COMPENSATION OF A POLYGON WITH A CENTRAL STATION

by

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This problem was dealt with in *Hydrographic Review*, Vol. VIII, N° 2, November 1931, on pages 211 et seq., by the introduction of simplifications in the general solution which reduces the sum of the squares of the errors to a minimum. In order to allow a comparison of the various methods which may be employed to be made, the general expression of the solution by the method of least squares will be given. This expression is fairly simple in form; it will be given without taking the weights of the observations into consideration and assuming that all the angles, and not the directions, have been observed, but there would be no difficulty in giving it if the case were otherwise.

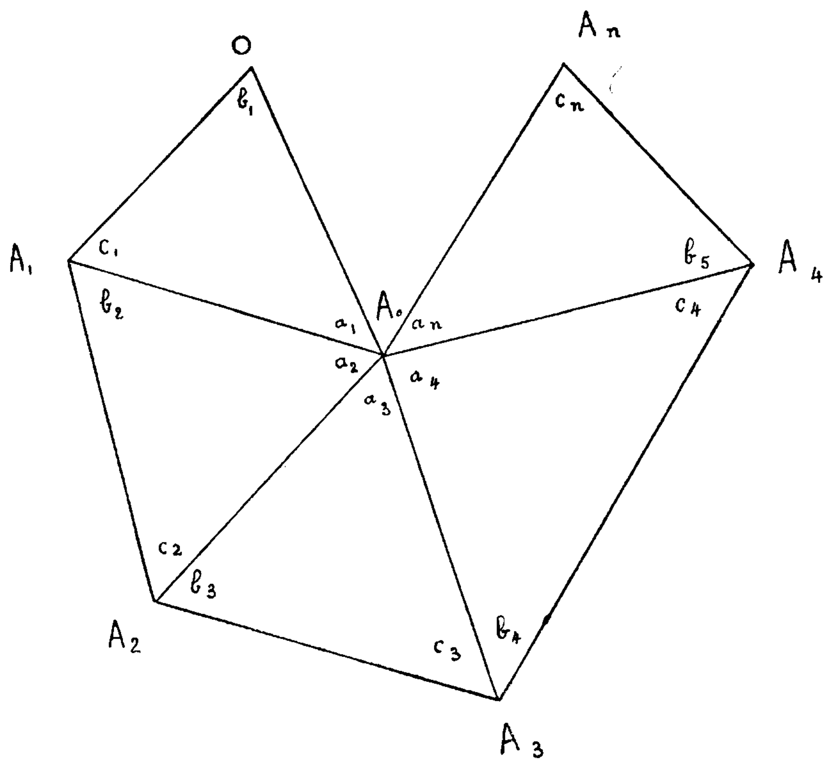

Fig. 1

Let A_0O be the departure base and A_0A_n the side reached, the length and direction of which is assumed to be known. Anyway A_0A_n may even coincide with A_0O.

First, the method set out on page 212 of *Hydrographic Review*, Vol. VIII, N° 2, will be applied, *i. e.* the angles of each triangle are corrected by a third of the difference of their sum from 360°, then the central angles are each corrected by an equal quantity $\frac{e}{n}$, in order to ensure closure about the point A_o, and lastly to each of the peripheral angles a correction $-\frac{e}{2n}$ is applied to maintain the closure of each triangle.

The angles thus provisionally adjusted in triangle n° p will be referred to as: a_p, b_p and c_p; and α_p, β_p and γ_p will be taken to represent the corrections yet to be applied to them in order that the sum of the squares of the errors: $\alpha_p + \frac{e}{n}$, $\beta_p - \frac{e}{2n}$ and $\gamma_p - \frac{e}{2n}$ should be the minimum.

We then have the equations for condition:

(1) $\alpha_p + \beta_p + \gamma_p = 0,$

(2) $\sum_1^n \alpha_p = 0,$

then the equation for the sides:

$$\frac{A_o A_n}{A_o O} \; \frac{\sin\,(c_1 + \gamma_1)\;\sin\,(c_2 + \gamma_2) \cdots\cdots\; \sin\,(c_n + \gamma_n)}{\sin\,(b_1 + \beta_1)\;\sin\,(b_2 + \beta_2) \cdots\cdots\; \sin\,(b_n + \beta_n)} = 1$$

Let ξ represent the quantity:

$$\xi = \frac{A_o A_n}{A_o O} \; \frac{\sin c_1 \;\; \sin c_2 \cdots\cdots \sin c_n}{\sin b_1 \;\; \sin b_2 \cdots\cdots \sin b_n} - 1$$

The equation for the sides then becomes, neglecting the 2nd order:

(3) $\xi + \sum_1^n (\gamma_p \cot c_p - \beta_p \cot b_p) = 0$

The condition that the sum of the squares of the errors be reduced to the minimum must be added to the equations (1), (2), and (3). This condition will obtain if the function:

$$\sum_1^n \left[\left(\alpha_p + \frac{e}{n}\right)^2 + \left(\beta_p - \frac{e}{2n}\right)^2 + \left(\gamma_p - \frac{e}{2n}\right)^2\right] - 2\sum_1^n \lambda_p \left(\alpha_p + \beta_p + \gamma_p\right)$$
$$- 2\mu \sum_1^n \alpha_p - 2\nu \left[\xi + \sum_1^n (\gamma_p \cot c_p - \beta_p \cot b_p)\right]$$

be derived.

If the derivatives, with reference to α_p, β_p and γ_p, be annulled we get the equations:

$$\alpha_p + \frac{e}{n} = \lambda_p + \mu$$

$$\beta_p - \frac{e}{2n} = \lambda_p - \nu \cot b_p$$

$$\lambda_p - \frac{e}{2n} = \lambda_p + \nu \cot c_p$$

Introduce these values of α_p, β_p and γ_p into equations (1), (2) and (3) and let d_p represent the quantity $\cot c_p - \cot b_p$, then :

$$(4) \qquad \lambda_p + \frac{1}{3}\mu + \frac{1}{3}\nu d_p = 0 \quad \& \ (4\ bis) \ \sum_1^n \lambda_p + \frac{1}{3} n \mu + \frac{1}{3}\nu \sum_1^n d_p = 0$$

$$(5) \qquad e = \sum_1^n \lambda_p + n\mu.$$

$$(6) \qquad \xi + \frac{e}{2n}\sum_1^n d_p + \sum_1^n \lambda_p d_p + \nu \sum_1^n (\cot^2 c_p + \cot^2 b_p) = 0.$$

Multiplying the first term of equation (4) by d_p and summing from 1 to n, we get :

$$\sum_1^n \lambda_p d_p = -\frac{1}{3}\mu \sum_1^n d_p - \frac{1}{3}\nu \sum_1^n d_p^2$$

Introducing this value into equation (6), and replacing d_p^2 by its value, this becomes :

$$(6\ bis) \quad \xi + \left(\frac{e}{2n} - \frac{\mu}{3}\right)\sum_1^n d_p + \frac{2}{3}\nu \sum_1^n (\cot^2 c_p + \cot^2 b_p + \cot c_p \cot b_p) = 0$$

Taking s_p to represent the quantity $\cot^2 c_p + \cot^2 b_p + \cot c_p \cot b_p$, and eliminating the quantity $\sum_1^n \lambda_p$ in the equations (4 bis) and (5), we get the two equations :

$$3e = 2n\mu - \nu \sum_1^n d_p$$

$$3\xi + \frac{3}{2}\frac{e}{n}\sum_1^n d_p = \mu \sum_1^n d_p - 2\nu \sum_1^n s_p$$

for determining μ and ν.

From these we get that :

$$\mu = \frac{3}{n}\left[\frac{1}{2} e - \frac{\xi \sum_1^n d_p}{4\sum_1^n s_p - \frac{1}{n}\left(\sum_1^n d_p\right)^2}\right]$$

$$\nu = \frac{-6\xi}{4\sum_1^n s_p - \frac{1}{n}\left(\sum_1^n d_p\right)^2}$$

and thence, by equation (4) :

$$\lambda_p = -\frac{e}{2n} + \xi \frac{2d_p + \frac{1}{n}\Sigma d}{4\Sigma s - \frac{1}{n}\left(\Sigma d\right)^2}$$

Then we get the following values for α_p, β_p and γ_p :

(7) $$\alpha_p = 2\left(d_p - \frac{1}{n}\Sigma\, d\right)\frac{\xi}{4\,\Sigma\, s - \frac{1}{n}\left(\Sigma\, d\right)^2}$$

(8) $$\beta_p = -\frac{\alpha_p}{2} + 3\left(\cot c_p + \cot b_p\right)\frac{\xi}{4\,\Sigma\, s - \frac{1}{n}\left(\Sigma\, d\right)^2}$$

(9) $$\gamma_p = -\frac{\alpha_p}{2} - 3\left(\cot c_p + \cot b_p\right)\frac{\xi}{4\,\Sigma\, s - \frac{1}{n}\left(\Sigma\, d\right)^2}$$

These expressions give the angles in parts of a radius. They may be obtained in seconds of arc by putting $\frac{\xi}{\sin 1''}$ instead of ξ in the three expressions.

On the other hand, ξ being but a very small quantity, it can be replaced by the Napierian logarithm of $1 + \xi$, or by the difference between the sums of the ordinary logarithms of the factors of the numerator and of the denominator.

Besides if, as was done in *Hydrographic Review*, Vol. VIII, N° 2, the difference for 1" in the *log. sine* be called *dabla*, then :

$$dabla_c = M \cot c \sin 1'',$$

M being the factor for transforming Napierian logarithms into ordinary logarithms.

In expressions (7), (8) and (9) ξ may be replaced by the difference between the sums of the logarithms of the factors of the numerator and of the denominator of $1 + \xi$ and the cotangents by the corresponding *dablas*; the *M* and *sin* 1" factors will be the same in both numerator and denominator and are thus eliminated.

Hence the calculation of these expressions becomes very simple and rapid, and it is easy to ascertain the error introduced by using a simplified method.

The first method, on page 212 of *Hydrographic Review*, Vol. VIII, N° 2, assumes that :

$$\alpha_p = 0 \qquad \beta_p = \frac{\xi}{\Sigma\,(\cot c + \cot b)} \qquad \gamma_p = \frac{-\,\xi}{\Sigma\,(\cot c + \cot b)}$$

The second method, called the "semi-rigorous" (page 227), assumes that:

$$\alpha_p = 0 \qquad \beta_p = \frac{\xi \cot b_p}{\Sigma\,(\cot^2 c + \cot^2 b)} \qquad \gamma_p = \frac{-\,\xi \cot c_p}{\Sigma\,(\cot^2 c + \cot^2 b)}$$

and has the disadvantage that it does away with the closure of the triangles, which has to be reintroduced by a second approximation.

Note. — It should be noted that the quantity Σd in formulae (7), (8) and (9) is nearly always very small and thus, frequently, the very close approximations :

$$\alpha'_p = \frac{\xi\, d_p}{2\,\Sigma\, s} \qquad \beta'_p = -\frac{\alpha'_p}{2} + \frac{3}{4}\,\frac{\xi\,(\cot c_p + \cot b_p)}{\Sigma\, s}$$

$$\gamma'_p = -\frac{\alpha'_p}{2} - \frac{3}{4}\,\frac{\xi\,(\cot c_p + \cot b_p)}{\Sigma\, s}$$ may be used.

COMPENSATION OF A QUADRILATERAL

It is fairly simple, likewise, to indicate the general formula for the corrections to be applied to the angles of a quadrilateral (all of which will be assumed to have been observed and to be of equal weight) by reducing the sum of the squares of the corrections to the minimum.

The quadrilateral might have been considered as a polygon with a central station as set out in *Hydrographic Review*, Vol. VIII, N° 2, page 219, and then formulae for such polygon might have been applied. But this no longer leads to the consideration of the various distinct angles only, but also of the sum of some of them and, hence, the squares of the sums of the corrections of these angles. The results, therefore, would not be those given below and besides they would have the serious disadvantage of varying according to the angle of the quadrilateral which is taken as the central point.

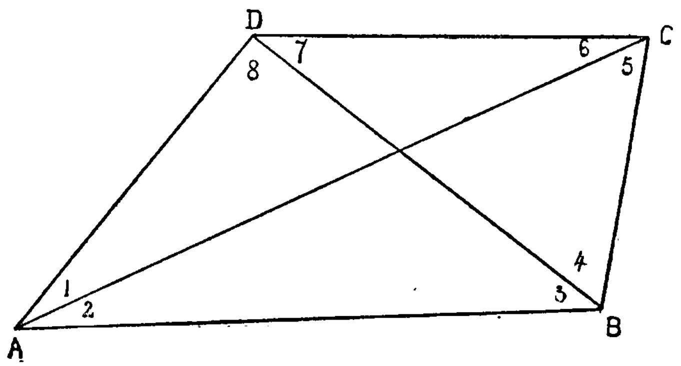

Fig. 2

First take the sum of the eight angles; let e_1 be the quantity to be added to bring the sum to 360°; then $\frac{1}{8} e_1$ is added to each angle.

If e_2 be the difference between the sum of angles 4 and 5 and the sum of angles 1 and 8, then $\frac{1}{4} e_2$ is added to angles 1 and 8 and $\frac{1}{4} e_2$ is subtracted from angles 4 and 5.

In the same way, if e_3 be the difference between the sums of angles 7 and 6 and of angles 2 and 3, $\frac{1}{4} e_3$ is added to each of the latter angles and subtracted from each of the former angles.

This is that which is done on page 218 of *Hydrographic Review*, Vol. VIII, N° 2. For the angles thus corrected, the names 1, 2, 3, 4, 5, 6, 7, and 8 will be used hereafter.

The equation for sides may be expressed:

$$\xi = \frac{\sin 1 \ \sin 3 \ \sin 5 \ \sin 7}{\sin 2 \ \sin 4 \ \sin 6 \ \sin 8} - 1.$$

Taking λ_1, λ_2, λ_3 and μ as the parameters and δ_1, δ_2, δ_3, ...δ_8 as the corrections yet to be applied to the angles, the quantity to be reduced to the minimum will be:

$$\left(\delta_1+\frac{e_1}{8}+\frac{e_2}{4}\right)^2+\left(\delta_2+\frac{e_1}{8}+\frac{e_3}{4}\right)^2+\left(\delta_3+\frac{e_1}{8}+\frac{e_3}{4}\right)^2+\left(\delta_4+\frac{e_1}{8}-\frac{e_2}{4}\right)^2$$
$$+\left(\delta_5+\frac{e_1}{8}-\frac{e_2}{4}\right)^2+\left(\delta_6+\frac{e_1}{8}-\frac{e_3}{4}\right)^2+\left(\delta_7+\frac{e_1}{8}-\frac{e_3}{4}\right)^2+\left(\delta_8+\frac{e_1}{8}+\frac{e_2}{4}\right)^2$$
$$-2\lambda_1\Sigma\delta-2\lambda_2\left(\delta_4+\delta_5-\delta_1-\delta_8\right)-2\lambda_3\left(\delta_6+\delta_7-\delta_2-\delta_3\right)$$
$$-2\mu\left(\xi+\delta_1 cot 1+\delta_3 cot 3+\delta_5 cot 5+\delta_7 cot 7-\delta_2 cot 2-\delta_4 cot 4-\delta_6 cot 6-\delta_8 cot 8\right)$$

The derivatives give the δ quantities which, introduced into the 3 equations:

$$\Sigma\delta=0,\qquad \delta_4+\delta_5=\delta_1+\delta_8\qquad \text{and}\qquad \delta_6+\delta_7=\delta_2+\delta_3,$$

allow the parameters:

$$\lambda_1=\frac{e_1}{8}-\frac{\mu}{8}\left(cot 1+cot 3+cot 5+cot 7-cot 2-cot 4-cot 6-cot 8\right)$$
$$\lambda_2=-\frac{e_2}{4}+\frac{\mu}{4}\left(cot 1+cot 4-cot 5-cot 8\right)$$
$$\lambda_3=-\frac{e_3}{4}+\frac{\mu}{4}\left(cot 3+cot 6-cot 2-cot 7\right)\quad \text{to be obtained.}$$

Then, introducing the values of the δ into the differential equation for sides we get:

$$\mu=\frac{-2\xi}{\begin{array}{l}(cot 1+cot 8)^2\\+(cot 4+cot 5)^2\\+(cot 2+cot 3)^2\\+(cot 7+cot 6)^2\end{array}+\left(\frac{cot 1+cot 2+cot 5+cot 6-cot 3-cot 4-cot 7-cot 8}{2}\right)^2}$$
$$=\frac{-2\xi}{P}$$

The δ corrections may be calculated, their expressions being, taking D as representing the quantity $\frac{cot 1+cot 2+cot 5+cot 6-cot 3-cot 4-cot 7-cot 8}{2}$:

$$\delta_2=+\frac{\xi}{P}\left(cot 2+cot 3+\tfrac{1}{2}D\right)\qquad \delta_3=-\frac{\xi}{P}\left(cot 2+cot 3-\tfrac{1}{2}D\right)\qquad \delta_2+\delta_3=+\frac{\xi D}{P}$$
$$\delta_4=+\frac{\xi}{P}\left(cot 4+cot 5-\tfrac{1}{2}D\right)\qquad \delta_5=-\frac{\xi}{P}\left(cot 4+cot 5+\tfrac{1}{2}D\right)\qquad \delta_4+\delta_5=-\frac{\xi D}{P}$$
$$\delta_6=+\frac{\xi}{P}\left(cot 6+cot 7+\tfrac{1}{2}D\right)\qquad \delta_7=-\frac{\xi}{P}\left(cot 6+cot 7-\tfrac{1}{2}D\right)\qquad \delta_6+\delta_7=+\frac{\xi D}{P}$$
$$\delta_8=+\frac{\xi}{P}\left(cot 8+cot 1-\tfrac{1}{2}D\right)\qquad \delta_1=-\frac{\xi}{P}\left(cot 8+cot 1+\tfrac{1}{2}D\right)\qquad \delta_8+\delta_1=-\frac{\xi D}{P}$$

As in the case of the polygon, ξ must be replaced by $\frac{\xi}{sin 1''}$ if the calculation be made with the sines and cotangents of the angles, but dablas and ordinary logarithmic sines may be used.

The first method set out on page 217 of *Hydrographic Review*, Vol. VIII, No 2 assumes that:

$$\delta_1 = \delta_3 = \delta_5 = \delta_7 = + \frac{\xi}{\Sigma\, cot} \qquad \delta_2 = \delta_4 = \delta_6 = \delta_8 = - \frac{\xi}{\Sigma\, cot}.$$

The semi-rigorous method, on page 224, assumes that:

$$\delta_1 = - \frac{\xi\, cot\ 1}{\Sigma\, cot^2} \qquad \delta_3 = - \frac{\xi\, cot\ 3}{\Sigma\, cot^2} \qquad \delta_5 = - \frac{\xi\, cot\ 5}{\Sigma\, cot^2} \qquad \delta_7 = - \frac{\xi\, cot\ 7}{\Sigma\, cot^2}$$

$$\delta_2 = + \frac{\xi\, cot\ 2}{\Sigma\, cot^2} \qquad \delta_4 = + \frac{\xi\, cot\ 4}{\Sigma\, cot^2} \qquad \delta_6 = + \frac{\xi\, cot\ 6}{\Sigma\, cot^2} \qquad \delta_8 = + \frac{\xi\, cot\ 8}{\Sigma\, cot^2}$$

Remark I. — The quantity D is usually small; it may frequently be neglected, which would greatly simplify the expressions for the δ and would be the same as if it were assumed that:

$$\delta_2 + \delta_3 = \delta_4 + \delta_5 = \delta_6 + \delta_7 = \delta_8 + \delta_1 = 0$$

Remark II. — The method of rigorous calculation set out here is identical, as to results, with that set out recently by Segundo Tenente Pires DE MATOS in an article entitled: "Compensaçâo de un quadrilàtero pelo método MAURY", which appeared in the *Anais do Club Militar Naval*, October, November, December 1931.

COMPENSATION OF A QUADRILATERAL IN THE CASE WHERE ONE OF ITS TRIANGLES HAS ALREADY BEEN DEFINITIVELY COMPENSATED.
(ANGLES OF EQUAL WEIGHT).

This case may happen when a secondary triangulation is connected with a triangle of a main triangulation, or when a new triangulation is connected to a triangle of an earlier triangulation, which has been definitively compensated.

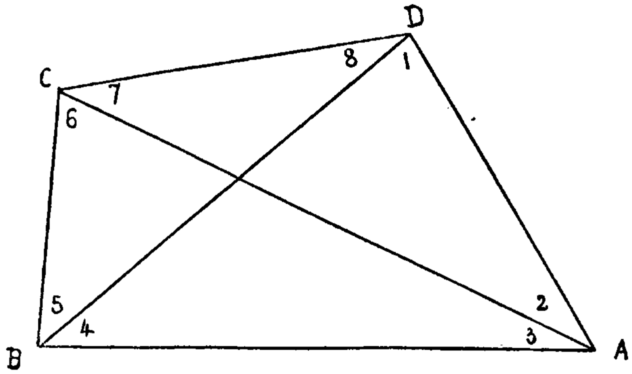

Fig. 3

Let $A\,B\,C\,D$ be the quadrilateral, the angles of which are numbered.

Triangle $B\,C\,D$ has been definitively compensated, *i. e.* the angles 5, 6 + 7 and 8 may not be altered.

The following preliminary corrections are made:

(1) Angles 6 and 7 are each increased by $\frac{\alpha}{2}$, so that

$$6 + 7 + \alpha = 180 - 5 - 8.$$

(2) Each of the angles, 1, 2, 3 and 4 is increased by $\frac{e_1}{4}$, so that

$$1 + 2 + 3 + 4 + e_1 = 180.$$

(3) To each of the angles $\underline{1}$, $\underline{2}$ and $\underline{7}$ thus corrected the quantity $\frac{e_2}{3}$ is added, this quantity being such that

$$\underline{1} + \underline{2} + \underline{7} + 8 + e_2 = 180,$$

and this same quantity is subtracted from angles $\underline{3}$, $\underline{4}$, and $\underline{6}$.

After these three preliminary corrections, the sums of the angles of every triangle of the quadrilateral are 180°. This result could have been attained otherwise, but, as shown by the calculation below, this method simplifies the ensuing operations more than any other.

The same steps will then be followed as in the calculation of the general compensation of a quadrilateral by the method of least squares, and further it will be assumed that all the angles are of equal weight and the same notation will be adopted. The angles to which the preliminary corrections have been applied will henceforth be referred to as 1, 2, 3, 4, 5, 6, 7 and 8.

In consequence of the previous definitive compensation of one of the triangles, δ_7 will be $= - \delta_6$ and there will be but two equations for the angles.

If the parameters be called λ_1, λ_2, and μ, the quantity to be reduced to a minimum is:

$$\left(\delta_1+\frac{e_1}{4}+\frac{e_2}{3}\right)^2 + \left(\delta_2+\frac{e_1}{4}+\frac{e_2}{3}\right)^2 + \left(\delta_3+\frac{e_1}{4}-\frac{e_2}{3}\right)^2 + \left(\delta_4+\frac{e_1}{4}-\frac{e_2}{3}\right)^2$$

$$+\left(\delta_6+\frac{\alpha}{2}-\frac{e_2}{3}\right)^2+\left(\delta_6-\frac{\alpha}{2}-\frac{e_2}{3}\right)^2-2\lambda_1\left(\delta_1+\delta_2+\delta_3+\delta_4\right)-2\lambda_2\left(\delta_1+\delta_2-\delta_6\right)$$

$$-2\mu\left(\xi+\delta_1\cot 1+\delta_3\cot 3-\delta_6\cot 7-\delta_2\cot 2-\delta_4\cot 4-\delta_6\cot 6\right)$$

The derivatives with reference to the five unknowns δ give these quantities which, if introduced into the two equations:

$$\delta_1 + \delta_2 + \delta_3 + \delta_4 = 0 \qquad \text{and} \qquad \delta_1 + \delta_2 - \delta_6 = 0$$

allow the parameters λ_1 and λ_2 to be found. The expression for these may be simplified by taking:

$$D = \cot 1 - \cot 2 - \cot 3 + \cot 4 - 2\cot 6 - 2\cot 7.$$

We thus get :

$$\lambda_1 = \frac{e_1}{4} - \frac{e_2}{3} - \frac{\mu}{2}\left(\frac{D}{6} + cot\,3 - cot\,4\right)$$

$$\lambda_2 = \frac{2}{3}\ e_2 - \mu\left(\frac{D}{3} + cot\ 6 + cot\ 7\right)$$

By introducing these values into the equations which give the δ' s, these quantities are obtained as functions of the single parameter μ.

The differential equation for sides then becomes :

$$(*)\quad \mu = \frac{-2\,\xi}{\frac{D^2}{6} + (cot\,1 + cot\,2)^2 + (cot\,3 + cot\,4)^2} = -2\frac{\xi}{P}$$

The values of the δ's then take the forms :

$$\delta_1 = -\frac{\xi}{P}\left(cot\,1 + cot\,2 + \frac{D}{6}\right) \qquad \delta_2 = +\frac{\xi}{P}\left(cot\,1 + cot\,2 - \frac{D}{6}\right) \qquad \delta_1 + \delta_2 = -\frac{\xi D}{3P}$$

$$\delta_3 = -\frac{\xi}{P}\left(cot\,3 + cot\,4 - \frac{D}{6}\right) \qquad \delta_4 = +\frac{\xi}{P}\left(cot\,3 + cot\,4 + \frac{D}{6}\right) \qquad \delta_3 + \delta_4 = +\frac{\xi D}{3\,P}$$

$$\delta_7 = +\frac{\xi}{P}\frac{D}{3} \qquad \delta_6 = -\frac{\xi}{P}\frac{D}{3} \qquad \delta_7 + \delta_6 = 0.$$

As has already been indicated, the difference between the sums of the vulgar logarithms of the *sines* of the even numbered angles and of the odd numbered angles is taken as ξ and the *dablas* replace the cotangents.

Remark I in the general case of quadrilaterals is not applicable here for quantity D no longer contains as many positive as negative terms and thus its value is generally not negligeable. It should be noted that the sum of the squares of the total errors is in very simple form :

$$\frac{2\,\xi^2}{P} + \frac{e_1^2}{4} + 2\frac{e_2^2}{3} + \frac{\alpha^2}{2}$$

The first method of compensation set out on page 212 of *Hydrographic Review*, Vol. VIII, N° 2, allows for the same preliminary corrections being made to angles 6 and 7, but instead of correcting each of the angles 1, 2, 3 and 4 by $\frac{e_1}{4}$, as is done here, the angles 1 and 4 are each corrected by $\frac{e_1}{3}$ and 2 and 3 by $\frac{e}{6}$ each.

Then, instead of correcting each of the angles 1, 2 and 7 by $+\frac{e_2}{3}$ and each of the angles 3, 4 and 6 by $-\frac{e_2}{3}$, the angles 2 and 7 are each corrected by $+\frac{e_2}{2}$ and 3 and 6 each by $-\frac{e_2}{2}$.

(*) The fact that the expressions for μ and for the δ's are independent of e_1 and e_2 is due to the selection of the three preliminary corrections introduced. Methods which use arbitrarily simplified values for δ independent of e_1 and e_2 will benefit greatly by applying these three corrections beforehand.

Angles	Corrected Angles. Angles corrigés	2. corr.	Corrected Angles. Angles corrigés	3. corr.	Corrected Angles. Angles corrigés	Log. Sin.	4. corr.	Definitive Angles Angles définitifs	Log. Sin.
1	42 46 35,5	— 1,05	42 46 34,45	+ 0,83	42 46 35,28	9,8319592	+ 10,17	42 46 45,45	9824
3	46 00 31,3	— 1,05	46 00 30,25	— 0,83	46 00 29,42	9,8569939	+ 19,57	46 00 48,99	70337
5	46 16 28,8		46 16 28,80		46 16 28,80	9,8589350	0	46 16 28,80	9350
7	35 48 30,2		35 48 30,20	+ 0,83	35 48 31,03	9,7672150	+ 17,02	35 48 48,05	2647
						9,3151031			2158
2	32 23 02,3	— 1,05	32 23 10,25	+ 0,83	32 23 02,08	9,7288322	— 27,19	32 22 34,89	7419
4	58 49 55,1	— 1,05	58 49 54,05	— 0,83	58 49 53,22	9,9322954	— 2,55	58 49 50,67	2921
6	28 53 09,4		28 53 09,40	— 0,83	28 53 08,57	9,6840047	— 17,02	28 52 51,55	9398
8	69 01 51,6		69 01 51,60		69 01 51,60	9,9702419	0	69 01 51,60	2419
	360 00 04,2	— 4,20	360 00 00,00			9,3153742		360 00 00	2157
	$e_1 = -4,2$					$\xi = -2711$			

Angles			Δ dabla		Δ dabla
3	46 00 30,25	1	22,7	2	33,2
4	58 49 54,05	4	12,8	3	20,3
5	46 16 28,80			2 × 6	76,4
6	28 53 09,40		35,5	2 × 7	58,4
	180 00 02,50				—188,3
	$e_2 = +2,5$				+ 35,5
				$D =$	—152,8

$\Delta_1 + \Delta_2 = 55,9$

$\Delta_3 + \Delta_4 = 33,1$

$(\Delta_1 + \Delta_2)^2 = 3124,81$

$(\Delta_3 + \Delta_4)^2 = 1095,61$

$\frac{D^2}{6} = 3891,31$

$P = 8111,73$

$$\frac{\xi}{P} = -\frac{2711}{8111,73} = 0,33421$$

$D^2 = 23347,84$

$$\frac{\xi}{P} \times \frac{D}{6} = +0,33421 \times 25,47 = +8,51$$

$\delta_2 = -55,9 \times 0,33421 - 8,51$

$\delta_4 = -33,1 \times 0,33421 + 8,51$

$\delta_6 = -17,02$

$\delta_1 = +55,9 \times 0,33421 - 8,51$

$\delta_3 = +33,1 \times 0,33421 + 8,51$

$\delta_7 = +17,02$

These angles thus corrected are used for calculating the final correction, for which the following quantity is arbitrarily adopted and which, with our notation, becomes for angles 2 and 3 :

$$\frac{2\,\xi'}{\cot 1 + \cot 4 + \cot 6 + \cot 7 + 2\cot 2 + 2\cot 3}$$

and, for angles 1, 4, 6 and 7, half of this value.

These last corrections differ from those which we deduced above by the rule of the least squares and called δ, not only in a considerable difference in the formulae but also in the values of the angles to which exactly the same preliminary corrections have not been applied.

The number ξ, particularly, which is a factor in the two expressions of the final corrections, is different and, taking ξ' as that which corresponds to the method in *Hydrographic Review*, Vol. VIII, N° 2, we get :

$$\xi' - \xi = \frac{e_1}{12}\left(\cot 1 + \cot 2 - \cot 3 - \cot 4\right) - \frac{e_2}{6}\left(2\cot 1 + 2\cot 4 + \cot 2 + \cot 3 + \cot 7 - \cot 6\right)$$

The above method of least squares will now be applied to the example given by Admiral NARES, after he had applied the corrections for weight. The numbers are those of our notation. (See table on previous page.)

COMPENSATION OF A QUADRILATERAL IN THE CASE WHERE ONE OF ITS TRIANGLES HAS ALREADY BEEN DEFINITIVELY COMPENSATED. (ANGLES OF UNEQUAL WEIGHT).

The same problem will now be examined for the case where the values of the angles are of unequal weight ; the weight being referred to as p (see Fig. 3).

Let s be the sum :

$$s = p_1\,p_2\,p_3 + p_2\,p_3\,p_4 + p_3\,p_4\,p_1 + p_4\,p_1\,p_2$$

and q the quantity :

$$q = s + (p_1 + p_2)\,(p_3 + p_4)\,(p_6 + p_7)$$

The following preliminary corrections are made (α, e_1 and e_2 being the same quantities as before) :

(1) Angle 6 is increased by $\dfrac{p_7}{p_6 + p_7}\,\alpha$ and angle 7 by $\dfrac{p_6}{p_6 + p_7}\,\alpha$.

(2) Angles 1, 2, 3 and 4 are increased by :

$$\frac{p_2\,p_3\,p_4}{s}\,e_1, \qquad \frac{p_3\,p_4\,p_1}{s}\,e_1, \qquad \frac{p_4\,p_1\,p_2}{s}\,e_1, \qquad \frac{p_1\,p_2\,p_3}{s}\,e_1,$$

respectively.

(3) The quantities:

$$\frac{p_2\ (p_6+p_7)\ (p_3+p_4)}{q}\,e_2, \qquad \frac{p_1\ (p_6+p_7)\ (p_3+p_4)}{q}\,e_2, \qquad \frac{s}{q}\,e_2$$

are added to the angles 1, 2 and 7 corrected as above, respectively, and the quantities:

$$\frac{p_4\ (p_6+p_7)\ (p_1+p_2)}{q}\,e_2, \qquad \frac{p_3\ (p_6+p_7)\ (p_1+p_2)}{q}\,e_2, \qquad \frac{s}{q}\,e_2$$

are subtracted from angles 3, 4 and 6, respectively.

The angles, thus corrected, will be referred to as 1, 2, 3, 4, 5, 6, 7 and 8 in future.

Working as before, the values λ_1 and λ_2 are found, taking:

$$D = p_2 \cot 1 - p_1 \cot 2 - \frac{p_1+p_2}{p_3+p_4}\left(p_4 \cot 3 - p_3 \cot 4\right) - \left(p_1+p_2\right)\left(\cot 6 + \cot 7\right)$$

$$\lambda_1 = p_3p_4p_1p_2\frac{e_1}{s} - p_3p_4\left(p_1+p_2\right)\left(p_6+p_7\right)\frac{e_2}{q} - \mu\left(p_3p_4\frac{D}{q} + \frac{p_4 \cot 3 - p_3 \cot 4}{p_3+p_4}\right)$$

$$\lambda_2 = \left(p_6+p_7\right)\frac{s}{q}\,e_2 - \mu\left[\left(p_3+p_4\right)\left(p_6+p_7\right)\frac{D}{q} + \cot 6 + \cot 7\right]$$

Then the value of μ will be:

$$\mu = \frac{-\xi\ (p_1+p_2)}{\frac{D^2}{q}\left(p_3+p_4\right) + \left(\cot 1 + \cot 2\right)^2 + \frac{p_1+p_2}{p_3+p_4}\left(\cot 3 + \cot 4\right)^2} = -\left(p_1+p_2\right)\frac{\xi}{P}$$

The values of δ thus become:

$$\delta_1 = -\frac{\xi}{P}\left[\cot 1 + \cot 2 + p_2\left(p_3+p_4\right)\frac{D}{q}\right] \qquad \delta_2 = +\frac{\xi}{P}\left[\cot 1 + \cot 2 - p_1\left(p_3+p_4\right)\frac{D}{q}\right]$$

$$\delta_3 = -\frac{\xi}{P}\frac{p_1+p_2}{p_3+p_4}\left[\cot 3 + \cot 4 - p_4\left(p_3+p_4\right)\frac{D}{q}\right] \qquad \delta_4 = +\frac{\xi}{P}\frac{p_1+p_2}{p_3+p_4}\left[\cot 3 + \cot 4 + p_3\left(p_3+p_4\right)\frac{D}{q}\right]$$

$$\delta_7 = +\frac{\xi}{P}\left(p_1+p_2\right)\left(p_3+p_4\right)\frac{D}{q} \qquad \delta_6 = -\frac{\xi}{P}\left(p_1+p_2\right)\left(p_3+p_4\right)\frac{D}{q}$$

$$\delta_1+\delta_2 = -\frac{\xi}{P}\left(p_1+p_2\right)\left(p_3+p_4\right)\frac{D}{q} \qquad \delta_3+\delta_4 = +\frac{\xi}{P}\left(p_1+p_2\right)\left(p_3+p_4\right)\frac{D}{q} \qquad \delta_6+\delta_7 = 0.$$

As before, ξ is replaced by the difference of the sums of the vulgar logarithms of the *sines* of the even numbered angles and of the uneven numbered angles, and the cotangents by the *dablas*.

The calculation can be done more rapidly by using these formulae than by solving the equations of condition again for each particular case.

The calculation of the example given by Admiral NARES in this Review is set out in the table on following page.

▣ ▣ ▣

Nº	Wght Poids	Angles	1. corr.		2. corr.		3. corr.		Log. Sin.	Δ	4. corr.	Defin. Angles. Angles définit.	Log. Sin.
1	0,4	42 46 47,5		47,5	− 8,26	39,24	+ 5,74	42 46 44,98	9,8319813	22,8	+ 19,94	42 47 04,92	9,8320267
3	1,3	46 00 34,2		34,2	− 2,54	31,66	— 3,89	46 00 27,77	9,8569905	20,3	+ 15,54	46 00 43,31	9,8570221
5		46 16 28,8		28,8				46 16 28,80	9,8589350			46 16 28,80	9,8589350
7	1,5	35 48 23,2	+ 4,3	35 48 27,5			+ 1,39	35 48 28,89	9,7672088	29,2	+ 8,85	35 48 37,74	9,7672346
									9,3151156				9,3152184
2	1,5	32 22 55,2		55,2	− 2,20	53,00	+ 1,53	32 22 54,53	9,7288071	33,2	— 28,80	32 22 25,73	9,7287115
4	1,5	58 49 58,3		58,3	− 2,20	56,10	— 3,38	58 49 52,72	9,9322947	12,8	— 6,69	58 49 46,03	9,9322862
6	3,8	28 53 10,4	+ 1,7	28 53 12,1			— 1,39	28 53 10,71	9,6840129	38,1	— 8,85	28 53 01,86	9,6839791
8		69 01 51,6		51,6				69 01 51,60	9,9702419			69 01 51,60	9,9702419
				e_1 = − 15,2				00,00	9,3153566				9,3152187
6+7 Obs.	5,3	64 41 33,6											
		39,6											
		α = + 6,0											

(1) 39,24
(2) 53,00
(7) 27,50
(8) 51,60
51,34 $e_2 = +8{,}66$

$\xi = -\ 2410$

$$\frac{1{,}5 \times 6}{5{,}3} = 1{,}7 \qquad \frac{3{,}8 \times 6}{5{,}3} = 4{,}3$$

	$\times \frac{e_1}{s}$		$\times \frac{e_2}{q}$		$\times \frac{e_2}{q}$
0,4 × 1,5 × 1,3 = 0,78	— 2,20	1,5 × 5,3 × 2,8 = 22,260	5,74	1,5 × 5,3 × 1,9 = 15,105	3,89
1,5 × 1,3 × 1,5 = 2,925	— 8,26	0,4 × 5,3 × 2,8 = 5,936	1,53	1,3 × 5,3 × 1,9 = 13,091	3,38
		28,196		28,196	
1,3 × 1,5 × 0,4 = 0,78	— 2,20				
1,5 × 0,4 × 1,5 = 0,90	— 2,54	s = 5,385	1,39		1,39
s = 5,385	— 15,20	q = 33,581	8,66		8,66
$\frac{e_1}{s} = -\frac{15{,}2}{5{,}385} = -2{,}82$		$\frac{e_2}{q} = +\frac{8{,}66}{33{,}581} = 0{,}258$			

$$1{,}5 \times 22{,}8 = 34{,}20 \qquad 0{,}4 \times 33{,}2 = 13{,}28$$

$$1{,}3 \times \frac{1{,}9}{2{,}8} \times 12{,}8 = 11{,}29 \qquad \frac{1{,}9}{2{,}8} \times 1{,}5 \times 20{,}3 = 20{,}66$$

$$+\ 45{,}49 \qquad 1{,}9 \times 67{,}3 = 127{,}87$$

$$-\ 161{,}81 \qquad -\ 161{,}81$$

$$D = -\ 116{,}32$$

$$\frac{D}{33{,}581} = -\ 3{,}46$$

$$\frac{D^2}{33{,}581} \times 2{,}8 = 1128{,}17$$

$$\Delta_1 + \Delta_2 = 56{,}0 \qquad (\Delta_1 + \Delta_2)^2 = 3136$$

$$\Delta_3 + \Delta_4 = 33{,}1 \qquad (\Delta_3 + \Delta_4)^2 \times \frac{1{,}9}{2{,}8} = 743{,}45$$

$$P = \frac{1128{,}17}{5007{,}62}$$

$$\frac{\xi}{P} = -\frac{2410}{5007{,}62} = -\ 0{,}481$$

$$\delta_1 = +\ 0{,}481\ (56{,}0 - 1{,}5 \times 2{,}8 \times 3{,}46) = +\ 19{,}94$$

$$\delta_2 = -\ 0{,}481\ (56{,}0 + 0{,}4 \times 2{,}8 \times 3{,}46) = -\ 28{,}80$$

$$\delta_3 = +\ 0{,}481\ (33{,}1 + 1{,}5 \times 2{,}8 \times 3{,}46) \times \frac{1{,}9}{2{,}8} = +\ 15{,}54$$

$$\delta_4 = -\ 0{,}481\ (33{,}1 - 1{,}3 \times 2{,}8 \times 3{,}46)\ \frac{1{,}9}{2{,}8} = -\ 6{,}69$$

$$\delta_6 = +\ (\delta_1 + \delta_2) = -\ 8{,}85$$

$$\delta_7 = -\ (\delta_1 + \delta_2) = +\ 8{,}85$$