USE OF THE PROPERTIES OF THE ARC CONTAINING AN ANGLE (POSITION LINE) IN NAVIGATION AND IN HYDROGRAPHIC SURVEYS.

by

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HISTORICAL.

For many centuries the methods employed in fixing the position of a vessel in sight of land and in plotting charts of the sea bottom were much less accurate than those used on shore by surveyors or by the officers who made surveys for maps. This is owing to the fact that special difficulties were encountered due to the mobility of the element on which the seaman worked. The adaptation of land surveying methods could consist only of taking simultaneous angles to the vessels or boats engaged in sounding from two or more points on shore. This method was successfully employed and is sometimes still used. C. F. BEAUTEMPS-BEAUPRÉ, Ingénieur hydrographe, expressed himself as follows on the subject in his *Méthodes pour la levée et la construction des Cartes et Plans Hydrographiques*, published in 1808, as an Appendix, at the conclusion of the account of the voyage of Rear-Admiral BRUNY-DENTRECASTEAUX (pages 22 and 23) :-

« Le premier moyen consiste à placer sur le rivage, à deux points dont la « position est déterminée, des observateurs, pour relever, avec un cercle azimu- « thal ou un graphomètre ou quelque autre instrument, le canot où l'on sonde, « toutes les fois qu'il mouille et fait un signal.

« Cette méthode est bonne, mais elle a le désavantage de ne pouvoir être « employée que quand il s'agit de sonder de très petits espaces de mer : elle « est aussi d'une longueur extrême. A combien de méprises d'ailleurs n'est-on « pas exposé lorsqu'on en fait usage ! Il faut mouiller à chaque instant ; les « signaux peuvent se confondre, les montres s'arrêter, les instruments se déran- « ger ; et le moindre inconvénient qu'il y ait à craindre, c'est de perdre le « fruit d'un travail très pénible. » (*)

Generally it is preferable to employ that method « qui consiste à observer, « avec une boussole, les gisements de deux ou d'un plus grand nombre « d'objets terrestres, de chaque point de la mer dont on veut fixer la posi- « tion... ; elle est d'une pratique facile, et il faudrait sans contredit la préférer « aux deux autres, si la boussole employée dans une embarcation légère était

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(*) "The first method consists in stationing observers on shore at two points whose positions have been fixed, to take bearings with an azimuth circle, pelorous or other instrument, of the boat which is sounding, whenever she anchors and makes a signal

"This method is good but has the disadvantages that it can only be employed where it is a question of sounding very small areas of the ocean; it is also a very long process. For the rest, how many mistakes are liable to occur in its use! It is necessary to anchor frequently; the signals may be misunderstood, the watches stop, the instruments get out of order, and the least inconvenience to be feared is the loss of the fruits of a very hard work".

« un instrument susceptible de donner les gisements avec quelque exactitude ;
« mais quel est le marin qui ne sache à quoi s'en tenir à cet égard ! » (*)

The following is the opinion of the British Hydrographic Surveyor DALRYMPLE with regard to this method in his *Essay on the most commodious Methods of Marine Surveying - London* 1771, and the method he proposes to replace it. It appears that it might be of interest to publish a fac-simile of the text, as the original work has become very rare :

Experience has fully convinced me, that Bearings, taken by *Compaſs*, cannot be ſafely truſted to, in making a correct Draught. I have found not only a Difference of 3° or more in different Compaſſes, but in the *ſame Compaſs* at different Times: I do not ſay the *Effect* had no *Cauſe*, but there was no ſenſible one which I could diſcover : And I have heard other People ſay, their Obſervations gave room to believe, there is a *caſual Deviation conſequent to the State of the Atmoſphere, or ſome other occult Influence.*

Sometimes the Obſervations made with the Compaſs will correſpond very well with each other ; it is not therefore my Intention to condemn the Uſe of it entirely ; but if the Sea be rough, it is very troubleſome taking many Bearings by the Compaſs, as it requires much Time to be ſure the Compaſs ſtands true ; ſo that the Uſe of the Compaſs, beſides other Inconveniencies, is attended with Delay, whereby the Lands loſe their reciprocal Situations, and by this Means, being as it were taken from different Stations, if the Objects be near, the Angles do not coincide in plotting the Draught. Beſides, the Rigging and Sails often intercept the Sight of the Objects from the only convenient Part of the Ship where the Compaſs can be placed.

Hadley's

(*) ...."which consists in observing, with a compass, the bearings of two or more objects on shore, at each point on the water of which one wishes to fix the position... ; this is very easy in practice and without question it would be preferred to the two others provided that the compass employed in light craft were an instrument capable of giving bearings with any degree of accuracy ; but every seaman fully realises the extent to which this can be counted on."

(*Exposé des travaux relatifs à la Reconnaissance hydrographique des Côtes Occidentales de France*, by M. BEAUTEMPS-BEAUPRÉ, Paris 1829, page 75).

fig. 2

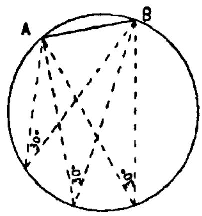
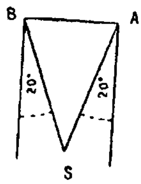


fig. 1



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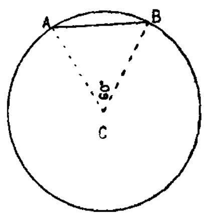

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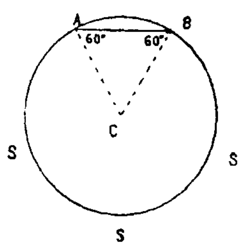

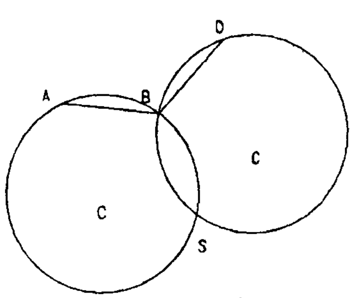

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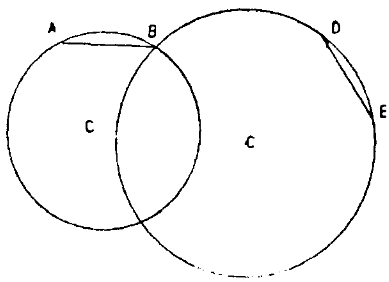

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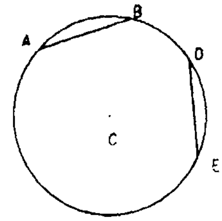

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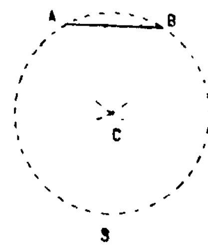

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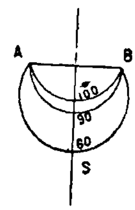

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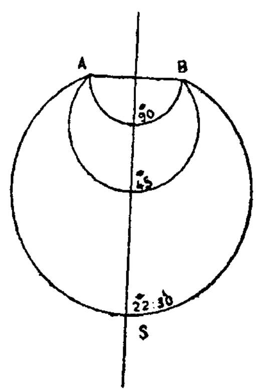

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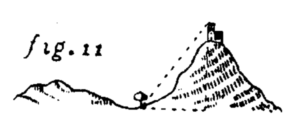

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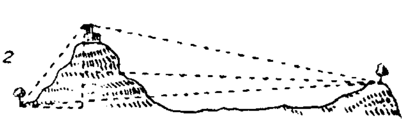

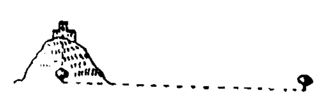

fig. 13

(3)

Hadley's Quadrant is as much preferable to the Compaſs for taking Angles in Facility, as Exactneſs. In the common Obſervation for finding the Latitude, the Quadrant being held upright, the Index is ſlided forward till the Image of the Sun, by Reflection, touches the Horizon ſeen by direct Viſion: For taking Angles, the Quadrant is held horizontal, and one Object by *Reflection* brought, by moving forward the Index, to coincide with another ſeen by *direct Viſion*.

If the Compaſs could be relied on, Bearings taken by it have one Advantage over the Angles taken by Quadrant, *viz.* That theſe Bearings, being the Angles from the Magnetick Meridian, are eaſily laid down by the Meridian in the Chart: for ſuppoſing A to bear N 20° E and B, N 20° W from an unknown Station; laying off the reverſed Bearings S 20° W from A, and S 20° E from B, the Interſection would give the Station, which cannot be found by *two Objects* with any other Inſtrument; all Inſtruments for meaſuring Angles, except the Compaſs, requiring *three* Objects to determine the Station: Indeed, the *North* Point of the Compaſs may be conſidered as a *third* Object when this Inſtrument is uſed.—The Compaſs is alſo very uſeful in ſetting Points as they come *in one*; but there are many Occaſions when a Compaſs cannot be uſed with any Accuracy, where a Quadrant is extremely commodious and equally exact. Fig. 1.

All Compaſs Obſervations made in Boats are liable to great Objection; the Motion of a Boat will ever prevent Exactitude in Compaſs

paſs Bearings, and the Extremities of Shoals and the Depths in intricate Channels, (for determining which, Boats are uſually employed) require the minuteſt Exactneſs. In ſuch Caſes, the Uſe of the Quadrant removes every Difficulty; for if the reciprocal Situations of any *three* Objects are known, the Compaſs Bearings are not neceſſary; and if there are not *three* Objects which can be uſed for this Purpoſe, the Bearing of the Boat may be taken from the Ship on making a Signal; and this, reverſed, will give the Bearing of the Ship from the Boat, which may then uſe the Ship as an Object, and lay off the other Angles equally, as if they could have been taken exactly by Compaſs.

All Bearings or Angles of very near Objects are liable to Incertainty from the Sheering of the Ship, as that Alteration will make a Difference in the Poſition of *very near* Objects. Obſervations made in Boats at Anchor are leſs liable to Error, as the Change of Place is ſmaller; and either in a Ship or Boat, the Obſervations made by Quadrant will be more exact than thoſe by Compaſs, as performed more expeditiouſly.

In Surveying, the *real Diſtance* is the direct Diſtance from one Place to another; the *apparent Diſtance* is the Angle under which two Objects are ſeen. The moſt uſeful Problem in Surveying is, "to "find a *Station*, by obſerved Angles of three or more Objects, whoſe "reciprocal Diſtances are known, but Diſtance and Bearings from "the Place of Obſervation unknown."

Two

(5)

Two Objects, A B, will appear under the fame Angle from every Part * of a Circle paffing through thefe Objects: And thefe Objects can only be feen under that Angle from fome Part of this Circle. Fig. 2.

Thefe two Objects A B, will appear from the Center of the Circle, under double the Angle under which they appear from the Circumference; for an Angle at the Center, is always equal to double the Angle at the Circumference. Fig. 3.

Suppofe A B to be feen under an Angle of 30°, from fome unknown Station, S; thefe Objects will be feen under an Angle of 60°, at the Center of a Circle paffing through A B and S. Fig. 4.

The three Angles of a Right-angled Triangle, are equal to two Right Angles, or 180°; and therefore the Difference between 180°, and double the Angle obferved will be the Sum of the two other Angles.

The Half of this laid off from each Object will interfect at the Center, and a Circle drawn from this Center through the two Ob-

* This is true *only* from *every* Part of the greater Segment of the Circle, for in the lefs Segment between A and B, they will not appear under this Angle: But although the Affertion is not *mathematically*, it is *practically* true, as the Object muft be feen from *behind* for the Cafe not to anfwer the Propofition; and therefore it feemed improper to perplex the *Practice* by *ufelefs Niceties*.

jects will be the Circle in ſome Part whereof the Obſervation muſt have been made.—Example, Subtract the Angle at Center C (always double the obſerved Angle) from 180°, and Half the Remainder will be the ∠ B A C = 60°, and ∠ A B C = 60°, which Angles laid off from A and B, the Interſection will be the Point C,

Fig. 4. which is the Center of a Circle paſſing through A B S.

It is therefore obvious, that as the Station S muſt be ſomewhere in this Circle, if the ſame Operation was repeated with the Angle, un-

Fig. 5. der which two Objects whoſe reciprocal Situations to A and B are known, that the Interſection of this Circle with the former, would give the Point of Obſervation S.

One of the Objects A or B may be uſed in the ſecond Operation,

Fig. 6. but it may be performed by two new Objects, D E.

But if the three or more Objects are in the ſame Circle, there will be no Solution, as the Center will be a common Center to both Circles; and therefore no Interſection to determine in what Part of the

Fig 7. Circle the Point S will fall.

It muſt alſo be obvious, that in uſing four Objects, the two Circles will interſect each other in two Places; but there can be no Difficulty of determining which Interſection is the Station.

Although this is a very ſimple Solution of the Problem by *Projection*, I think the following by the Logarithm Tables is preferable.

Sin.

(7)

Sin. ∠ A S B : Rad. : : A B : Diameter of Circle A B S, Half of which Sum is the Diſtance from A to C and from B to C.

This Diſtance laid off from A and B, by a Scale of equal Parts, the Interſection will be the Point C, which is the Center of a Circle paſſing through A B S. Fig. 8.

Mr. Michell, in a Paper on this Subject, ſays, if the Line drawn from A to B, be *biſected* by an *indefinite* Right Line *perpendicular* to A B; the *Tangent* of the Angle between the *obſerved Angle* and 90°, laid off on this *indefinite Line*, from the Point of *Biſection*, will on that Line give the Center of a Circle paſſing through A B and S, or Place of Obſervation. It muſt be obvious if the obſerved Angle be 90°, the Point of *Biſection* will be the Center of the Circle; if the obſerved Angle be more than 90°, the Difference between it and 90° muſt be laid off on the indefinite Line on that Part *beyond* the Line A B. Fig 9.

This Method of Mr. Michell appears preferable to all others, becauſe the Centers of the Circles or Segments correſponding to every obſerved Angle are eaſily deduced, after the firſt Operation *of drawing the indefinite Line perpendicular* to A B, by marking on this *Line* the Tangent of the Angle between the Angle obſerved and 90°; whereas in the other Modes, the ſame Operation is to be performed to find a Center to every Circle without any Aſſiſtance being derived from the former Centers found. Beſides, in Mr. Michell's having ſound the Center to the Segment of one Angle, the Interſection of the

the indefinite Line by that Segment, gives the Center of a Segment correſponding to Half that Angle. Thus the *Point*, where the Segment of 90° interſects the *indefinite Line*, is the Center of a Segment of 45°, or *Half the Angle*. Where the Segment of 45° interſects the *indefinite Line*, is the Center of a Segment of 22° 30′, &c.

Fig. 10.

It has been already obſerved, that no Inſtrument is ſo commodious for taking Angles as Hadley's Quadrant. It is uſed with equal Facility at Maſt-head as upon Deck, and therefore the Sphere of Obſervation is by this Inſtrument much extended. For, ſuppoſing many Iſlands are viſible from Maſt-head, and only *one* from Deck, no uſeful Obſervations can be made by any other Inſtrument; becauſe Compaſs Bearings from Maſt-head can only be taken very vaguely, and a ſmall Error in the Bearing of a diſtant Object makes a great Error in its Poſition; but by the Quadrant the Angles may be taken at Maſt-head from the *one* viſible Object with the utmoſt Exactneſs. Beſides, taking Angles from Heights, as Hills, or a Ship's Maſt-head, is almoſt the only way of exactly deſcribing the Extent and Figure of Shoals.

It has been objected to the Uſe of Hadley's Quadrant for ſurveying in general, "that *it* does not meaſure the *horizontal* Angles, "by which *alone* a Plan can be laid down." This Objection is *true* in *Theory*, but may be removed in *Practice* by a little Caution, which, in the Obſervations made from *Heights*, is very requiſite.

If

(9)

If an Angle is meaſured between an Object on an Elevation, and another near it in a Hollow, the Difference between the *Baſe*, which is the horizontal Angle, and the *Hypothenuſe*, which is the Angle obſerved, may be very great; but if theſe Objects are meaſured not from each other, but from ſome very diſtant Object, the Difference between the Angles of each from the diſtant Object will be very nearly the ſame as the horizontal Angle. Beſides, a Correction may be made by meaſuring the Angle, not between an Object on a Plane and an Object on an Elevation, but between the Object on a Plane, and ſome Object in the ſame Direction as the elevated Object, of which the Eye is ſufficiently able to judge. Fig. 11. Fig. 12. Fig. 13.

It would appear therefore that to DALRYMPLE belongs the honour of first having advocated the employment of the method which consists « à mesurer du canot même dans lequel on travaille à la mer, et au moyen d'instruments à réflexion, les angles compris entre trois objets terrestres ou un plus grand nombre. » (*)

This is the first indication we are able to find of the employment at sea of the properties of the arc of the circle containing an angle (three point problem). It is readily apparent from the manner in which the author goes into detail in the foregoing quotation that the method did not exist solely in his mind, but that he actually applied it.

The problem of the determination of a position by the intersection of two arcs (lines of position) containing the angles had been known for long and SNELLIUS, who gave the problem his name, had used it for measurements made on land. We are indebted to the Chief of the Hydrographic Service of the Netherlands for the following information. :- The solution of this problem is found in the publication *Eratosthenes Batavus* by SNELLIUS published in Leiden in 1617, of which one copy, used by SNELLIUS and containing marginal notes in his own hand, is now in the library of the University of Leiden. SNELLIUS employed this method for connecting his house with Leiden and determined its latitude and longitude by triangulation between Alkmaar and Bergen op Zoom.

The delay which occurred in the adoption in boats of this very valuable method must be attributed to the lack of an instrument permitting the angles to be measured with accuracy at sea.

(*) "in observing from the boat itself in which one is working on the sea, by means of "reflecting instruments, the angles comprised between three or more objects on shore".

(*Exposé des travaux relatifs à la Reconnaissance Hydrographique des Côtes Occidentales de France*, by M. BEAUTEMPS-BEAUPRÉ, Paris, 1829, page 75.)

We know that the measuring instruments used at sea were constructed at first for the sole purpose of measuring angles in a vertical plane for astronomical purposes. After the *Astrolabe* (*), which was a suspended graduated circle and which did not make use of the sea horizon,the *Balustre* or *Jacob's Ladder*, which did use that horizon, was introduced into Europe in 1499. Towards 1594, *Davis' Quadrant* appeared. This could be used in conjunction with a plumb-line or the sea horizon and it had an accuracy of about one third of one degree.

The double reflection instruments were invented to facilitate the measurement of lunar distances, which required a greater accuracy than that obtainable by the instruments then in use if this method was to be employed to solve the problem of the determination of differences of longitude, which at the time preoccupied the whole maritime world.

It was certainly Sir Isaac NEWTON who first conceived a double reflection instrument, but he did not make his invention generally known. HADLEY, without knowing this, announced to his colleagues of the Royal Society of London on 13th May, 1731, the principle of the quadrant which he claimed to have invented during the summer of 1730 and which retained his name for some time. At the next meeting, on 20th May, Dr HALLEY recalled the invention of NEWTON, who had described his instrument to the Royal Society on 16th August 1699. At the same time as HADLEY, Thomas GODFREY of Philadelphia constructed a similar instrument towards the end of the year 1730.

The use of HADLEY's quadrant does not appear to have spread very rapidly ; it is only towards 1750 that it is found in use in the vessels of the East India Company where it was employed for taking altitudes and lunar distances. The other navies adopted it much later.

DALRYMPLE had the happy idea of using the quadrant for the measurement of horizontal angles probably a short time after its introduction on board the vessels in which he served, and thus brought about enormous progress in measurements at sea. Further, as we have seen, he advocated the use of calculations in order better to utilise the full extent of the accuracy of which the instrument was capable. He returns to this question at the end of his *Essay* to advise the observer to be particularly careful, when sounding, to measure, if possible, the angles between three points whose positions had been determined with very great accuracy and which are separated from each other by at least 20°. He also recommends the use of more points if three cannot be found which fulfil these conditions.

From this time on observers had a method for use at sea which was comparable in accuracy with those which had long been in use on shore and an instrument which was particularly well adapted to their needs for it is in no way influenced by the movements of the vessel, further it is generally required to measure angles in the horizontal plane only and these do not require any correction. It does not appear, however, that its employment became rapidly general, as BEAUTEMPS-BEAUPRÉ made use of it for the first time when he accompanied Rear-Admiral DENTRECASTEAUX, who was sent in search of LA PÉROUSE in 1791. In his *Méthodes*, cited above, he states that it

(*) *See*: "*Gli Strumenti a riflessione per misurare angoli*", by G.-B. MAGNAGHI, 1875, from which we have extracted most of the data concerning ancient instruments.

was by reading the memorandum entitled *Essay on Nautical Surveying* published by the celebrated hydrographic surveyor DALRYMPLE in 1771, that he first conceived the possibility of substituting this method for the ordinary procedure used up to that time. He describes it at length in his publication, going into numerous details on the subject, and also gives four methods for the graphical solution of the POTHENOT problem (the last three avoid the use of circumferences). He then gives a method of calculating the point of intersection of two circumferences and, as a prudent procedure, recommends that a third angle be taken at each station, which will serve as a check on the position obtained by the first two. All the recommendations which he makes for observers are still excellent and there is scarcely a better manual for the technique of boat sounding.

Further, BEAUTEMPS-BEAUPRÉ dealt with this question again in his *Exposé des travaux relatifs à la Reconnaissance Hydrographique des Côtes Occidentales de France, Paris,* 1829, where his very detailed account of the best methods for making marine surveys shows the great practical experience he had acquired in such operations. From the very beginning he employed the BORDA reflecting circle. The reflecting circle was invented by the astronomer Tobia MAYER who described it in 1767. It was then improved by MENDOZA, MAGELLAN (a descendant of the famous navigator) and finally by BORDA who, in 1778, published the *Description et usage du Cercle à Réflexion.* The aim of Tobia MAYER and his successors, in replacing the sextant by a circle, was to be able to employ the method of repetition, which admits of much greater accuracy in the measurement of angles for astronomy or geodesy. But BEAUTEMPS-BEAUPRÉ did not employ the method of repetition at sea, accuracy to within one minute being generally adequate for operations afloat; he took only single angles and in order to work more rapidly he replaced the telescope by a sighting hole. He adopted the circle in preference to the sextant because this instrument is light, better balanced and capable of measuring larger angles.

VARIOUS USES OF THE ARC CONTAINING AN ANGLE.

Being very greatly impressed by the errors in compass bearings, even those taken on board ship (which were then entirely built of wood), BEAUTEMPS-BEAUPRÉ recommended the use of the arc of position method (segment of circle containing an angle) even for navigation:

« Quand il est nécessaire de faire le point, de jour, avec une grande préci-
« sion, les navigateurs doivent observer les angles entre trois des objets terres-
« tres qui sont en vue, et opérer avec ces angles de la même manière que
« nous opérons pour placer nos points de station sur les plans de construc-
« tion. » (Exposé de 1829, page 56).

« Nous pensons que les navigateurs doivent employer cette construction de
« préférence à toute autre, pour faire le point ; d'abord parce qu'elle indique
« de suite si l'on peut compter sur un résultat exact avec les angles observés,
« et ensuite parce que rien n'est plus facile que de trouver le point de station,
« même sans le secours du compas, quand une fois les centres des cercles, sur
« les circonférences desquelles se trouve ce point, sont marqués sur la carte.

« Tout cela peut se concevoir aisément et s'exécuter de même ; et nous « persistons à croire que les navigateurs, dès qu'ils auront fait usage de nos « méthodes, renonceront à l'emploi des gisements observés à la boussole pour « faire le point, toutes les fois qu'ils navigueront le long des côtes et qu'ils « auront de bonnes cartes entre les mains. » (Exposé page 57). (*)

This procedure is actually employed in certain navies when navigating in sight of the coast, but it cannot be said that its use is very general even now.

About fifty years later the French Ingénieur Hydrographe BOUQUET DE LA GRYE, in his *Pilote des Côtes Occidentales de France,* indicated the curved courses along the arc containing a given angle (danger angle) for passing through certain dangers. A similar course may be used for avoiding a danger which does not uncover. In this manner the effects of a possible error in the variation of the compass or in the set and drift of the current are eliminated. Such courses are easier to follow, for vessels with large turning circles, than the broken lines joined by curves which cannot be accurately determined.

Be that as it may, even if the efforts to extend the employment of the subtended angle beyond the narrow confines of hydrographic surveys have not been entirely successful, it may nevertheless be said that it has become the basis of all work at sea.

Not only is the subtended angle used for fixing the positions of soundings, but it has become current practice in Hydrography to follow the arc of the subtended angle. If it is desired to proceed to a rocky head or to a definite point, for instance in order to insert a line of soundings between others, two suitable angles at which well-known land-marks may be seen from this point are measured on the chart or on the sheet and, on approaching the vicinity of the point to which one is proceeding, one places oneself on the arc of one of the subtended angles and follows along this until one is located upon the second arc also.

Very extended use is made of this procedure by surveyors of certain Hydrographic Services. We know that in order to obtain close lines of soundings, instead of drawing arbitrary lines on the chart or the sheet, which are often difficult to follow exactly, one generally prefers, whenever it is possible, to follow the directions given by fixed objects in transit. Instead of following a series of straight lines, often converging on one of the fixed objects, one

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(*) "When it is necessary to fix the position, by day, with very great accuracy, navigators should observe the angles between three of the objects in view on shore and use these angles in the same way as we use them to fix our stations on the plotting sheet". (*Exposé* of 1829, p. 56)

"We believe that navigators should employ this construction in preference to all others in obtaining a fix, first because it indicates at once whether an accurate result can be counted on with the angles observed and, second, because nothing is easier than to obtain a fix, without a compass, once the centres of the circles, on the circumferences of which the fix is located, are marked on the chart.

"All of this is easily understood and readily done, and we persist in believing that once navigators have made use of this method, they will renounce the use of compass bearings for fixing the position whenever they are navigating along coasts and they have good charts at their disposal." (*Exposé*, page 57).

may follow a series of circular arcs defined by angles between two objects whose positions are well determined.

To pass from one curved line to the next, it will suffice to change the angle which defines the arc by a quantity which can be measured on the sheet. The advantages of working thus are that the objects on which one relies to define the course may be chosen for their clearness and visibility, that the subtended angle (which must be read off accurately at each fix), may be utilised in plotting, and that the operator is thus freed from any worry on the subject of the suitable arrangement of his lines of soundings ; he is certain to preserve a perfectly regular interval between successive lines of soundings and, if the results of the soundings make it necessary, he can easily insert other lines between those already taken. It is no more difficult when using such curved courses than when using straight courses to arrange them at right angles to the depth lines, as is always desirable.

This method has been very generally employed in the French Hydrographic Service since 1892 ; it has been found to be extremely practical both for the Chief of the Hydrographic Survey, by facilitating the distribution of the work, and for the surveyors, *particularly* if the latter are inexperienced.

In any case it is as easy to follow the arc of a circle as to keep on a transit. The two objects serving to define the arc containing the subtended angle should be kept in coincidence in the telescope of the reflecting instrument (either reflecting circle or sextant) of which the index is clamped at the selected angle. The observer's impressions in this case are exactly the same as though he were keeping on a transit by maintaining the two points which mark it in coincidence, and thus he should have no hesitancy with regard to the manœuvres which must be executed. The point to be considered the nearer is the one which would be encountered first by following the arc containing the angle in the direction of the course, the other is taken as the further point. The procedure, which is very easy in a small boat, becomes even more so in a larger vessel (it is evident that one will only try to follow an arc of large radius). The officer directing the work may easily, if he desires, allot to a subordinate the duty of keeping the vessel on her course — seamen become accustomed to this work very quickly.

ACCURACY OF THE INTERSECTION OF TWO ARCS OF POSITION.

The problem of the employment of the intersection of two arcs of position for the determination of the position has attracted much attention for a long time. We will not give its history here as this may be found in the second volume of *Jordans Handbuch der Vermessungskunde*, 8th edition, page 386. This problem has been designated sometimes by the names of the principal mathematicians who have dealt with it :- the SNELLIUS, COLLINS, or POTHENOT problem, sometimes as the "chart problem" and at times as the "inverse intersection". (*)

It would seem that there is nothing further to be said about it, yet, in

(*) *Hydrographic Review*, Vol. VI, N° 1, May 1929, page 112.

the last half century, new methods of calculation have been proposed to solve the problem, either because it was desired to compare this method with that of direct intersection or because it appeared desirable to leave apparent the geometrical loci the intersection of which defines the position of the point. On the other hand the study of the degree of accuracy in the position obtained is most important. It determines the conditions of choice of the angles which should be taken and employed when their number is limited perforce to two or three as a result of the mobility of the observer. It is also essential that this accuracy be known when an abundance of angles taken produces a "cocked-hat" within which the position of the point must be located. For this reason we shall note some of the properties which are not generally included in treatises on this subject, or which are only incompletely dealt with.

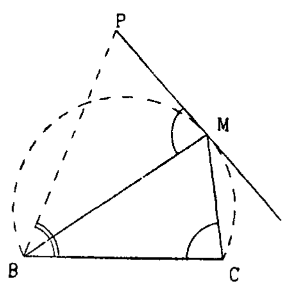

Fig. 1

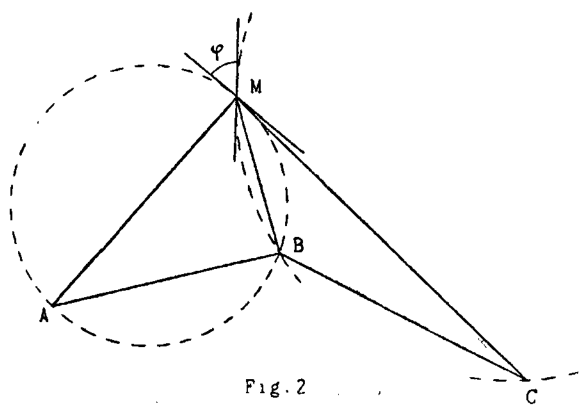

Fig. 2

If at a point M (see fig. 1) we measure the angle which subtends the chord BC, and if the maximum error (expressed in radians) which may be expected in this angle under the conditions of the experiment be called ε, we know that in the vicinity of point M the maximum distance of the segment of the circle obtained from the measured angle, to the true segment, will have a value of :-

$$\pm \frac{\overline{MB} \times \overline{MC}}{\overline{BC}} \varepsilon$$

The tangent to the arc of the circle at the point M makes an angle equal to C with the side MB. In the vicinity of the point M we may replace the arc of the circle by a short straight line lying in the same direction as the tangent, and we may consider this short straight line as obtained from a bearing taken from a point on this tangent at a distance :

$$\frac{\overline{MB} \times \overline{MC}}{\overline{BC}}$$

from the point M, and observed with an error ε.

If at B we draw the straight line BP which cuts BM at an angle equal to the angle $\widehat{CBM}$, and which is on the opposite side of BC with respect to BM, it is easily seen that the length $\overline{PM}$ is equal to $\frac{\overline{MB} \times \overline{MC}}{\overline{BC}}$. The arc of the circle in the vicinity of the point M may therefore be considered exactly as a bearing taken from point P.

Let us assume that the position of point M (see fig. 2) is determined by the intersection of two arcs containing the angles $\widehat{AMB}$ and $\widehat{BMC}$ ("inverse intersection"); these may be replaced in the vicinity of point M by two bearings of which the intersection (known as "forward intersection") is made at an angle of

$$\varphi = A + C$$

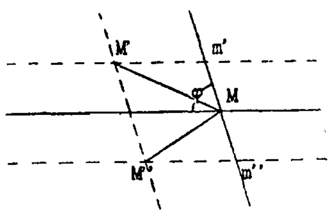

Fig. 3

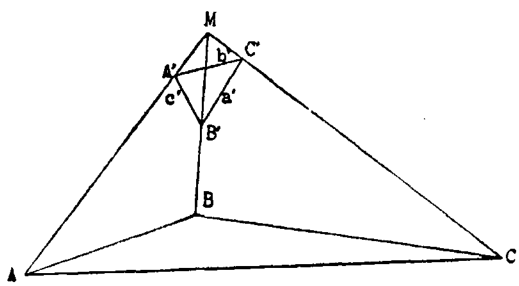

Fig. 4

If the two angles at M both have the same error ε, the error in the position of the point M will be represented, in absolute value, by the vectors $\overline{MM'}$ or $\overline{MM''}$ (*see* fig. 3) depending on whether the two errors ε have the same or opposite signs.

We have also :-

$$Mm' = \frac{\overline{MA} \times \overline{MB}}{\overline{AB} \sin \varphi} \varepsilon \qquad m'M' = \frac{\overline{MB} \times \overline{MC}}{\overline{BC} \sin \varphi} \varepsilon$$

An interesting expression for these quantities, due to Dr O. Eggert, is found in *Jordans Handbuch der Vermessungskunde*, Vol. 2, 8th edition, page 384. It may be derived in the following manner (*See* fig. 4) :-

Lay off from the point M on the sides MA, MB and MC, the lengths MA', MB' and MC' equal respectively to $\frac{\lambda}{MA}, \frac{\lambda}{MB}, \frac{\lambda}{MC}$ (λ being any whole number).

The equations :-

$$\frac{MA'}{MB} = \frac{MB'}{MA} = \frac{A'B'}{AB} = \frac{\lambda}{MA.MB}$$

show that the straight lines $A'B'$, $A'C'$ and $B'C'$ are "anti-parallel" (*) respectively to the straight lines AB, AC, and BC and that the angle B' of the triangle $A'B'C'$ is equal to the angle φ included between the two arcs of circles of which B is the common point (**)

Let us call the sides of this triangle a', b', c' (*see* fig. 3); then we may write :-

$$Mm' = \frac{\lambda\ \varepsilon}{c'\ sin\ \varphi} \qquad m'M' = \frac{\lambda\ \varepsilon}{a'\ sin\ \varphi}$$

Calculating MM' we have :-

$$\overline{MM'}^2 = \frac{\lambda^2\ \varepsilon^2}{a'^2\ c'^2\ sin^2\ \varphi}\ (a'^2 + c'^2 + 2a'c'\ cos\ \varphi).$$

The quantity in brackets is equal to four times the square of the median $m_{b'}$, corresponding to the side $A'C'$, or $4\,m_{b'}^2$: the denominator is equal to four times the square of the area of the triangle $A'B'C'$, or $4\ s'^2$.

Therefore :

$$\overline{MM'} = \frac{\lambda\ \varepsilon}{s'}\ m_{b'}.$$

We find similarly :-

$$\overline{MM''} = \frac{\lambda\ \varepsilon}{s'} \cdot \frac{b'}{2}.$$

The two displacements in the position of the point M which may be caused on account of errors in the measured angles are proportional to $m_{b'}$ and $\frac{b'}{2}$ In reality, if, as we have assumed, ε is the maximum error possible in each angle, we know only that the maximum possible displacement of M is equal to the largest of the two vectors MM' and MM''. As we are not seeking to calculate the mean error, but rather to find the maximum displacement to which M might be subjected, we must assume that the accuracy of the intersection of the two arcs is defined by the greatest of the two quantities MM' and MM''. The conclusions which we will reach will be the same if, instead of considering the greatest of these displacements, we adopt as the criterion their arithmetical mean or the mean of their squares.

The length $m_{b'}$ is equal to $\frac{b'}{2}$ when the angle B', *i. e.*, φ, is a right angle; $m_{b'}$ is greater than $\frac{b'}{2}$ if φ is less than 90°; $m_{b'}$ is less than $\frac{b'}{2}$ if φ is greater than 90°. The same holds good for the relations between the quantities MM' and MM''.

(*) *i. e.* Such that *opposite* interior angles are equal to each other.

(**) The lines such as $A'B'$ are the inverse figures of the arcs of circles intersecting at M which we are studying. We know that the angles are conserved in inversion. The three angles of the triangle $A'B'C'$ are those at which the three arcs of the circles intersect at point M.

CHOICE OF THE TWO ANGLES.

Of the three angles $\widehat{AMB}$, $\widehat{BMC}$ and $\widehat{AMC}$ which may be taken at the point M, we can now choose the two angles which should be measured if the position of the point M is to be defined as accurately as possible by the intersection of the arcs containing the two angles measured. (*)

Depending on the selection made, the two arcs will intersect at an angle equal to one of the angles A', B' or C' of the triangle $A'B'C'$.

Assume that one of these angles is obtuse (fig. 5), B' for instance. The three vectors which characterise the maximum error in the position of the point M, will have the lengths :-

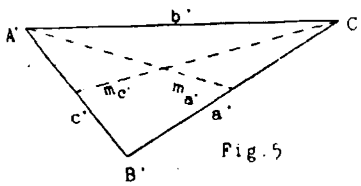

Fig. 5

$$\frac{\lambda\,\varepsilon}{s'}\,\frac{b'}{2},\qquad \frac{\lambda\,\varepsilon}{s'}m_{a'},\qquad \frac{\lambda\,\varepsilon}{s'}m_{c'}\,.$$

But the medians $m_{a'}$ and $m_{c'}$ are greater than $\frac{b'}{2}$, since :-

$$m_{a'}^2 = \left(\frac{b'}{2}\right)^2 + \frac{c'^2 + 2\,b'c'\cos A'}{4},\qquad m_{c'}^2 = \left(\frac{b'}{2}\right)^2 + \frac{a'^2 + 2\,b'a'\cos C'}{4},$$

in which expressions the quantities $\cos A'$ and $\cos C'$ are positive.

Thus we shall have the least error by choosing the two arcs which intersect at an angle equal to B' ; or in other words by not employing the angle $\widehat{AMC}$.

It should be noted that this is the angle corresponding to the longest side $A'C'$ of the triangle $A'B'C'$.

If none of the angles of the triangle $A'B'C'$ is obtuse, the three characteristic vectors will have the lengths :-

$$\frac{\lambda\,\varepsilon}{s'}\,m_{a'}\ ,\quad \frac{\lambda\,\varepsilon}{s'}\,m_{b'}\ ,\quad \frac{\lambda\,\varepsilon}{s'}\,m_{c'}.$$

We must choose the two arcs which correspond to the shortest median. We know that this is the one corresponding to the longest side. We have :-

$$m_{a'}^2 = \frac{a'^2 + b'^2 + c'^2}{2} - \frac{3}{4}\,a'^2.$$

(*) It is not our purpose here to advocate the taking of two angles only, but to guide the choice in cases where only two angles can be taken. It is always preferable to take a third angle as a check.

Further, it has often been recommended, when the operator is taking angles alone from a boat or vessel under way, to choose two angles such that one gives an arc as parallel as possible to the course being followed and the other at right angles to this course, as far as possible, the second angle being taken at the instant the sounding is taken. Generally this excellent method does not permit a choice between three angles : it is necessary only to make certain that the accuracy of the intersection of the two arcs used is adequate and for this purpose the considerations in the previous chapter should be taken into account.

The angle which should not be employed will again be the one corresponding to the longest side of the triangle $A'B'C'$.

But in the triangle $A'B'C'$ the longest side is opposite the greatest angle; and since the angles of this triangle are equal to those made by the intersection of the arcs containing the subtended angles, we see that of the three arcs it is necessary to choose the two which intersect at the greatest angle; which amounts to saying that the angle of intersection should be as near 90° as possible.

CONCLUSION.

When it is necessary to determine the position of the point M by the intersection ot two arcs subtending the angles taken between the three points A, B, C, the angles chosen from among the three possible angles should be such that the arcs containing the subtended angles intersect at an angle as near to 90° as possible.

The following procedure will permit them to be found:-

Starting from point M (which is always approximately known) lay off on the chart or the sheet, in the directions MA, MB and MC, the lengths MA', MB', MC', inversely proportional to the lengths of MA, MB and MC. Of the three angles that which should not be taken is the one opposite the longest side of the triangle $A'B'C'$.

Instead of considering, as we have just done, the maximum displacement to which the position of M might be subjected as a result of an error in reading the angle, an unknown error of which we only know the maximum possible value, let us take into consideration the average displacement given by the theory of least squares for the point M when the two angles are successively affected by all possible errors, positive or negative, included between zero and the maximum value ε.

Then the theory of least squares shows that the mean displacement of the point M is proportional to $\sqrt{\overline{MM'}^2 + \overline{MM''}^2}$. Therefore according to the expressions given above for MM' and MM'', it is proportional to:-

$$\sqrt{m_{b'}^2 + \frac{b'^2}{4}}$$

But this quantity is proportional to $\sqrt{a'^2 + c'^2}$; the smallest of the three values which can be obtained is thus the one in which the longest side of the triangle $A'B'C'$ is discarded. Thus we reach the same rule given in the preceding section.

The quantity $\dfrac{MM' + MM''}{2}$ is least also if we discard the longest side of the triangle $A'B'C'$. It is in fact proportional to

$$m_{b'} + \frac{b'}{2},$$

a quantity the square of which may be written:-

$$\frac{a'^2 + b'^2 + c'^2}{2} - b'\left(\frac{b'}{2} - m_{b'}\right).$$

If b' is the longest side, $m_{b'}$ will be the shortest median, hence $\frac{b'}{2} - m_{b'}$ will be at maximum, as will be the term $b'\left(\frac{b'}{2} - m_{b'}\right)$.

Therefore in each case, regardless of the criterion adopted, we reach the same result, *viz.*: eliminate the angle corresponding to the longest side of the triangle $A'B'C'$, which means in effect that we choose the two arcs which will intersect at the angle nearest to 90°.

It appears that this rule might be serviceable, particularly for novices, in assisting them to make the choice of angles and to acquire the ability to judge by eye and later make the selection without resorting to graphic construction.

Notes.

1) The construction of the triangle $A'B'C'$ in fig. 4, which we employed for selecting the best angles to take between the points A, B and C, may also be used to show those cases in which the two angles will determine the position of the point M with insufficient accuracy.

An error ε in the angle subtending the chord AB in an arc results in a displacement equal to

$$\frac{\lambda\ \varepsilon}{A\ B'} \text{ of the arc in the vicinity of the point } M.$$

It is essential therefore that the length $\frac{A'B'}{\lambda}$ should not be too small — but this does not necessarily mean that the angle $\widehat{AMB}$ may not be small. It may be small without being unsuitable if, at the same time, the lengths MA and MB are very different. The angle $\widehat{AMB}$ may even be equal to zero (*i. e.* the case of a transit, or alignment) and $\frac{A'B'}{\lambda}$ of a high value. There is no need therefore for an *a priori* rejection of all angles under 20° as recommended by DALRYMPLE at the conclusion of his *Essay*.

2) It is well known that if the points A, B, C and M are on the same circumference, the position of the point M cannot be determined by taking the angles between the points A, B and C. In this case the triangle $A'B'C'$ is reduced to a straight line, its area becomes zero and the expressions given for MM' and MM'' become infinite.

3) If one of the points A, B, C is very much closer to the point M than the other two, say the point A, for instance, the sides $A'B'$ and $A'C'$ will certainly be larger than $B'C'$ and we see at once that only one of the angles $\widehat{AMB}$ and $\widehat{AMC}$ should be chosen and combined with the angle $\widehat{BMC}$.

CASE OF MORE THAN THREE OBJECTS.

Another form of error in the position of the point M as defined by the intersection of the arcs of the subtended angles permits us to lay down general rules for the problem of the selection of the two angles.

If the triangle $A'B'C'$ is obtuse (fig. 5), b' being the side opposite to the obtuse angle B' and h' the height corresponding to this side, we have seen that the angles $\widehat{AMB}$ and $\widehat{BMC}$ should be selected and that the maximum error to be feared is :-

$$\frac{\lambda \varepsilon b'}{2 s'}$$

But $2s'$ is equal to $b'h'$, and we have therefore a new expression for the error

$$\frac{\lambda \varepsilon}{h'}$$

If the triangle $A'B'C'$ is not obtuse, we have seen that the maximum error to be expected is

$$\frac{\lambda \varepsilon}{s'} m_{b'}$$

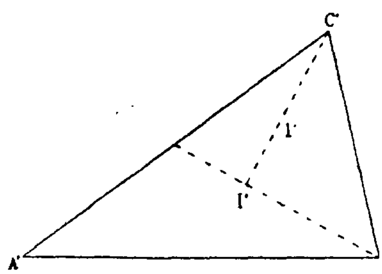

Fig. 6

if $m_{b'}$ is the median corresponding to the longest side (*see* fig. 6). Let l' be the distance of the points A' or C' from this median, and as s' is equal to $l'm_{b'}$, we have a new expression for the error :-

$$\frac{\lambda \varepsilon}{l'}.$$

If the operator has four points A, B, C, D, in sight, there are 15 ways in which these may be combined in pairs for the determination of the position by two angles. Each triangle formed by joining three points, two by two, gives three ways — with four points we may therefore form four triangles which give twelve combinations. There remain three other combinations and in these the same point should not be used twice. For instance, let the two arcs be AMB and CMD. We form the triangle by drawing a straight line equal and parallel to $C'D'$, to one of the extremities of the inverse straight line $A'B'$. We have also two others with $A'C'$ and a parallel to $B'D'$; then $A'D'$ and the parallel to $B'C'$.

In each of these first four triangles the most favourable combination will appear from the rules given above : the length *h'* or *l'* will be constructed depending on whether the triangle is obtuse or not. (*)

In each of the last three one combination only is to be considered, namely, that which corresponds to the two straight lines with which we have constructed the triangle. We draw a straight line with respect to the third line closing the triangle (which is not necessarily the longest) such as *h'* or *l'*, depending on whether the angle included between the first two lines is obtuse or acute (by drawing the second straight line in a suitable manner we may always arrange matters so that the angle shall be obtuse ; this allows one to avoid the use of lengths other than *h'* and the median need therefore not be drawn). Finally we obtain 7 lengths, and the most favourable combination of angles will correspond to the greatest length.

The two arcs containing the subtended angles thus selected may, in cases where there are more than three points, not be those which intersect at the angle nearest to 90°.

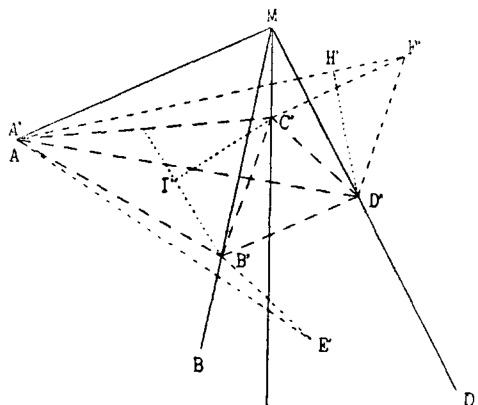

fig. 7

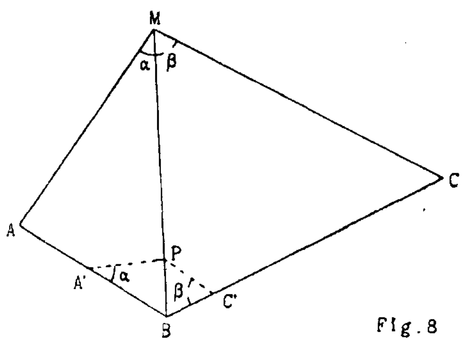

Fig.8

In fig. 7 we have applied this procedure. We have formed the first four triangles *A'B'C'* ; *B'C'D'* ; *A'C'D'* and *A'B'D'*. Among the combinations of angles offered for selection, the most favourable will be that in which the length $l' = I'C'$ corresponds to the angles $\widehat{AMB}$ and $\widehat{BMC}$. The other three triangles are :- *A'B'E'*, *A'C'F'* and *A'D'F'*, of which the greatest of the three heights to be taken into consideration is *D'H'*. This quantity is somewhat greater than *C'I'* ; therefore the combination of angles $\widehat{AMD}$ and $\widehat{BMC}$ is to be preferred. The two arcs containing the subtended angles will intersect at *M* at an angle equal to $\widehat{A'D'F'}$.

(*) If the station is to be plotted with a station pointer, these first four triangles only need be considered.

CALCULATION OF THE INTERSECTION.

A consideration of the inverse figures of the segments of circles will also provide the means for calculating the point M by direct intersection. The method is indicated in the work of JORDAN cited above, page 386.

From the point M the angles α and β are measured between the known points A, B and B, C.

Let us lay off on BA and BC the lengths $\overline{BA'}$ and $\overline{BC'}$ respectively equal to $\frac{\lambda}{\overline{BA}}$ and $\frac{\lambda}{\overline{BC}}$ and draw straight lines through the points A' and C' at angles α and β to BA and BC. These straight lines are the inverse with respect to B of the segments of the circles ABM and CBM. They meet at point P situated on the straight line BM such that $\overline{BP} = \frac{\lambda}{\overline{BM}}$

We can calculate the positions of the points A' and B', and then by the method of "forward" intersection calculate the point P. The point M will be found on the straight line BP at a distance :-

$$\overline{BM} = \frac{\lambda}{\overline{BP}}$$

Note.

To complete this article we give a list of some articles dealing with this question :-

a) "Etude de la précision du Point par deux segments capables", by Ingénieur Hydrographe en chef DRIENCOURT, *Annales Hydrographiques,* 1919-20, pages 179 et seq.

b) "Sul punto rilevato", by Lieutenant JACHINO, *Rivista Marittima,* July 1919, pages 8 et seq.

c) "La determinazioni di posizione della Navigazione costiera", by Captain Luigi TONTA, *Rassegna Aeronautica Illustrata,* Roma, January 1920 et seq.

d) "Un problema di Geometria pratica", by Captain Luigi TONTA, *Rivista Marittima,* April 1921.

In this there will be found a study of the curves of equal accuracy of the arcs containing the angles constructed on one and the same base BC. These curves, which are the loci of the points such that $\overline{MB} \times \overline{MC}$ is constant, are CASSINI ovals.

They enable the various conditions of the problem to be discussed ; but the results given above do not appear to us to have yet been mentioned.

May 1930.