TRAITE DES PROJECTIONS DES CARTES GEOGRAPHIQUES A L'USAGE DES CARTOGRAPHES ET DES GEODESIENS.

(TREATISE ON THE PROJECTIONS OF GEOGRAPHICAL MAPS FOR THE USE OF CARTOGRAPHERS AND GEODESISTS).

by

L. DRIENCOURT and J. LABORDE.

Review and Remarks

by Ingénieur Hydrographe Général P. DE VANSSAY DE BLAVOUS, Director.

The problem of the projections of geographical maps has claimed the attention of many mathematicians and cartographers; but among the very numerous works which deal with this question there are few which give a comprehensive theory of it and a logical grouping of the innumerable solutions proposed. One such work had been carried out by Ing. Hyd. GERMAIN, whose *Traité* published in 1865 became a classic in France; the edition is long since out of print. In 1881 there appeared the *Mémoire sur la Représentation des Surfaces* by TISSOT of whom ZÖPPRITZ said that he paid with interest France's share in the debt contracted by the mathematicians and geographers of all nations with the German LAMBERT. The French edition to-day is quite out of print; E. HAMMER published a good German translation of it with some additions in 1887.

We must then first congratulate the authors of the new publication on having given us the essential parts of these works, TISSOT's *Mémoire* almost in its entirety and the useful numerical tables contained in it. But at the same time the general arrangement has been rendered uniform and appears in an absolutely logical manner; it is based on TISSOT's idea of an indicatrix which immediately furnished the key to the whole study of deformations and leads directly to the general formulae proper to the conformal and equivalent projections. It gives distinct weight to the importance, for the study of conformal projections, of the use, as variables, of meridional parts and of conformal latitude, which enables us to render the formulae independent of the form of the terrestrial meridians. The reader will be able to satisfy himself how necessary is this function of the latitude, not only for MERCATOR's projection but in most of the problems connected with conformal projections, and consequently how important was the publication of the 5-decimal tables by the International Hydrographic Bureau.

In the study of equivalent projections, equivalent latitude plays the same part as conformal latitude in that of conformal projections.

The study of projections consists above all in the determination of the proportions h and k of the lengths along the meridian and the parallel, and the radius of curvature of the curves which represent these meridians and parallels on the projection.

The classification and study of the definitions and properties of projections fills the first part of the book. This classification is that of TISSOT's

Mémoire (supplement); it seems well worth retaining, in spite of some rather unprepossessing names, if it is desired to assign categories to the innumerable projections which have been invented, the majority of which the authors of this treatise have quoted without however being able to claim absolute completeness.

The first pages of Part II reproduce the history of the projections according to Tissot. And an important chapter deals with the choice of a method of projection for representing an entire hemisphere in an open or a closed frame or a complete globe on a single map. In the latter case we notice the particular interest presented by the group of spherical conformal projections of Lagrange, and among them that which enables the entire sphere to be represented within one circumference. The reader will also find mention of doubly periodic conformal projections, an application of elliptical functions to map-making of which the chief authors are Guyou and Adams.

The author next passes to the detailed study of the more important projections for use in connection with world-maps and planispheres. He treats with great competence the question of the use of the gnomonic projection for great-circle sailing and for radiogoniometric position finding from fixed stations. He also examines the use of this projection for aerial navigation; the needs of this type of navigation could be satisfied by six charts obtained by gnomonic projection upon the faces of a cube circumscribed about the sphere. The other projections recommended for aerial navigation (besides that of Mercator which will always be very useful in combination with the use of Favé's tracing of orthodromic curves, completed by *lines* of equal azimuth), are the stereographic polar projection for high latitudes, and special charts for a given itinerary which would be on an oblique cylindrical projection (charts by L. Kahn : see *Hydrographic Review*, Vol. V, Nº 2, page 39), or on Lambert's conformal conical projection; the latter are studied also in great detail in chapter IV. In the following review of the principal systems of projection we shall note the particularly advanced study of perspective projections.

Chapter IV of Part II furnishes, by means of numerical tables, the reasons for the choice of projection to be used for the representation of parts of the world intermediate between a hemisphere and a particular district.

Chapter V is devoted to a search for the most appropriate system of projection for the representation of a particular district. It reproduces textually for the most part the second chapter of Tissot's *Mémoire*, which contains his chief study of projections with the minimum of deformation.

Part III studies systems of plane rectangular co-ordinates for both geodetic and topographic operations by the use of projections limited to the terms of the third order in areas not exceeding 400 kilometres (215 n. miles) in diameter. It is a development of the memoire by Ing. Hyd. Courtier which appeared in the *Annales Hydrographiques* of 1912, and which had not previously had the notice it deserved. The adoption of a restricted area enables one to be satisfied with projections which are only conformal to within the third order and considerably simplifies the calculations. We gave a very rapid sketch of these theories in the first pages of an article in the *Hydrographic Review*, Vol. VII, Nº 1, pp. 13-16; here they will be found with all the development and precision of which they admit.

The first three parts are the work of Ing. Hyd. en Chef L. DRIENCOURT, the fourth is particularly the work of Colonel LABORDE. He expounds therein the use of rigidly conformal projections in very extended areas. Though some of the projections studied are well known (stereographic projection, projections of GAUSS and MERCATOR, conical projection of LAMBERT), the exposition of them is entirely new, based on the theory of surfaces, on the properties of orthogonal systems and on the new idea of an auxiliary surface, called *the indicative surface of conformal representation,* which furnishes a simple geometrical picture of the various deformations.

By a generalisation of the method followed by TISSOT, the author in the first two chapters studies the problem of the deformations without explicitly bringing in the analytical relationship between the co-ordinates ; a fact which gives great generalisation to his study and leads it to the most interesting results for the conformal representation of the whole of a closed surface in a finite portion of the plan, to the study of critical points, and of simple and doubly periodic projections (of GUYOU, PEIRCE and ADAMS). For the plane conformal representation of the sphere the solution which he calls that of *separate variables* logically presents itself and leads, as an application, to the hitherto unused construction of a planisphere, which allows the best representation of the regions under consideration and the removal to infinity of arbitrarily chosen points.

For small scale Atlas maps the author recommends the following conformal projections : LAMBERT'S conical projection, stereographic projection, LAGRANGE'S projections.

Chapter III presents the analytical theory of conformal representation by the study of isothermal systems and symmetrical co-ordinates. We are thus led to LAGRANGE'S *projections* with circular meridians and parallels, as well as to the more general projections known as *double circular* and to new *doubly periodic* projections derived from the projections of GUYOU, PEIRCE and ADAMS.

Chapters IV, V and VI constitute the practical part from the point of view of the application of conformal projections to geodetic calculations. The author shows the advantage of *rigidly* conformal projections which admit of the employment of very extended areas with no greater complication in the working out of the corrections.

After dealing with the classic problem of the representation of the ellipsoid on the sphere, which enable further consideration to be confined to the plane representation of the sphere, the author examines under the title of *general solution N° 1* a system of projection which he has applied in Madagascar and which we have already mentioned in Vol. VII, N° 1, pp. 16-31, of the *Hydrographic Review.*

Next comes the study, from the point of view of geodesy, of the particular projections derived from *general solution N° 1* which are identical or practically very closely allied to the well-known stereographic, GAUSS, MERCATOR and LAMBERT'S conical projections, and finally the presentation of *general solution N° 2,* intended for more extended areas of application than those to which *general solution N° 1* is applicable, and where the foregoing particular projections, the first three of which are in fact only particular cases of *general solution N° 2*, are inconvenient for the shape of the area of application. *General solution N° 2* is the application of the solution known as that of *separate variables*, already mentioned in Chapter II.

Chapter V is a very complete study of the corrections necessitated by the application of the projections studied in Chapter IV. It is demonstrated that the use of these corrections is consistent with the greatest care for precision in geodetic application and that the same precautions are necessary for projections with a limited area and for projections with a very extended area. But, while *general solution N° 1* is only applicable with complete accuracy within a radius of 1,000 kilometres (535 n. miles) about the origin, the other projections studied in Chapter IV can be so up to 3,000 kilometres (1600 n. miles) which allows a uniform system of conformal representation to be used to cover the surface of the largest states without change of origin.

Chapter VI shows the practical application made by the author of *general solution N° 1* in the geodetic and topographic survey of the island of Madagascar.

REMARKS ON LABORDE'S GENERAL SOLUTION N° 2.

LABORDE has given the name of *general solution N° 2* to a group of projections characterised by the fact that the variables are separated in the expression of the "ratio of similitude" (linear modulus). The simplicity of the expression of this relationship results in the fact that corrections can be readily calculated with great precision up to 3,000 kilometres (1600 n. miles) from the origin. While making the elliptic indicatrix of this projection correspond to TISSOT's conic, we have at the same time a projection with the minimum of linear deformation. LABORDE has shown that two points I and J on a great circle of the sphere, following a path along the minor axis of this elliptic, are at infinity on the projection. On the sphere they are at distances from the central point O equal to $\pm (\pi - C)$. The angle C is then linked to the axes of TISSOT's conic by the relation :

$$\frac{a^2 - b^2}{a^2 + b^2} = n = tan^2 \frac{C}{2}$$

Taking the major and the minor axis of TISSOT's conic as the axes of co-ordinates ξ, η and calling u and v the co-latitude reckoned from the point I taken as the pole, and the longitude reckoned from $I\,O$, the expression of the co-ordinates of a point is given by the equations :

$$(1) \quad \begin{aligned} e^{\frac{\eta\sqrt{n}}{R}} cos\left(\frac{\xi\sqrt{n}}{R}\right) &= cos\,C + sin\,C\,cos\,v\,cot\,\frac{u}{2} \\ e^{\frac{\eta\sqrt{n}}{R}} sin\left(\frac{\xi\sqrt{n}}{R}\right) &= sin\,C\,sin\,v\,cot\,\frac{u}{2} \end{aligned}$$

It is not without interest to point out that this projection is the same one that was shown in 1930 in a different and rather elegant form by YUNG in Nos 16 and 18 of the *Zeitschrift für Vermessungen.*

This author, calling A a real or imaginary number and Z the complex number $X + iY$ studied the projections in which the expression

$$\frac{1}{A} \tan \frac{AZ}{2} \tag{2}$$

depends only on the geographical position of the point in question with regard to the chosen origin.

We shall give A the following general form, and will take R equal to unity to simplify the figures :

$$A = \sqrt{n}\, e^{-i\theta} = \sqrt{n}\left(\cos \theta - i \sin \theta\right)$$

To show the identity of this projection with general solution No 2 it will suffice, in expression (2), to replace X and Y by values defined by equations (1) and the relations :

$$\begin{aligned} \xi &= Y \sin \theta + X \cos \theta \\ \eta &= Y \cos \theta - X \sin \theta, \end{aligned} \tag{3}$$

and to state that the expression (2) takes a form which depends merely on the geographical positions of the point.

We have in fact :

$$\tan \frac{AZ}{2} = i\, \frac{1 - e^{iAZ}}{1 + e^{iAZ}}$$

$$iAZ = \sqrt{n}\left(-\eta + i\,\xi\right)$$

and (4) :

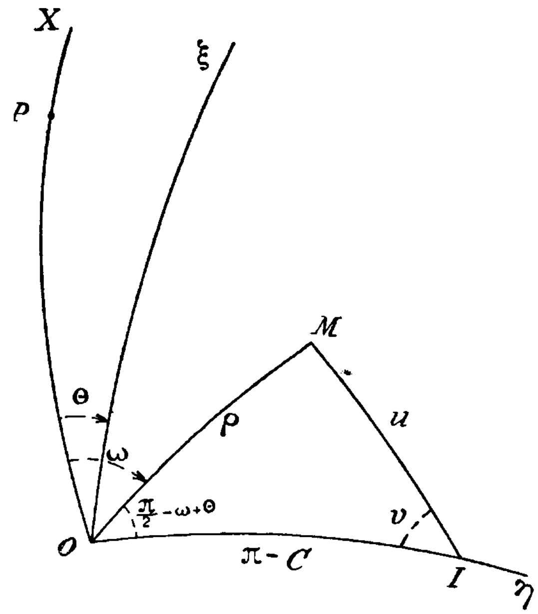
$$\frac{1}{A} \tan \frac{AZ}{2} = \frac{\left[2e^{\eta\sqrt{n}} \sin (\xi\sqrt{n}) \cos \theta + (1 - e^{2\eta\sqrt{n}}) \sin \theta\right] + i\left[2e^{\eta\sqrt{n}} \sin (\xi\sqrt{n}) \sin \theta - (1 - e^{2\eta\sqrt{n}}) \cos \theta\right]}{\sqrt{n}\left[1 + 2e^{\eta\sqrt{n}} \cos (\xi\sqrt{n}) + e^{2\eta\sqrt{n}}\right]}$$

Let us replace, in equation (4), the quantities $e^{\eta\sqrt{n}}$, $\sin (\xi \sqrt{n})$ and $\cos (\xi \sqrt{n})$ by their values drawn from equations (1), and the co-ordinates u and v by the co-ordinates ρ and ω, the distance and bearing on the sphere of the point M with respect to the origin, by means of the relations :

$$\begin{aligned} &\cos u = -\cos \rho \cos C + \sin \rho \sin C \sin (\omega - \theta), \\ &\cot v \cos (\omega - \theta) = \cot \rho \sin C + \cos C \sin (\omega - \theta), \\ &\sin u \sin v = \sin \rho \cos (\omega - \theta), \end{aligned}$$

and we find :

$$\frac{1}{A} \tan \frac{AZ}{2} = \tan \frac{\rho}{2} (\cos \omega + i \sin \omega), \tag{5}$$



which is the definition of YUNG'S projections. We see that this second element depends only on the position of the point M on the sphere with respect to O, and is independent of the value of A, i.e. of n and θ. *General solution N° 2* is thus identical with YUNG'S projections. We see besides that the stereographic projection falls in the class of YUNG'S projections. It is sufficient to make n equal to zero. The co-ordinates of the point M in the stereographic projection will be :

$$x' = 2 \tan \frac{\rho}{2} \cos \omega, \qquad y' = 2 \tan \frac{\rho}{2} \sin \omega,$$

and thus we have :

$$\frac{1}{A} \tan \frac{AZ}{2} = \frac{x' + iy'}{2}$$

This observation enables us to calculate the co-ordinates ξ, η of *general solution N° 2* by starting from any one of YUNG'S projections and without passing through the calculation of the co-ordinates u and v.

One could for instance start from the stereographic co-ordinates, or more simply from the co-ordinates ρ and ω (Guillaume POSTEL or HATT projection). Equalising the real and the imaginary parts of equations (4) and (5), we have :

$$(6) \quad \tan\left(\xi\sqrt{n}\right) = \frac{\sin C \sin \rho \cos(\omega - \theta)}{\cos C + \cos \rho}, \quad \frac{e^{2\eta\sqrt{n}} - 1}{e^{2\eta\sqrt{n}} + 1} = \frac{\sin C \sin \rho \sin(\omega - \theta)}{1 + \cos C \cos \rho}$$

We could also have started from the co-ordinates of GAUSS, whose projection belongs to the YUNG group, by making θ equal to zero and n equal to I. One would then have, if x and y are GAUSS's co-ordinates :

$$\tan\left(\xi\sqrt{n}\right) = \sin C \frac{2\,e^{v} \sin x \cos\theta + (e^{2v} - 1)\sin\theta}{2e^{v}\cos x + (e^{2v}+1)\cos C}$$

(7)

$$\frac{1 - e^{2\eta\sqrt{n}}}{1 + e^{2\eta\sqrt{n}}} = \sin C \frac{2\,e^{v}\sin x \sin\theta + (e^{2v} - 1)\cos\theta}{2\,e^{v}\cos x \cos C + e^{2v} + 1}$$

It is even unnecessary to calculate the GAUSS's co-ordinates; we get the result more rapidly by calculating simply the CASSINI-SOLDNER co-ordinates x and y', which are those most easily deduced from the geographical positions (on the sphere).

We then get :

$$\tan\left(\xi\sqrt{n}\right) = \sin C \frac{\sin x \cos\theta \cos y' + \sin\theta \sin y'}{\cos x \cos y' + \cos C}$$

(8)

$$\frac{1 - e^{2\eta\sqrt{n}}}{1 + e^{2\eta\sqrt{n}}} = \sin C \frac{\sin x \sin\theta \cos y' - \cos\theta \sin y'}{1 + \cos x \cos y' \cos C}$$

Generally speaking, if one has calculated the co-ordinates of the point M on any one of YUNG's projections, defined by the quantity A (n, θ), it will suffice, to pass to another projection defined by the value A_1 to put the expression $\frac{1}{A}\tan\frac{AZ}{2}$ in the form $f + i\,\varphi$, f and φ being real expressions as in (4) ; and we shall then have the co-ordinates of solution A_1 (n_1, θ_1) by the formulae (9) :

$$\tan \xi_1\sqrt{n_1} = 2\sqrt{n_1}\,\frac{f\cos\theta_1 + \varphi\sin\theta_1}{1 - n_1(f^2+\varphi^2)} = 2\sqrt{nn_1}\,\frac{2\,e^{\eta\sqrt{n}}\sin(\xi\sqrt{n})\cos(\theta - \theta_1) + (1 - e^{2\eta\sqrt{n}})\sin(\theta - \theta_1)}{(n - n_1)(1 + e^{2\eta\sqrt{n}}) + (n + n_1)\,2\,e^{\eta\sqrt{n}}\cos(\xi\sqrt{n})}$$

$$\frac{1 - e^{2\eta_1\sqrt{n_1}}}{1 + e^{2\eta_1\sqrt{n_1}}} = 2\sqrt{n_1}\,\frac{f\sin\theta_1 - \varphi\cos\theta_1}{1 + n_1\,(f^2+\varphi^2)} = 2\sqrt{nn_1}\,\frac{-2\,e^{\eta\sqrt{n}}\sin(\xi\sqrt{n})\sin(\theta - \theta_1) + (1 - e^{2\eta\sqrt{n}})\cos(\theta - \theta_1)}{(n + n_1)(1 + e^{2\eta\sqrt{n}}) + (n - n_1)\,2\,e^{\eta\sqrt{n}}\cos(\xi\sqrt{n})}$$

It seems that YUNG only envisaged the calculation of the projections which he defined by the development in series of the expression of the tangent, and he showed that the first two terms of the development of the tangent lead to what LABORDE has called *solution N° 1*. The result is that solutions N° 1 and N° 2 do not differ from one another in practice up to a certain distance from the origin and the superiority of solution N° 2 beyond this distance results above all in the greater facility with which the corrections necessary for geodesy may be precisely calculated. Solution N° 2 leads also to the construction of new planispheres by the possibility of throwing any region which one desires to infinity.

The new exposition given to this question by Colonel LABORDE brings out in a remarkable way all the advantages of a projection which, while adapting itself the best to the shape of the country to be represented, permits corrections to be calculated with all the precision required by geodesy of the first order up to a distance of 3,000 kilometres (1600 n. miles) from the central point.