NOTES ON NAUTICAL CARTOGRAPHY.

ORTHODROMIC CHARTS FOR FIXING THE POSITION BY RADIO-GONIOMETRIC BEARINGS.

by

CAPTAIN L. TONTA, DIRECTOR.

1.- In accordance with recent usage (MAURER, *Annalen der Hydrographie*, etc., 1919, page 75) we shall use the term *orthodromic* to describe those cartographic projections in which the *orthodromes* — the geodetic lines of the sphere — are represented by straight lines.

Every orthodromic chart may be termed a gnomonic projection or a homographic transformation thereof. (1)

It is particularly interesting and at the same time a very simple matter to make the transformation by which a gnomonic projection is transformed into an *orthodromic projection, conformal at two points.*

This is the term applied to the orthodromic projection which differs from the gnomonic projection in that the angles are reproduced faithfully at two given points (*the fundamental points*) and not at a single point only (*the central point*, or *the centre of the projection*).

This projection, which is well-known in Germany, is also called the "*doppelazimutaler gnomonischer Kartenentwurfe*" (IMMLER) ; in England it is called the "*two point azimuthal, or double azimuthal, projection*" (CLOSE, MCCAW).

2.- Given the two fundamental points, P and Q, *on the terrestrial sphere*, let us consider the orthodrome PQ and its middle point C. Let Δ_0 be the spherical distance separating the points P and Q from C.

$$PC = QC = \frac{PQ}{2} = \Delta_0$$

The gnomonic projection, constructed by taking point C as a centre, is that which, suitably transformed, gives rise to the orthodromic projection conformal at the two given fundamental points. Let p and q be the projections of P and Q on this gnomonic projection and let γ be the plan of this latter.

The transformation is effected by taking a straight line Y perpendicular to the straight line pq on γ, passing a plane ω through Y inclined at the angle Δ_0 to γ, and then projecting *orthogonally* the given gnomonic projection onto the plane ω.

This is equivalent to carrying out the following operations :-

a) Consider on the plane of the gnomonic projection a system of axes of orthogonal coordinates, X and Y, the axis of the *abscissae* being parallel to the straight line pq joining the fundamental points ;

b) Consider on the plane of the new projection a system of axes of orthogonal coordinates, X' and Y' ;

c) Make the point A' ($x' = x \cos \Delta_0$; $y' = y$) on the new projection correspond to each point A (x, y) of the original gnomonic projection.

In short it is a question of multiplying the abscissae (taken parallel to the straight line pq) of the gnomonic projection by the cosine of the spherical distance Δ_0, without changing the ordinates.

Above all it is evident that in the new method of representation the orthodromes will be straight lines, as in the original projection. It remains to be demonstrated that the transformed projection is conformal for the two given fundamental points.

3.- For this purpose may we recall certain *general properties of the representations of surfaces*, enunciated by TISSOT (*Mémoire sur la représentation des surfaces et les projections des cartes géographiques*). We cannot believe that these properties are unknown, but we must note that they are frequently overlooked, to the great detriment of the simplification and elegance of the demonstration.

As is customary, we shall call the *modulus of deformation* or of *linear reduction*, or more simply the *linear modulus*, the ratio $\frac{ds_1}{ds}$, according to which the infinitely small linear elements are augmented or reduced on the representation in passing from the portrayal of the surface S to the portrayal of the surface S_1. According to TISSOT, we may say :-

a) Generally, the linear modulus depends on the *position* of the point at which the element ds begins and on the *direction* of this same element.

b) At each point there are two directions which give maximum and minimum values of the modulus ; these directions are perpendicular to each other *on the two surfaces* and are called the *principal directions* ; in the same manner the *maximum modulus* m_1 and the *minimum modulus* m_2, which correspond respectively to them, are called the *principal moduli.*

c) The representation of the surface S_1 of an infinitely small *circular* figure described on the surface S becomes an *ellipse* which, taking the radius ds of the circle as unity, has its axes oriented in the principal directions and they are equal respectively to the moduli m_1 and m_2. This ellipse is called the *indicating ellipse* because it serves to indicate several properties of the representation of the surface S on the surface S_1. (2)

d) When the two principal moduli are equal to each other, this means that at the point where this occurs, the modulus is *independent* of the direction of the element. The indicating conic is then reduced to a circle and it follows that the corresponding infinitely small figures on the two surfaces are similar. The angles which have their apices at such point are maintained unchanged in the representation ; in other words at these points the *representation is conformal.*

4.- In the gnomonic projection, assuming the plane of the projection to be a plane tangent to the sphere, each linear element ds of the terrestrial surface proceeding from the *centre* (or point of tangency) C is reproduced at its true length and consequently :-

$$m_1 = m_2 = 1.$$

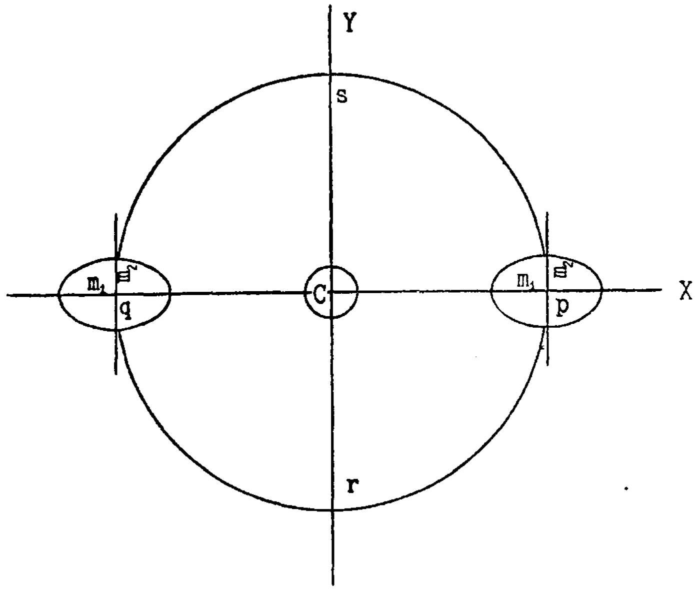


Fig. 1

Therefore, at the centre of the projection, the indicating conic becomes a circle (fig. 1) and the angles are retained unchanged. At every other point the similarity between corresponding infinitely small figures *does not exist* and consequently we have *angular deformations* ; the principal directions (directions of the axes of the indicating ellipse) coincide with the radius and with the tangent to the circle described about C as centre ; the principal moduli (the semi-axes of the ellipse) depend solely on the spherical distance Δ which separates the point under consideration from the centre of the projection C. They have the following values :-

$$m_1 = \frac{1}{\cos^2 \Delta} \qquad m_2 = \frac{1}{\cos \Delta}$$

Consequently, at points equi-distant from C we obtain equal indicating ellipses, and further, for each couple of points diametrically opposed with respect to C, such as by construction, the points p and q which correspond to the fundamental points P and Q, the axes of these ellipses lie in the same direction and their major axis coincides with the diameter pq.

Let m'_1 and m'_2 be the values of the principal moduli at p and q and let Δ_0 be the spherical distance which separates P and Q from C.

It is evident that if we select the diameter pq (or a straight line parallel to it) as the axis of the abscissae x, and a perpendicular (for instance, the diameter rs) as the axis of the ordinates, y, and if we multiply the values of x by the coefficient

$$\frac{m'_2}{m'_1} = \cos \Delta_0$$

without altering the values of y (which is equivalent to a uniform contraction in the direction of the straight line pq), the indicating ellipses at p and q will be transformed into a circle of radius m'_2.

At the same time the indicating conic at C, which was originally a circle, becomes an ellipse in which the *semi-major axis* = 1 (lying in the direction of rs) and the *semi-minor axis* = $\cos \Delta_0$ (oriented in the direction pq) ; *i. e.*, it is similar to the original indicating ellipses at p and q. Consequently the transformed projection is no longer conformal at C, but on the contrary, is conformal at the two fundamental points, as we desired to demonstrate.

5.- The linear and angular deformations at every other point of the *transformed projection* may be determined in a very simple and elegant manner by considering the modifications which the original indicating ellipse undergoes as a result of the transformation. Having thus determined the magnitude and position of the axes of the transformed indicating conic, we know the moduli and the principal directions, that is, the elements necessary to calculate the new linear and angular deformations (in this connection *see* note 2).

It is clear that this determination is particularly easy on the straight lines corresponding to the orthogonal diameters pq and rs (produced indefinitely) ; but even at other points in the representation the determination is reduced to a simple problem in analytical geometry, which is hardly worth discussion.

6.- Figure 2 represents the conformal orthodromic projection for the following two fundamental points :-

$$P \left\{ \begin{array}{ll} \text{latitude} & 10^\circ \text{ N.} \\ \text{longitude} & 60^\circ \text{ W.} \end{array} \right. \qquad Q \left\{ \begin{array}{ll} \text{latitude} & 50^\circ \text{ N.} \\ \text{longitude} & 0^\circ. \end{array} \right.$$

This is a partial reproduction of the illustrating example given by W. IMMLER in his article "*Ein doppelazimutaler gnomonischer Kartenentwurf*" in the *Annalen der Hydrographie* 1919, pages 22 *et seq.*

The orthodrome PQ has a length of 63° 17.2', and its middle point is

$$C \left\{ \begin{array}{l} \text{latitude } 33^\circ\ 30.6' \text{ N.} \\ \text{longitude } 36^\circ\ 55.9' \text{ W.} \end{array} \right.$$

$$\Delta_0 = PC = QC = \frac{63^\circ\ 17.2'}{2} = 31^\circ\ 38.6'.$$

7.- It is clear, that in view of the principles enunciated by TISSOT, the transformation of the gnomonic projection into the orthodromic projection conformal at two points becomes a very simple problem and, in a manner of speaking, it works out quite spontaneously. (3)

So far as we know, this cannot be said of the majority of demonstrations published till now ; it would almost seem as though the remark with reference to the demonstrations, "*that was accomplished with tolerably heavy artillery*" ("*Das ist mit ziemlich schwerem Geschütz gearbeitet*", MAURER in the *Zeitschrift für Vermessungswesen*, 1st. Jan. 1922, page 15) is very appropriate.

It is superfluous to state that the numerous investigations made on the subject of this new construction have been undertaken in view of the construction of *radio-goniometric charts*. We would not venture to assert that a

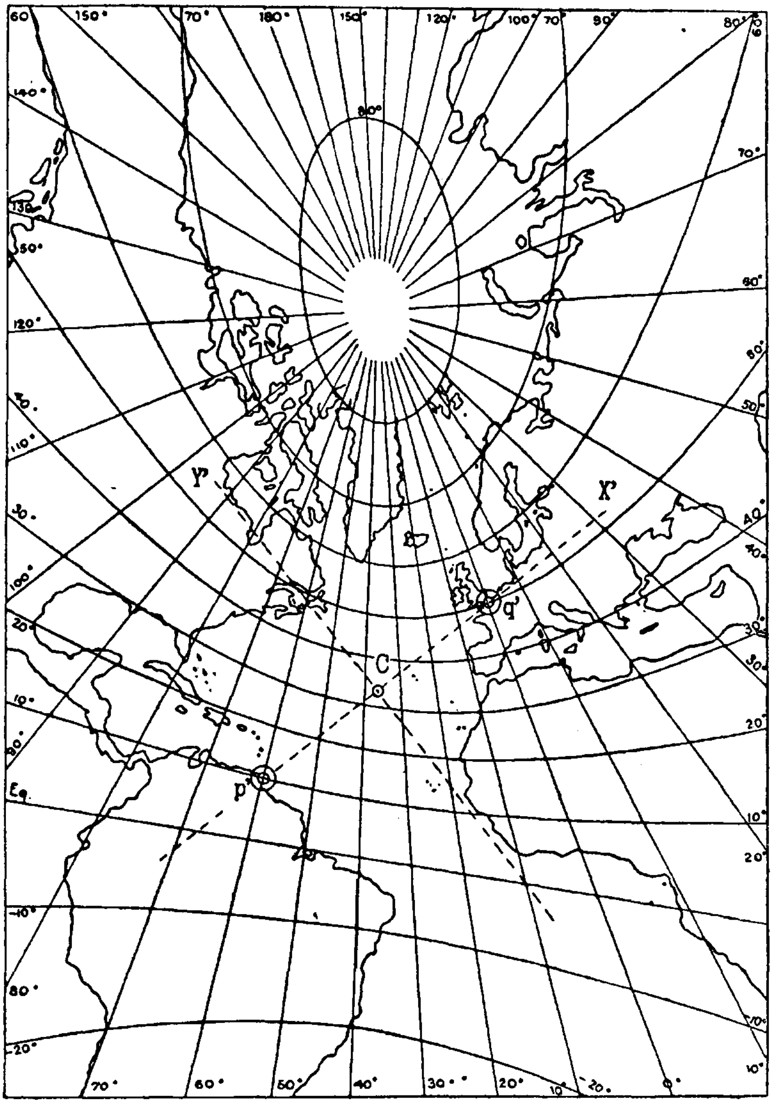


Fig. 2

practical solution of this problem requires that a special projection be drawn for each pair of radio-goniometric stations ; on the contrary, we are convinced that the same object may be attained "by employing very much lighter artillery". If we are not mistaken this conclusion has the sanction of present-day practice in cartography. In fact, and regardless of such scientific discussions, none of the published radio-goniometric charts now on sale are constructed on the new orthodromic projection. All known charts may be taken in fact to be simple *gnomonic projections, either with or without modified compass roses.*

Modified compass roses are not inserted except at points where the distance of the radio-goniometric stations from the centre of the projection is sufficiently great to cause an appreciable angular deformation. This is the case, for example, in the gnomonic chart "*C*" issued by the Canadian Hydrographic

Office (Ottawa), entitled "Strait of Belle Isle to Cape Sable" which covers a region of extensive landing areas.

If, however, the region represented is not very extensive and consequently angular deformations are negligible, the insertion of a special compass rose is unnecessary, and, for the solution of the radio-goniometric problems at least, it may be assumed that the chart is conformal at every point.

To give some idea of the importance of the deformation of the compass rose at a given point on a gnomonic projection, it may be stated that the *maximum alteration* ε in azimuth (*see* note 2) which occur at various distances Δ from the centre of the projection, is of the following value :-

$\Delta = 5^\circ$	$\varepsilon = 0^\circ\ 07'$	$\Delta = 35^\circ$	$\varepsilon = 5^\circ\ 41'$
10°	$0^\circ\ 26'$	40°	$7^\circ\ 33'$
15°	$1^\circ\ 00'$	45°	$9^\circ\ 44'$
20°	$1^\circ\ 47'$	50°	$12^\circ\ 16'$
25°	$2^\circ\ 49'$	55°	$15^\circ\ 10'$
30°	$4^\circ\ 07'$	60°	$18^\circ\ 26'$
		etc., etc...	

NOTES.

1. *a*) FIORINI. (*Le projezioni delle Carte Geografiche,* Bologna, 1881, page 134) cites an investigation made by the mathematician BELTRAMI (*Annali di Matematica pura ed applicata,* Tomo VII, Roma 1865, page 185) in which the following problem is examined :- Transfer of points on a surface to a plane in such a manner that the geodetic lines shall be represented by straight lines. In this article BELTRAMI demonstrated the following theorem :- The only surfaces capable of being represented on a plane in such a manner that every one of its points shall correspond to a point and that every geodetic shall correspond to a straight line are those surfaces the curvature of which is constant throughout (positive, negative or nil). When the curvature is nil the law of correspondence does not differ from the ordinary law of homography. When the constant curvature is *not nil* (spherical surface) this law becomes that of the central (or gnomonic) projection of the sphere and of its homographic transformations.

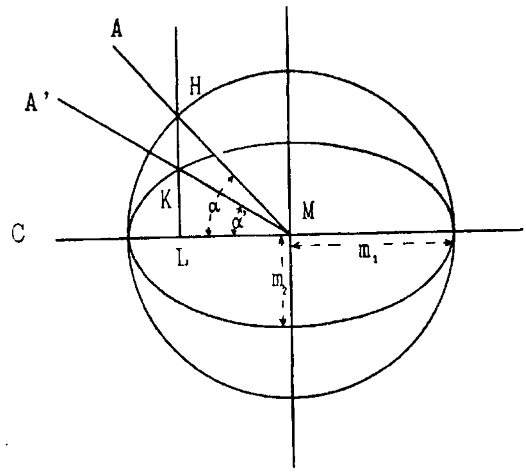
b) Two figures are said to be *homographic* (CHASLES) when for the points on a straight line in the first figure there exist corresponding points on a straight line in the second figure, and the converging lines in the first figure correspond to the converging lines in the second figure ; in other words when the two figures possess the same descriptive properties.

The *homographic transformation* is the operation carried out in passing from one figure to the other. The homographic transformation does not change the degree of the curves, etc...

2. Further it serves to determine the *angular deformations* of the method of representation by making a simple and elegant construction ; this construction is definitely based on the consideration of the indicating ellipse.

Let α be the angle which a linear element ds, originating at the given point M on the surface S, makes with the principal direction corresponding to the major axis of the ellipse (m_1) and let α' be the corresponding angle on the representation S_1 ; we may demonstrate that

$$(*)\quad \tan \alpha' = \frac{m_1}{m_2} \tan \alpha$$



Consequently, given the angle α, we may construct the angle α' as follows :- Draw the indicating ellipse of which the semi-axes m_1 and m_2 are known (or a similar ellipse), and draw the circle of which the major axis is the diameter. Let M be its centre (see figure). Draw a straight line MA at the given angle α from the principal direction MC ; let H be the point where MA cuts the circle. From H drop the perpendicular HL onto the diameter MC, which cuts the ellipse at K. The straight line MA' drawn to join M and K meets the straight line MC at the required angle α'. The angle $A'MA = \alpha - \alpha'$ and is the amount of the *azimuthal alteration relative to the direction* α.

Let ε be the *maximum value* of the azimuthal alteration, we have

$$(^{**}) \quad \sin \varepsilon = \frac{m_1 - m_2}{m_1 + m_2}$$

It is on this principle that the construction of the *modified compass rose* is based for a given point of a given cartographic projection, when the values of the *principal moduli* m_1 and m_2 and the *principal directions* are known (for this purpose it is well to make use of the theory of the projection under consideration). In actual practice the determination of α' is accomplished by calculation, that is by numerical solution of the formula (*). The graphic construction which has been described above then serves only as an auxiliary to facilitate the interpretation of the numerical results obtained by calculation, without ambiguity.

As stated before, the principles enunciated here are valid for all types of projections. The formulae (*) and (**) applied to the *gnomonic projection* give :-

$$(^{***}) \quad \tan \alpha' = \frac{1}{\cos \Delta} \tan \alpha$$

$$(^{****}) \quad \sin \varepsilon = \tan^2 \frac{\Delta}{2}$$

in which Δ is the spherical distance separating the point under consideration from the centre of the projection (point of tangency of the plane of the projection with the sphere).

3. It may not be without interest to note that *not only the gnomonic projection, but that also all zenithal projections* (which are conformal at the centre C of the projection and in which the principal directions coincide with the radius and with the tangent to the circle described about C as a centre) *may be transformed into conformal projections for two fundamental points* by a method analogous to that which we have just described.