψ' we get:

$$\xi_1' = \frac{\xi' - f\tan\frac{\psi'}{2}}{\cos\psi' + \frac{\xi'}{f}\sin\psi'} \qquad \eta_1' = \frac{\eta'}{\cos\psi' + \frac{\xi'}{f}\sin\psi'}$$

Finally, transferring the origin to the new principal point, we have:

$$X' = \frac{\xi' - f\tan\psi'}{1 + \frac{\xi'}{f}\tan\psi'} \qquad Y' = \frac{\eta'}{\cos\psi'\left(1 + \frac{\xi'}{f}\tan\psi'\right)}$$

If then we neglect the terms of order greater than the first:

$$X' = x' - \theta y' \frac{f}{p'} - \psi' f \left(1 + \frac{x'^2}{f^2}\right)$$

$$Y' = y' + \theta f \frac{x'}{p'} - \psi' \frac{x' y'}{f}$$

As a check, Y should be equal to Y'.

Advantage may also be taken of this property to calculate X' and Y' first, their calculation being more simple, and then calculate $\frac{Y}{X}$ the value of which is exactly:

$$\frac{X}{Y} = \frac{y \cos\omega \cos\psi \cos\theta + \frac{f}{p}(x - p)\sin\theta + y \frac{\cos\omega}{\cos\theta} \sin\psi \tan\psi + \frac{f}{p} y \sin\psi \sin\theta \tan\theta}{x \cos\omega - y \sin\omega - f \tan\psi}$$

By neglecting some terms of the third order, as we did in the calculation of the quantities θ, ψ, ψ', ω and ω', we have only to calculate:

$$\frac{Y}{X} = \frac{x \cos\omega \cos\psi \cos\theta + \frac{f}{p}(x - p)\sin\theta}{x \cos\omega - y \sin\omega - f \tan\psi} + \frac{y}{x} \sin^2\psi$$

$$\frac{X'}{Y'} = \frac{x' \cos\omega' \cos\psi' - y' \sin\omega' - f \sin\psi'}{y' \cos\omega' + x' \sin\omega'}$$

$$Y = Y' = \frac{y' \cos\omega' + x' \sin\omega'}{\cos\psi' + \frac{x'}{f} \sin\psi' - \frac{y'}{f} \sin\omega' \sin\psi'}$$

APPENDIX III

The construction described in III and on fig. 5 gives rise to several remarks when the principal point O lies on the circumference of a circle cir-

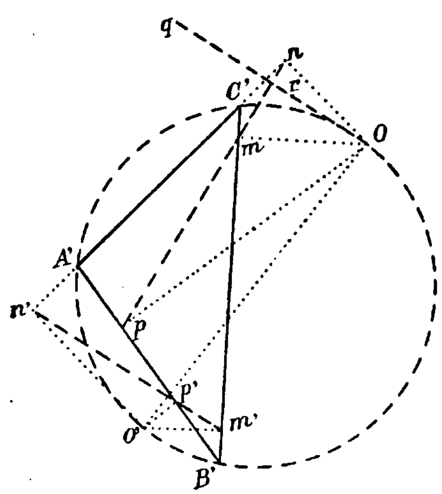

Fig. 17.

cumscribed about the triangle $A'B'C'$. In that case the feet m, n and p of the perpendiculars dropped from O onto the sides of the triangle lie on the same straight line, called SIMPSON's line. (See fig. 17).

The lines which have been designated by or and oq thus coincide and we can no longer employ this method for finding the tilt of the plate with respect to the plane of reference.

This brings to light also certain curious properties possessed in that case by the projecting pyramid $SA'B'C'$.

Consider a plane which makes a very small angle ε with $A'B'C'$, the intersection of which is a line inclined at an angle φ to the line orq (perpendicular to nmp). The angles in the section of the pyramid, made by this plane, differ from those of the triangle $A'B'C'$ by the quantities:

$$\delta A' = -\frac{\overline{np}}{f}\,\varepsilon \sin\varphi$$

$$\delta B' = \frac{\overline{mp}}{f}\,\varepsilon \sin\varphi$$

$$\delta C' = \frac{\overline{mn}}{f}\,\varepsilon \sin\varphi$$

We may state:

$$\frac{\delta A'}{-\overline{np}} = \frac{\delta B'}{\overline{mp}} = \frac{\delta C'}{\overline{mn}}$$

The relations between the quantities $\delta A'$, $\delta B'$ and $\delta C'$ are then no longer indeterminate. Since they are independent of ε and of φ the determination of $\delta B'$ and $\delta C'$ will not permit us to determine ε by the method employed in III. This is due to the fact that this method neglects quantities of the second order. There are an infinite number of planes making a small angle with $A'B'C'$ which satisfy the conditions: those in which φ is equal to 90^o will correspond to the smallest value of ε; those for which φ is zero will result in no modification of the angles A', B' and C'. Every plane which meets the plane $A'B'C'$ at a very small angle and which intersects it along a line perpendicular to SIMPSON's line will produce in the pyramid a triangle similar to the triangle $A'B'C'$. Further, the triangle will be identical with the triangle $A'B'C'$ if the two planes intersect along the SIMPSON line $m'n'p'$ relative to the point O', the symmetrical to the point O with respect to the centre of the circumscribed circumference. (It may easily be demonstrated that $m'p'n'$ is perpendicular to $n\,m\,p$ and the above property is readily deduced from the considerations discussed in the French *Annales Hydrographiques* of 1927).

We have thus a demonstration of the properties of the *dangerous cylinder*; we have already given two others in *Hydrographic Review* IV-2, page 92 and VII-1, page 105. We give also the following demonstration, due to Dr FINSTERWALDER, which is remarkable for its elegance :

If a second triangle, identical to $A'B'C'$, could be placed on the projecting pyramid in such a manner as to be infinitely close to the first, the projections of the angles on the plane $A'B'C'$ will be points on the straight lines OA' OB' and OC'. These may be considered as the new positions of the angles of the triangle $A'B'C'$ after it has moved in its plane without the lengths of its sides being changed by a quantity greater than an infinitesimal amount of the second order. The instantaneous centre of rotation corresponding to this movement will lie at the meeting point of the perpendiculars to the straight lines OA', OB' and PC' erected at A', B' and C'; now it is evident that the straight lines will meet in a point provided only that the point O lies on the circumference of the circle circumscribed about the triangle $A'B'C'$. The instantaneous centre will therefore be the point O' of fig. 17.