SYSTEMS OF PLANE PROJECTION FOR PLOTTING HYDROGRAPHIC SURVEYS.

by

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The problem of representing the earth's surface on a plane is quite different according to whether it is regarded from the viewpoint of the construction of charts or that of the plotting of geodetic, topographic or hydrographic work.

The construction of charts generally involves the representation of rather large areas, sometimes very large ones, in certain parts of which it is inevitable that the distances, areas and angles are appreciably modified, since, as is well known, the surface of the earth cannot be applied with accuracy to a plane. Among the innumerable systems of representation which have been proposed up to the present, the one would be selected which will result in the least alteration in the values (angles, distances and areas) which it is particularly desired to measure on it. Thus, for navigation, where the angle of the course and the direction of a bearing are of prime importance, the MERCATOR projection is ordinarily preferred because it is more convenient than others for their measurement and laying off.

STATEMENT OF THE PROBLEM. — But this question is entirely outside the scope of the subject of this article, wherein only the employment of rectangular coordinates for plotting surveys and the systems of plane projections which facilitate their use will be considered. Two conditions are necessary for the easy and accurate plotting of a survey:

1°) It is essential that all *graphic construction* which must be made from the distances and angles measured in the field or at sea, may be drawn on the plan without the necessity of modifying them *in any way*.

This requires that no modification appreciably exceeding 1' should have to be made to the angles and that no length which has to be employed should be modified by an amount equal to 1/10 or 2/10 of a millimetre at the scale of construction. In fact, the procedure would be much too complicated if it were necessary to modify all such measurements, which are considerable in number.

2°) A survey necessitates certain accurate measurements, the precision of which is not fully utilized in graphic construction. When lengths of the order of one metre cannot be neglected, and the same is true for angles of the order of a second or even of tens of seconds, a graphic construction becomes inadequate; therefore it becomes necessary to resort to calculation. This is the case with all of the operations in the various orders of triangulation.

These calculations can be made, and they frequently are carried out, based on the ellipsoid of reference by well-known but very long methods. Thus geographical positions for the various triangulation points are obtained which have then to be reduced to the plane projection chosen for graphic representation, since it is necessary to plot these points on the construction sheet. If the network of meridians and parallels has been previously laid off according to the rules of transformation for the plane projection adopted, then the calculated geographic positions can be immediately plotted and in this case it will suffice if the projection fulfills the conditions which have just been stated under N° 1).

In this case the plane rectangular coordinates are scarcely necessary except for drawing in the meridians and the parallels and the border lines of the charts.

But calculations on the ellipsoid of reference are very complicated and it is much simpler to be able to make all the calculations relative to the triangulations as though all the measurements were made on a plane surface and thus avoid having to solve problems other than simple ones in plane geometry. This procedure is perfectly feasible, with complete accuracy, provided only that the distances and angles measured are modified according to the system of plane transformation adopted.

COINCIDENCE OF THE ELLIPSOID WITH A TANGENT PLANE. — The most simple solution, and one which is still employed by some Offices, consists in avoiding any modification of the lengths or angles by limiting the use of rectangular coordinates to an area not exceeding about 50 kilometres from the point of origin selected. If we attempt to apply a segment of a sphere to a plane surface, the angular opening augmentations which must then be applied to the segment and the width of this opening are given in the following table (*See* fig. 1) (*See :* COURTIER : *Annales Hydrographiques* 1912, page 24) :

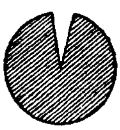

Fig. 1

Radius of the Segment.	*Angular Opening.*	*Width of Opening.*
7 km. 3	0". 3	0.01 m.
15 km. 7	1". 3	0.10 m.
33 km. 9	~~2". 2~~ 6".1	1.00 m.
72 km. 9	~~10". 0~~ 28".3	10.00 m.
100 km. 0	53". 1	25.74 m.

In order that the angular and linear alterations necessary in this representation may be neglected in calculation, it will be necessary to adopt a new origin at every 100 km, at most; and each time to calculate in the two systems a certain number of points equidistant from the two successive origins. To this complication is added the difficulty of introducing into calculations in plane geometry, points whose coordinates are referred to two different origins. Consequently this prevents the employment of these calculations for triangulations other than those of the very last order. Anyway, it is inevitably necessary ro return to geographic positions, which alone provide a coherent system of coordinates for the entire group of points, the positions of which are the results of calculations.

METHOD OF APPROXIMATE POSITIONS FOR CALCULATING POSITIONS. — About 40 years ago the French Hydrographic Office revised its methods of calculating triangulations on a plane, as a result of the work of Ingénieur Hydrographe HATT (*See*: Des coordonnées rectangulaires et de leur emploi dans la triangulation - Publication N° 746, Serv. Hydr. de la Marine, 1893).

The basic idea on which this new method was elaborated is the same which led the navigator to adopt the position line of MARCQ ST. HILAIRE in place of the older method of direct calculation of latitude and time.

The calculation of the position of a point by rectangular coordinates is preceded by the determination of an approximate position by means of rapid graphic construction on the plotting-sheet itself. The calculation then consists in determining the equations, with respect to this approximate position as origin, of the lines which represent the bearing and the tangent to the arc subtended by the observed angle. The intersection of two such lines gives the position of the station. The method of calculation permits the employment of as many lines as are desired in fixing the position and consequently of making use, with equal facility, of all the angles taken to the station from points already known, as well as all the angles observed from the station between known points. The lines thus obtained are plotted on a very large scale (between 1/10 and 1/100 depending on circumstances) on a sheet of cross-section paper, and definite location of the position within the "cocked-hat" which they form is easily made after rapid appreciation of the probable accuracy of each line.

The simplicity of this method led to the employment of a larger number of calculated points, thus attaining greater accuracy for the less important points and greater security due to the precise location of topographical features, and made it certain that well fixed points could always be found to complete a hydrographic survey, should need arise for such further work in later years.

The great facility given by this method for utilizing all observations and for estimating their accuracy soon led to the desire to employ it in the calculation of secondary triangulations and even in those of the first order. Such calculation yields the degree of accuracy required for this triangulation if the modifications which must be applied, before they are employed, to the angles taken in the field can be accurately calculated, in accordance with the system of plane projection adopted.

Thus it was necessary to find systems of projection which conform to these conditions over as large an area as possible. First, several projections which are in general use for the plotting of surveys will be examined briefly and then the improvements which have been proposed in recent articles will be examined.

a) CASSINI-SOLDNER PROJECTION OR ORTHOGONAL CO-ORDINATES. — Having selected an initial meridian on the ellipsoid of reference and a point of origin O on this meridian, the position of any point M may be determined by the distance y of the geodetic drawn perpendicular to the initial meridian and by the distance x along the meridian from the point of origin to the

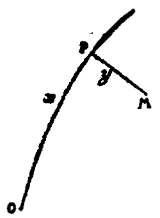

Fig. 2

foot of the perpendicular p (See figure 2). By drawing on the projection sheet two rectangular axes OX and OY, in which OX represents the meridian, the position of the point is defined by :-

$$X = x \qquad Y = y$$

This system of co-ordinates is due to the astronomer Cesar-François CASSINI, who used it for the large map of France which he commenced in 1745 ; SOLDNER took up the study of this system in 1805 and later it was applied in the construction of the maps of Bavaria and Wurtemberg. (*)

To make a study of a plane projection the *angular* and *linear alterations* about a point will be given. The limit approached by the difference in bearing (reckoned from N. towards E.) between the line which joins a point to another on the plane and the geodetic which joins them on the ellipsoid, when this second point draws indefinitely close to the first, will be termed the *angular alteration* at that point.

If l' and l represent the lengths of the straight line and of the geodetic, the limit approached under such conditions by the quantity $\frac{l'-l}{l}$ is called the *linear alteration* and will be designated by μ. The term *linear modulus*, designated by m, will be applied to the limit approached by the ratio $\frac{l'}{l}$. Thus $m = 1 + \mu$

On the CASSINI projection the maximum angular alteration is :

$$\frac{y^2}{4 R_0^2 \sin 1''} \;;$$

and the maximum linear alteration is :

$$\mu = \frac{1 y^2}{2 R_0^2} \;,$$

R_0 being the total radius of the curvature $\sqrt{N_0 \rho_0}$ of the ellipsoid at the point of origin and neglecting the terms of a higher order than the second in the equation of deformation.

The angular and linear alterations may reach the following absolute values :-

for y =	*Angular Alterations.*	*Linear Alterations.*
50 km.	~~1.6~~" 3",2	31×10^{-6}
100 km.	~~6.3~~" 12",7	123×10^{-6}
150 km.	~~14.3~~" 28",5	276×10^{-6}
200 km.	~~25.3~~" 50",7	491×10^{-6}

(*) If the point M is defined by its latitude φ and its longitude L (reckoned from the initial meridian) then :

$$\sin \frac{y}{R_0} = \cos \varphi \sin L \qquad \cot\left(\frac{x}{R_0} + \varphi_0\right) = \cot \varphi \cos L$$

It should be noted that the modifications which must be applied to the observed angle is the algebraic difference between the two relative angular alterations to its sides, and consequently may attain twice the value given in this table.

Since the alteration depends on the square of the distance from the meridian, the projection is particularly suitable for a narrow strip of territory extending a short distance on each side of a meridian but over a length which may be very great.

It is evident that even with no very great values of y the correction to be applied to the angles cannot be neglected in the calculation and, for over 200 km., may not even be neglected in graphic construction.

b) LAMBERT CYLINDRICAL ORTHOMORPHIC PROJECTION. (Conformal Projection of GAUSS or the Inverse MERCATOR Projection). — The CASSINI projection is not conformal (*orthomorphic*, or *autogonal* or *isogonic*) *i. e.*, the angular alteration is not nil, or, what amounts to the same thing, the *linear alteration* is not independent of the orientation of the element considered.

LAMBERT, and later GAUSS, have shown that a conformal projection may be obtained by starting with the orthogonal co-ordinates x and y of CASSINI and making the following transformation :

$$X = x \qquad Y = R_o \operatorname{Log} \operatorname{tang}\left(\frac{\pi}{4} + \frac{y}{2R_o}\right)$$

Here it is a question of the Naperian logarithm and the earth is considered as being coincident with a sphere of the same total curvature at the origin. This restriction is not obligatory ; SCHREIBNER, and later KRÜGER, have given the complete developments of Y as a function of the latitude and of the difference of longitude, taking the ellipsoid into consideration. However, the above formula suffices for the comparison in view. This projection was used in the survey for the maps of Hanover.

As a first approximation the terms of the 4th order may be neglected and then :-

$$Y = y + \frac{y^3}{6R_o^2}$$

In this projection the *angular alteration* is nil, for the projection is conformal ; the linear alteration is :-

$$\mu = \frac{y^2}{2R_o^2}$$

it remains the same regardless of the orientation of the element. It is equal to the maximum linear alteration given by the CASSINI-SOLDNER projection. Thus the conditions favourable to its use are the same.

c) LAMBERT CONICAL ORTHOMORPHIC PROJECTION. — This projection is conformal and extends on both sides of the parallel selected as the origin. The parallels are represented as concentric circles, and the meridians as straight lines radiating from the centre of these circles.

The co-ordinates X and Y of plane transformation may be expressed as functions of the co-ordinates x and y of CASSINI. Here, neglecting terms of the 4th order :-

$$X = x + \frac{x^3}{6R_0^2} - \frac{xy^2}{2R_0^2}$$

$$Y = y + \frac{x^2 y}{2R_0^2}$$

The linear alteration will be :- $\mu = \frac{x^2}{2R_0^2}$

The same linear alterations will be found here as in the preceding projections, except that the axis of the x's takes the place of the axis y. This projection therefore is suitable for the representation of a strip of territory perpendicular to the meridian.

d) STEREOGRAPHIC PROJECTION. — The stereographic projection also is a conformal projection. The CASSINI co-ordinates x and y may be converted into X and Y by the approximate formulae, to within the 4th order :-

$$X = x + \frac{x^3}{12R_0^2} - \frac{xy^2}{4R_0^2}$$

$$Y = y + \frac{y^3}{12R_0^2} + \frac{x^2 y}{4R_0^2}$$

The linear alteration is :-

$$\mu = \frac{x^2 + y^2}{4R_0^2}$$

Its value depends only on the distance of the position from the origin; thus the projection is particularly suitable for the representation of a territory of circular form.

e) TISSOT'S PROJECTIONS. — TISSOT, in his "Mémoire sur la représentation des surfaces et les projections de cartes géographiques", published in 1881, shows that *it is impossible to find a system of representation which reduces the linear alteration below the second order.*

This is, in fact, the order of the linear deformations which have been found in paragraphs *a*), *b*), *c*), and *d*).

In addition, TISSOT gives the general formula for the projections which have the double property of producing *angular alterations of the third order and linear alterations of the second order only.*

In order to retain the same notations, as far as possible, this general formula will not be presented in the form given to it by TISSOT, but in that deduced by Ingenieur Hydrographe COURTIER (*See: Annales Hydrographiques* 1912, page 40) as a function of the CASSINI co-ordinates x and y :-

$$(1) \qquad \begin{aligned} X &= x + \frac{A}{3}\frac{x^3}{R_0^2} - B\frac{x^2 y}{R_0^2} - A\frac{xy^2}{R_0^2} + \frac{B}{3}\frac{y^3}{R_0^2} \\ Y &= y + \frac{B}{3}\frac{x^3}{R_0^2} + A\frac{x^2 y}{R_0^2} - B\frac{xy^2}{R_0^2} + \left(\frac{1}{2} - A\right)\frac{y^3}{3R_0^2} \end{aligned}$$

The equation for the linear alteration of the point the co-ordinates of which are x and y is :-

$$(2) \qquad \mu = A \frac{x^2}{R_0^2} - 2B \frac{xy}{R_0^2} + \left(\frac{1}{2} - A\right) \frac{y^2}{R_0^2}$$

The quantities A and B are arbitrary constants, the expressions for X and Y are only expanded to the third order inclusive.

Tissot did not consider it necessary that the system of projection should be absolutely conformal, provided that the angular alteration were always very small.

In fact it is well known that the use of a conformal projection does not obviate the necessity for modifying the observed angles before employing them for calculations in plane geometry. The normal sections of the geoid passing through the point of observation and the points observed when transformed on the projection, are curves which intersect at the same angle as the normal sections, provided that the projection is conformal, but this angle is not ordinarily the same as that which we employ in calculations in plane geometry : *i. e.* that of the straight lines joining the point of observation to the observed points on the projection. (*)

In a non-conformal projection the modification to be applied to the observed angles may be considered as the resultant of two terms, the first being due to the non-conformal projection : *i. e.* the angular alteration, and the second due to the inevitable difference between the angle made by the transformed geodetic lines and the straight lines drawn on the projection. The Tissot projections are conformal to within the third order ; that is, on them the angular alteration is of the third order. Tissot then considered this quantity as negligible, since he looked at it from the point of view of map construction.

But it is permissible in the equations for X and Y of the Tissot projections to develop the expressions beyond the third order and consequently to approach as close as desirable to the conformal condition. Among the general projections which are well known there are several which give a development identical to that of Tissot's equations up to terms of the third order inclusive. They have the same properties as the Tissot projections and instead of calculating X and Y by the complete theoretical formulae for these projections, it is always permissible to calculate them by means of a development of the type indicated above, and to stop at the terms the values of which are negligible for the purpose intended.

Tissot's limiting conic section. — If the equation (2) for the linear alteration (limited to the second order) be considered it will be seen that it

(*) It is known that the normal section makes an angle with the geodetic equal to

$$\frac{l^2}{12} \frac{e^2 \cos^2\varphi \sin 2V}{a^2}$$

(l being the length of the geodetic and V its bearing). This correction will be assumed to have been made ; in general it does not reach a value of 0.01" ; and thus the geodetic lines only need be considered.

has the same value everywhere on a conic section for which, with the same approximation, the equation :-

$$(3) \qquad A\frac{X^2}{R_0^2} - 2B\frac{XY}{R_0^2} + \left(\tfrac{1}{2}-A\right)\frac{Y^2}{R_0^2} = C$$

For a given survey the projection shows the *minimum* deformation if the greatest value of the quantity μ is the least possible. This leads to a search for the conic section which, while enclosing the entire area of the survey, will give the minimum value for μ. It is known, if D be the length of the semi-diameter of the conic section (3) which bi-sects the axes, that :-

$$(3\ a) \qquad C = \frac{D^2}{4R_0^2}$$

It suffices therefore to chose from among the conic sections which can enclose the area, that in which the diameter bisecting the axes is the smallest. When this has been determined its centre should be taken as the origin of the coordinates and the quantities A and B will be fixed. On every homothetic conic section the linear alteration will be constant and equal to a value of μ which is proportional to the square of the axes. (*)

In practice this conic section cannot be a hyperbola (with which it would be necessary to include the conjugate hyperbola). Generally it will be an ellipse, but may be a circle or a system of two parallel straight lines.

If the projections studied in paragraphs *a*), *b*), *c*) and *d*) be examined from this point of view, it will be seen that :-

a) The CASSINI projection is not a projection of the TISSOT type, its equations cannot become the same as the equations (1) and the angular alteration therein is of the second order.

(*) If the 3rd order be neglected, no type of conformal projection (at least of within the third order) can give a value less than that given by the TISSOT minimum deformation projection for the difference between the maxima and minima of μ.

By means of an artifice which may be called the scale correction, due to TISSOT and which will be discussed later, under the name of coefficient K_0, in dealing with the LABORDE projection, the linear alteration at the periphery may be reduced by half by making an equal and contrary correction at the origin. This has not been done here in order to simplify the exposition, but two projections should not be compared when this scale correction is applied to one and not to the other. (*See :* "Some Investigations in the Theory of Map Projections", by A. E. YOUNG, page 49).

This scale correction does not affect the angular alteration which is the modification of greatest interest in this particular study.

TISSOT has clearly defined what he terms the *projection of minimum deformation.*

Indeed other kinds of definition may easily be conceived. In 1861 Sir George AIRY adopted another under the name of *Zenithal projection by Balance of Errors,* which consists in reducing to a minimum the sum of the squares of the linear alterations in the directions of the radius vector and its perpendicular, integrated over the entire circular surface represented. CLARKE improved its application and A. E. YOUNG generalized the method in the work cited above. This criterion gives very interesting results for cartographic representation of very large surfaces. But the projections to which it leads are not all conformal and even when they are, they *always* show linear alterations at the periphery which are greater than those of the TISSOT projections, except when they coincide with the latter.

b) The GAUSS projection is a projection of the TISSOT type, it corresponds to :-

$$A = B = 0$$

The equation of the TISSOT conic section becomes :-

$$\frac{Y^2}{2R_0^2} = C^{te}$$

It represents a system of two straight lines parallel to the axis of the *X's* and is, as seen above, a strip extending along a meridian.

c) The LAMBERT conical projection corresponds to :-

$$A = \frac{1}{2} \quad B = 0$$

The equation of the TISSOT conic section is :-

$$\frac{X^2}{2R_0^2} = C^{te}$$

It represents a system of two straight lines parallel to the axes of the *Y's* and is the strip perpendicular to a meridian which was referred to above.

d) The stereographic projection corresponds to :-

$$A = \frac{1}{4} \quad B = 0$$

The equation of the TISSOT conic section represents a circle :-

$$\frac{X^2 + Y^2}{4R_0^2} = C^{te}$$

This projection is particularly convenient for representing an area of circular shape.

Remark :- The axes of the TISSOT conic section will coincide with the axes of the coordinates when $B = 0$; the equations (1) and (2) will then be simpler. But the simplification of equation (1) is not of very great importance for, in practice, the calculation for the transformation of coordinates is carried out for a very small number of points only. The simplification of equation (2) is not very great and does not justify the abandonment of a more perfect projection when such projection is of advantage owing to the geographical configuration of the country. Equation (2) which gives the value of μ is of great importance; the method of deduction therefrom of the principal term of the modification to be applied to the observed angles will be set out below.

LABORDE PROJECTION. — If the equation be developed only up to the third order inclusive, as was done by TISSOT, the field of application of the projection is necessarily limited, in practice it cannot be employed when the values of *D* are greater than 180 or 200 kilometres, for then the exact formulae for deformation become very complex and the corrections to be applied to the angles cease to be negligible for the calculation of the less important points and shortly thereafter even for graphic construction. Thus it is impossible to use these projections for the direct calculation of a triangulation of the 1st order, but they are of great service in plotting hydrographic surveys provided that the origin is changed every 300 or 400 kilometres.

Commandant LABORDE, when he was Chief of the Geographical Section in Madagascar, made a very interesting mathematical study of conformal projections which led him to propose a new formula for projections for plotting surveys. He applied it in practice with great sucess to the surveys of Madagascar (*See: La nouvelle projection du Service Géographique de Madagascar* by Commandant LABORDE, April 1928, Tananarive). In this article a few of the principal results of the theories can be indicated without giving the proofs.

The projection proposed by Commandant LABORDE is a strictly conformal projection. It is also a projection exhibiting minimum deformation following the ideas of TISSOT. The linear and angular alterations, as well as the modifications to be applied to the angles, are reduced to a minimum and, further, can be calculated by strictly accurate formulae.

This permits the field of application to be extended to more than 1.000 kilometres from the origin without any appreciable complications being introduced into the calculations by this extension and consequently it makes the use of this projection possible even for accurate triangulations of the first order.

Reverting to the ideas of LAMBERT, GAUSS and SCHREIBER which are applied under the name *Doppelprojektion* by the Landesaufnahme of Prussia (*), LABORDE first drew up a conformal projection of the ellipsoid on the sphere of the same radius of total curvature at the origin.

On the ellipsoid, let φ be the latitude and L the longitude (reckoned from the meridian selected as the initial meridian); on the sphere let ψ and λ be the coordinates given by the transformation.

$$(4) \quad \operatorname{tg} \psi_0 = \sqrt{\frac{\rho_0}{N_0}} \operatorname{tang} \varphi_0$$

λ is deduced from L by the formula :-

$$(5) \quad \lambda = \alpha L, \qquad \alpha = \frac{\sin \varphi_0}{\sin \psi_0}$$

ψ is deduced from φ by the formula :-

$$(6) \quad \operatorname{Log} \operatorname{tang}\left(\frac{\pi}{4} + \frac{\psi}{2}\right) = \alpha \operatorname{Log} \operatorname{tang}\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) - \alpha \frac{e}{2} \operatorname{Log} \frac{1 + e \sin \varphi}{1 - e \sin \varphi} + \text{constante}$$

in which *Log* indicates Naperian logarithms and of which the constant is determined by the condition that ψ_0 and φ_0 are related by formula (4).

The coordinates ψ and λ are transformed into the coordinates u and v of CASSINI-SOLDNER (*see*: *a*).) by the formulae :-

$$(7) \quad \sin \frac{v}{R_0} = \cos \psi \sin \lambda, \qquad \cot g \left(\frac{u}{R_0} + \psi_0\right) = \cot g \psi \cos \lambda$$

or $\frac{u}{R_0} = \psi - \psi_0 + \nu$ with $\sin \nu = \sin \psi \operatorname{tang} \frac{\lambda}{2} \operatorname{tang} \frac{v}{R_0}$

The GAUSS conformal projection is then considered (*see*: *b*).) by the formulae :-

(*) *See*: JORDAN-EGGERT "Handbuch der Vermessungskunde", third vol., 7th edition, page 593.

$$x = u$$

$$(8) \qquad y = R_0 \operatorname{Log} \operatorname{tang}\left(\frac{\pi}{4} + \frac{v}{2R_0}\right)$$

Finally the definite projection will be given by :-

$$(9) \qquad \begin{cases} X = x + \frac{A}{3}\frac{x^3}{R_0^2} - B\frac{x^2y}{R_0^2} - A\frac{xy^2}{R_0^2} + \frac{B}{3}\frac{y^3}{R_0^2} \\ Y = y + \frac{B}{3}\frac{x^3}{R_0^2} + A\frac{x^2y}{R_0^2} - B\frac{xy^2}{R_0^2} - \frac{A}{3}\frac{y^3}{R_0^2} \end{cases}$$

Remark :- The formulae (9) differ from those given above for the TISSOT projections by the fact that y is not the CASSINI coordinate, but a coordinate of the GAUSS conformal projection, which, as was seen in *b*), may be expressed to within the fourth order by $y' + \frac{y'^3}{6R_0^2}$, if y' is the CASSINI coordinate. This transformation makes the equation (1) identical with equations (9). But further, from the fact that progression was through the intermediary of a strictly conformal projection of the ellipsoid onto a sphere, and also that strictly accurate formulae have always been employed for the transformation, the result is that the formulae (9) do not represent the first terms of a development, but the exact formulae for a strictly conformal projection.

LINEAR MODULUS. — The equation for the linear modulus (relation of infinitely small lengths on a plane and on the ellipsoid) will be given rigorously by :-

$$m = \alpha R_0 \frac{\cos\psi}{N\cos\varphi \cos\frac{v}{R_0}} \sqrt{1 + 2A\frac{x^2-y^2}{R_0^2} - 4B\frac{xy}{R_0^2} + (A^2+B^2)\left(\frac{x^2+y^2}{R_0^2}\right)^2}$$

If this be developed as a function of the quantities X and Y, then :-

$m = 1 + A\frac{X^2}{R_0^2} - 2B\frac{XY}{R_0^2} + \left(\frac{1}{2} - A\right)\frac{Y^2}{R_0^2} + \cdots$ the terms of the 3rd order and beyond. The factor μ of equation (2) will be found; the constants A and B will thus be determined, according to the idea of TISSOT, by means of a limiting conic section on which μ will be as small as possible. If θ be the angle which the major axis of the conic section makes with the axis of the X's and n the ratio $\frac{a^2-b^2}{a^2+b^2}$ between the lengths a and b of the axes, then :-

$$A = \frac{1-n\cos 2\theta}{4} \qquad B = \frac{n\sin 2\theta}{4} \qquad \frac{1}{2} - A = \frac{1+n\cos 2\theta}{4}$$

A lies between 0 and 1/2 ; B between 0 and 1/4

If μ' is the value of μ on the limiting conic section, it may be reduced by half, as was shown by TISSOT, by reducing the scale of the entire projection in the proportion $1 - \frac{\mu'}{2}$. For this purpose, it suffices to multiply the radius of the sphere on which a conformal representation of the ellipsoid has been made by $1 - \frac{\mu'}{2}$, which quantity will be called K_0 and the quantity $K_0 R_0$ will be called R. The value of m at the origin will be $1 - \frac{\mu'}{2}$

instead of 1 and that of m on the limiting conic section will be $1 + \frac{\mu'}{2}$ neglecting the terms of the third order and above.

Then the strictly accurate equation :-

$$(10)\quad m = \alpha R \frac{\cos\psi}{N \cos\varphi \cos\frac{y}{R}} \sqrt{1 + 2A\frac{x^2-y^2}{R^2} - 4B\frac{xy}{R^2} + (A^2+B^2)\left(\frac{x^2+y^2}{R^2}\right)^2}$$

or its development :-

$$(10a)\quad m = K_0\left[1 + A\frac{X^2}{R^2} - 2B\frac{XY}{R^2} + \left(\frac{1}{2} - A\right)\frac{Y^2}{R^2} + \dots\right.$$ the terms of the 3rd order and above) will be obtained.

The advantages offered by the LABORDE projection over the *Doppelprojektion* of the *Landesaufnahme* can now be appreciated. The quantities x and y of formula (9) are, in fact, the coordinates of the *Doppelprojektion* ; the two projections are identical if $A = B = 0$. It was seen above that, in this case, the limiting conic section of TISSOT becomes a system of two straight lines parallel to the axis of the X's.

Let an area be considered in which the limiting conic section $A\,B\,C\,D\,E\,F$ is represented in figure 3. Lay off on the bisectors of $X\,O\,Y$ lengths equal to the diameters bisecting the axes of the conic section and through their extremities draw IH and JG parallel to the axis OX. The linear alteration which was $\frac{\mu}{2}$ in the LABORDE projection over the entire contour of the ellipse will be μ in the *Doppelprojektion* on the straight lines IH and JG, and will therefore be twice as great at the common points A, B, D and E.

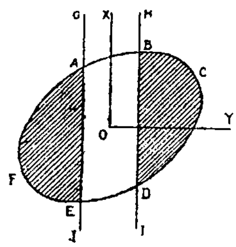

Fig. 3

Besides, since the Y of the vertex C (in the figure) is nearly double that of the straight line IH, the value of the linear alteration at C which, in the LABORDE projection, is always $\frac{\mu}{2}$, will be equal, in the *Doppelprojektion*, to $4\,\mu$; *i. e.* about eight times as great. (*)

ANGULAR CORRECTIONS.— The modification which should be applied to the bearing of a straight line MP of length l, starting from a point M on the plane (the angle between the transformed geodetic and its rectilinear chord) may be developed as a function of the curvature of the transformed geodetic (*see* fig. 4). If the curvature at the extremity P_3 of the first third of the transformed geodetic be called Γ_3, the correction of the bearing may be expressed by :-

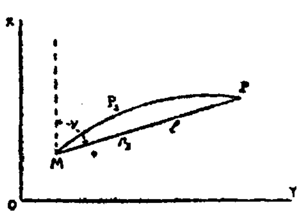

Fig. 4

$$(11)\quad c'' = \frac{\Gamma_3\, l}{2 \sin 1''}$$

(*) There would not be the least difficulty in applying the same factor to the DOPPEL projection and thereby reducing its maximum linear alteration by half, but the linear alteration would still be 4 times as great at F and C as in that of the LABORDE projection.

neglecting the terms in l^3 which do not attain a value of one three-hundredth of a second when $l = 60$ kilometres and $OM = 3000$ km.

Also, the curvature Γ may be expressed as a function of the linear modulus and of its derivative m'_n taken in a direction normal to the curve.

$$(12) \quad \Gamma = \frac{m'_n}{m}$$

By taking the derivative in its relation to X and Y, the following equation for Γ is obtained :-

$$(13) \quad \Gamma = \frac{1}{m}\left(m'_X \sin V - m'_Y \cos V\right)$$

This formula is the rigorous expression.

It is very important to calculate the value of c'', as this value is required for correcting all the observed angles. For the plotting of the surveys, this is the only calculation which is necessary after the coordinates X and Y of the two points which serve as the base for the survey have been obtained. The plane coordinates of all the points may then be obtained by simple calculations in plane geometry. Therefore the need for this correction is emphasized.

Taking m_a as the value of m limited to terms of the 2nd order (*see 10 a*) :-

$$m = m_a + \varepsilon$$

Taking the derivatives of m_a in relation to X and Y and adding the index 3 to those values which refer to the position situated on the transformed geodetic at one third of the distance between the station and the point of which the angle is taken, the principal term for the angular correction will be :-

$$(14) \quad c'' = \frac{k_0 \, l}{m_3 \, R^2 \sin 1''}\left[\left(AX_3 - BY_3\right)\sin V + \left(BX_3 + AY_3 - \tfrac{1}{2}Y_3\right)\cos V\right]$$

In this equation we may always conveniently take as X_3 and Y_3 the coordinates of the point p_3 (fig. 4) on the straight line in place of those of the point P_3 on the curve (anyway it will be easy to take small segment $p_3 \, P_3$ into account).

Tables or diagrams to facilitate the calculation of c'' may then be prepared. Its approximate value may be obtained very rapidly in the following manner :-

Let curves be constructed on a small scale chart of the area (1:1,000,000 for instance) corresponding to various equidistant values of μ. These are the homothetic ellipses of the limiting conic section of TISSOT. The values of $\frac{c}{\sin 1''}$ are inserted above them. The ellipses will be closer together the greater their axes, for according to the formula ($3a$) :-

$$dD = \frac{2R^2}{D} d\mu$$

Let these ellipses be considered as contour lines of which the altitudes have the values of $\frac{c}{\sin 1''}$ which correspond to them. Mark on this chart the station point M and the observed point P, then through the point p_3 at the first third of the line which joins them drop a perpendicular ab such that

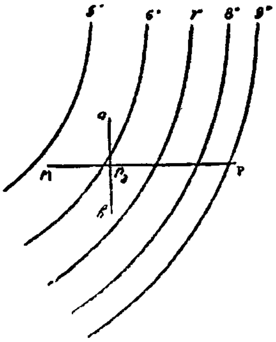

Fig. 5

$$a p_3 = p_3 b = \frac{MP}{4} \qquad (\textit{see}: \text{fig. } 5)$$

Calculate the difference in altitude between a and b; commencing with a which is such that $\overline{p_3 a}$ bears $V - \frac{\pi}{2}$

This difference divided by $1 + \mu$ will give a value of c'', with its sign, near enough for most of the calculations.

In the figure: altitude of a 5.7''
altitude of b 6.8''
difference — 1.1''

Here $1 + \mu$ may be replaced by 1, then $c'' = -1.1''$.

Further, merely from the form of the figure, it will be seen whether the two corrections relative to the two sides of the angle should be added or subtracted.

At a given point p_3, the closer the straight line MP approaches the direction of the tangent to the ellipse passing through the point, the greater correction c'' will be. (*)

To obtain great accuracy, c'' should be calculated by formula (14) instead of obtaining it graphically as in fig. 5; then a quantity γ'' must be added which is due to the fact that c'' was obtained by getting m_s and not m. The value of γ'' is :-

$$\gamma'' = \frac{l \varepsilon'_n}{2 m \sin 1''}$$

ε'_n being the derivative of ε taken at the point P_3 in the direction $V - \frac{\pi}{2}$

It may always be assumed that $m = 1$ in the equation for γ''.

In order to obtain γ'', it is sufficient to calculate the values of ε for a certain number of points and to construct the topographic surface of which the curves of equal altitude are the curves of equal values of $\frac{\varepsilon}{\sin 1''}$. The operation is then continued as in fig. 5. It will scarcely ever be necessary to have to resort to direct calculation of ε'_n.

If it be desired, the development of ε according to the powers of X and of Y may always be employed, and thereafter the formula (13). Now :-

(15)

$$\varepsilon = -e^2 \sqrt{\frac{\rho_0}{N_0}} \sin 2\varphi \frac{X^3}{R^3} + (7B^2 - 4A^2)\frac{X^4 + Y^4}{6R^4} + B(22A - 1)\frac{X^3 Y}{3R^4} + \left(6A^2 - 5B^2 - \frac{1}{2}A\right)\frac{X^2Y^2}{R^4} - 22AB\frac{XY^3}{3R^4} + \left(\frac{1}{4} - A\right)\frac{Y^4}{6R^4}$$

This equation is entirely of the 4th order of magnitude, since the first term, which contains the factor e^2, is at least of the 4th order. There are

(*) To be strictly accurate, it is necessary to take the slope of the topographic surface of fig. 5 at the point P_3 in the direction ab and to multiply it by $\frac{l}{2}$

therefore no terms of the 3rd order in the equation for the linear modulus m, nor in that of the angular correction. It follows therefore that γ'' need scarcely ever be calculated.

REMARKS ON THE LABORDE PROJECTION.—

a) In the case where the terms of the 2nd order only are taken another geometrical representation to the value of c'' may be given.

From formulae (11) and (13) it may be deduced that:-

$$c'' = \frac{\ell}{2m \sin 1''}\left(m'_x \sin V - m'_Y \cos V\right)$$

The locus of the extremity of a radius vector through the point P_3 (X_3, Y_3) in the direction V and of length equal to c'' (with its sign) is, when V alone varies, a circumference passing through P_3 and normal at this point to the regular curve μ = a constant, which passes through the same point. The equation of the diameter of this circle is $\frac{\ell}{2m \sin 1''}\sqrt{m'^2_{X_3} + m'^2_{Y_3}}$ or

$$(16) \qquad \frac{\ell}{R(1+\mu)\sin 1''}\sqrt{(A^2+B^2)\frac{X_3^2+Y_3^2}{R^2} + \frac{Y_3}{R}\left[\left(\frac{1}{4}-A\right)\frac{Y_3}{R} - B\frac{X_3}{R}\right]} \quad (*)$$

If the expressions $\frac{1-n\cos 2\theta}{4}$ and $\frac{n \sin 2\theta}{4}$ which contain the bearing θ of the major axis of the ellipse and the relation $\frac{a^2-b^2}{a^2+b^2}$ of its axes be substituted for A and B, the equation for this diameter may be expressed:-

$$\frac{\ell}{2R^2(1+\mu)\sin 1''}\sqrt{\left(\frac{1-n}{2}\right)^2(X_3^2+Y_3^2) + n\left(X_3 \sin\theta - Y_3\cos\theta\right)^2}$$

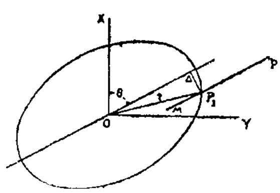

Fig. 6

It should be noted that $X_3^2 + Y_3^2$ represents the square r^2 of the distance of the point P_3 from the origin of the coordinates and $X_3 \sin\theta - Y_3\cos\theta$ represents the distance Δ of this point from the major axis of the ellipse (*see* fig. 6). Thus the value of the diameter of the circle considered is:-

$$\frac{\ell}{2R^2(1+\mu)\sin 1''}\sqrt{\left(\frac{1-n}{2}\right)^2 r^2 + n\Delta^2},$$

(*) **The following formula permits the exact maximum angular correction to be computed:**

$$\frac{2\ell}{R(1+H-\varepsilon)\sin 1''}\sqrt{\frac{p^2\cos^2\lambda + q^2\sin^2\lambda}{4\alpha^2\cos^2\psi} + \frac{A^2+B^2}{H}\frac{x^2+y^2}{R^2} + \frac{x\left[Ap\cos\lambda - Bq\sin\lambda + (A^2+B^2)\frac{x^2+y^2}{R^2}p\cos\lambda\right] - y\left[Bp\cos\lambda + Aq\sin\lambda + (A^2+B^2)\frac{x^2+y^2}{R^2}q\sin\lambda\right]}{\alpha R H\cos\psi}}$$

assuming

$$p = \sin\varphi - \alpha\sin\psi \qquad \varepsilon = (A^2+B^2)\left(\frac{x^2+y^2}{R^2}\right)^2$$

$$q = \alpha - \sin\varphi\sin\psi$$

$$H = 1 + 2A\frac{x^2-y^2}{R^2} - 4B\frac{xy}{R^2} + (A^2+B^2)\left(\frac{x^2+y^2}{R^2}\right)^2$$

Here λ, φ, ψ, x and y refer to point P_3.

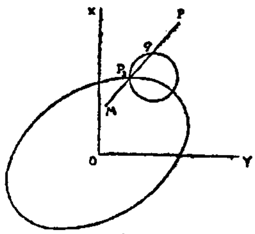

Fig. 7

If this circumference be plotted on the plotting sheet drawn to a convenient scale giving the desired accuracy, it meets the straight line MP at a point q. The segment $P_3 q$, with its sign, represents the value of the angular correction which should be applied to the bearing V (*see :* fig 7). Thus this correction will be seen to be at its maximum for a point P_3 in the direction of the tangent to the ellipse.

It should be remembered that the bearing ω of the tangent to the ellipse may be calculated by the formula :-

$$\tan\omega = -\frac{m'_X}{m'_Y} = \frac{AX_3 - BY_3}{BX_3 - (\frac{1}{2} - A)Y_3} = \frac{n(X_3 \cos 2\theta + Y_3 \sin 2\theta) - X_3}{n(Y_3 \cos 2\theta - X_3 \sin 2\theta) + Y_3}$$

b) When the conic section becomes two parallel straight lines of bearing θ, such as may frequently occur in hydrography, the equations for the μ and the diameter of the circle are greatly simplified.

In these cases :- $n = 1 \quad A = \frac{\sin^2\theta}{2} \quad B = \frac{\sin\theta\cos\theta}{2} \quad \frac{1}{2} - A = \frac{\cos^2\theta}{2}$

$$\mu = \frac{(X \sin\theta - Y \cos\theta)^2}{2R^2}$$

If Δ is the distance of the point X, Y to the straight line of bearing θ passing through the origin

$$(17) \quad \mu = \frac{\Delta^2}{2R^2}, \quad m = K_0\left(1 + \frac{\Delta^2}{2R^2} + \ldots \text{ terms of the 3rd order and above}\right)$$

$$\text{diam.} = \frac{l\Delta_3}{2R^2(1+\mu)\sin 1''}$$

The circumference which serves to measure c'' becomes very easy to plot, its diameter lies on the bearing θ and has a value equal to a constant $\frac{1}{2R^2 \sin 1''}$ multiplied by $l\,\Delta_3$ (because $1 + \mu$ may be replaced by 1).

It will be seen that c'' increases only in proportion to Δ_3, whereas μ increases proportionally to the square of the distance Δ.

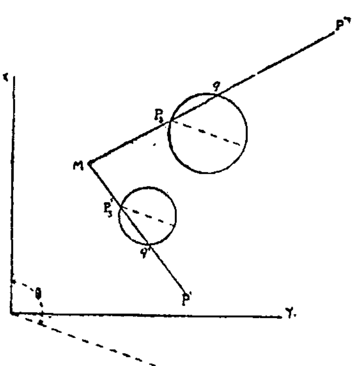

Fig. 8

Fig. 8 shows the angular corrections to be applied to the bearings of the sides MP and MP' of an angle. These are $P_3 q$ and $P'_3 q$. The correction to the angle M will be :-

$$P'_3 q' - P_3 q$$

If the angle M be equal to 60° and if its sides are equal, which is the case with perfectly formed triangles, the maximum correction value of an angle M will be the same as the equation (17) of the diameter and will correspond to an angle M the interior bisector of which is perpendicular to the direction θ.

c) In the case also where the limiting conic section is a circumference, the equations are simplified. In this case :-

$$n = 0 \qquad A = \frac{1}{4} \qquad B = 0$$

$$\mu = \frac{z^2}{4R^2} \qquad m = K_o\left(1 + \frac{z^2}{4R^2} + \cdots\right)$$

The diameter of the circle of the angular correction will be equal to :-

$$\frac{\ell_{1,3}}{4R^2(1+\mu)\sin 1''}$$

When the limiting conic section is a circumference, it is simpler to employ the following system of projection :-

1° Apply the ellipsoid to the sphere (formulae 4,5 and 6).

2° By the rigorous formulae of spherical trigonometry calculate the azimuthal coordinates ω and ρ of each point of the sphere of coordinates ψ and λ.

3°) Apply the ordinary theory of the stereographic projection of the sphere. (*)

d) If equation (16), which gives the maximum value at a point of the angular correction of a direction, be considered, it is evident that in view of the prime importance of the angular corrections, it is advisable that thsi value should be made as small as possible ; it will be the same all over the conic section for which the equation is :-

$$(18) \qquad (A^2+B^2)X^2 + \left[(1-A)^2 + B^2\right]Y^2 - BXY = \text{constante}$$

It would appear natural to give this conic section the form of the limiting conic section of TISSOT of which the characteristics n and θ are determined by the shape of the area to be represented.

For this purpose, the values :-

$$A = \frac{1}{4} - \frac{1-\sqrt{1-n^2}}{4n}\cos 2\theta$$

$$B = \frac{1-\sqrt{1-n^2}}{4n}\sin 2\theta$$

should be adopted for A and B.

Thus a projection of *minimum angular correction* will be obtained, whereas the values of A and B given above correspond to a *projection of minimum linear deformation.*

In functions of the lengths of the axes of the ellipse, the value of n is :-

$$n = \frac{a^2-b^2}{a^2+b^2}$$

Taking :- $$p = \frac{a-b}{a+b}$$

the new expressions of A and B may be written in a form analogous to the old :-

$$A = \frac{1-p\cos 2\theta}{4} \qquad B = \frac{p\sin 2\theta}{4}$$

(*) It may be noted that, when this system of projection is applied to the case wherein the origin is the pole, it will turn out to be the projection described by Capt. L. TONTA under the name of *Lambert Conformal Polar Projection* in *Hydrographic Review* Volume VI, N° 1, May 1929

It may be seen therefrom that the TISSOT conic sections of equal linear deformation have the same directions for the axes as the limiting conic section, but are not homothetic with respect to it. They have as characteristics p and θ instead of n and θ. Their flattening is less. That among them which has the same minor axis b as the limiting conic section has a major axis a' such that:-

$$a'^2 = ab$$

If D is the semi-diameter bisecting the axis of the conic section (18) passing through a point M, the equation for the maximum angular correction of a bearing is:-

$$\frac{lD}{2R^2(1+\mu)\sin 1''} \cdot \frac{1}{\sqrt{2}\sqrt{1+\sqrt{1-n^2}}}$$

The adoption of equation (18) for the limiting conic section has the effect of diminishing the value of the maximum angular correction for points which lie between the minor axis of the ellipses and the diameters bisecting the axes, but of augmenting it for points lying in the other sectors.

The maximum correction possible in one direction was:-

$$\frac{lb(1+n)}{4R^2(1+\mu)\sin 1''} = \frac{lb(1+p)\frac{1+p}{1+p^2}}{4R^2(1+\mu)\sin 1''}$$

now it will not be more than:-

$$\frac{lb(1+p)}{4R^2(1+\mu)\sin 1''}$$

The relation between the last value and the preceding is:-

$$\frac{1+p^2}{1+p} = \frac{a^2+b^2}{a(a+b)}$$

The maximum linear alteration which obtained previously over the whole limiting conic section and was equal to:-

$$\frac{a^2 b^2}{4R^2(a^2+b^2)}$$

will now reach a value at the extremity of the major axis of the limiting conic section equal to:-

$$\frac{a^2 b}{4R^2(a+b)}$$

The relation between this value and the preceeding is:-

$$\frac{a^2+b^2}{b(a+b)} = \frac{1+p^2}{1-p}$$

The most favourable case is that where $p = \sqrt{2}-1 = 0{,}41422$, $n = \frac{\sqrt{2}}{2}$ $\frac{a}{b} = \sqrt{2}+1$

The first relation is then $2(\sqrt{2}-1) = 0{,}82844$, the second is equal to 2. It is evident that but very slight advantage is obtained at the cost of considerable increase in the linear alteration and that it would be preferable to keep to the method of LABORDE. Besides it should be noted that in the case where the limiting conic section is a circumference or a system of two parallel straight lines, its homothetics are at the same time loci of equal linear alteration and of equal maximum angular correction; the two methods are therefore the same in the two cases.

e) To summarize : the principal value of c'' is easily obtained by carrying out the following operations. On a small scale chart (1 : 1,000,000 for instance) plot a series of ellipses homothetic to the limiting conic section, as in fig. 5. Then write above them the value of $\frac{c}{\sin 1''}$ and write the same values at the locus of their foci.

Then plot a second series of homothetic ellipses according to equation (18). It would be convenient to give them the same minor axes as the preceding ones; their major axes will have a value of $a' = \frac{a^2}{b}$; the value $\frac{(1-n)\,b}{4\,R^2 \sin 1''}$ is then inscribed on each curve of this series.

Then plot the point P_3 on this chart and note the altitudes inscribed on the two ellipses which pass through it (interpolating between the ellipses which are plotted). The exterior bisector of the straight lines joining P to the two foci marked with the altitude of the ellipse $\frac{c}{\sin 1''}$ will give the direction of the diameter of the small circle in fig. 7.

The altitude of the other ellipse, multiplied by the distance l will give the length of this diameter. The graphic procedure indicated in par. *a*) may then be readily applied to determine the angular correction to be applied to an angle $\widehat{PMP'}$.

EXAMPLE OF THE CALCULATION. — An example will be given to show that the calculations to be made in employing this type of projection in hydrography are much less complicated than would appear at first sight and that the trouble they give is amply compensated by the advantage derived from avoiding changes of origin.

Let the west coast of Italy be taken as an example and let a system be chosen which will permit all the hydrographic surveys from the frontier with France to the Gulf of Taranto to be plotted without change of origin. It would be possible to adopt a single system for the whole of Italy, but this, which has many advantages for the Geographical Survey, is inconvenient for a survey which is interested in the coasts only. In fact, such survey being always of the outer boundaries of the State would be situated in the region where the maximum deformations occur.

By choosing a special projection for the coasts, however, the coast line is often in a region where the corrections are very small.

For the west coast of Italy a limiting conic section may be adopted which results in two parallel straight lines. This is a type of projection which may often be adopted in hydrography and which introduces an appreciable simplification in calculation.

Let a fictitious point be taken as origin, of which the geographical position is :-

$$\varphi_0 = 41°\text{-}40'\ N.$$

$$L_0 = 12°\text{-}30'\ E.$$

The formulae (4) and (5) and table IV of Publication N° 21 give :-

$$\psi_0 = 41°\text{-}36'\text{-}46''.9387$$

$$\log \alpha = 0.00045721$$

In the formula (6) the value of the constant is found by replacing φ and ψ by φ_0 and ψ_0.

For this purpose, take, from Publication N° 21, the meridional part corresponding to φ_0 on the ellipsoid, multiply it by α and subtract the result from the meridional part for ψ_0 on the sphere. The difference is the constant expressed in minutes of arc. The calculation is rapid; it is well to work with the second differences in order to make certain of having the 5th decimal accurate :-

constante = 8'.18817

Taking :- $\theta = 133°,5 \qquad n = +1$

$\log A = 9.42009442 \qquad \log B = 9.39734441\ (-)$

$A = 0.263084 \qquad B = -0.2496574$

$\frac{1}{2} - A = 0.236916$

On straight parallel lines situated 100 kilometres on each side of the direction θ, the linear alteration μ will be equal to about 0.0001, *i. e.*, there will be an increase in length of 0.1m. for each kilometre. This will be reduced by half by employing the factor which was called $K_0 = 1 - \frac{\mu'}{2} = 0.99995$

$\log K_0 = 9.99997828$	$\log A = 9.42009442$	$\log B = 9.39734441\ (-)$
$\log R_0 = 6.80453845$	$\log R^2 = 13.60903346$	$\log R^2 = 13.60903346$
$\log R = 6.80451673$	$\log \frac{A}{R^2} = 5.81106096(-20)$	$\log \frac{B}{R^2} = 5.78831095 - (-20)$

Now calculate the coordinates X and Y of a point situated at the North-West limit of the area to be represented.

Monte Grammondo (*See :* Special Publication N° 24, page 78).

$$\begin{cases} \varphi = 43°\text{-}50'\text{-}29''.68\ N. \\ L = 7°\ 30'\ 38''.27\ E. \\ L - L_0 = -4°\text{-}59'\text{-}21''.73 \end{cases}$$

Formula (6) gives, using Table I of Meridional Parts of the ellipsoid for φ

2916'.59242	$\log L - L_0 = 4.25434817$
$\log = 3.46487574$	$\log \alpha = 0.00045721$
$\log \alpha = 0.00045721$	$\log \lambda = 4.25480538$
3.46533295	$\lambda = -4°\text{-}59'\text{-}40''.6496$
2919.66452	
const. = 8.18817	
2927.85269	(Table VI) $\psi = 43°\text{-}47'\text{-}03''.370$

By formulae (7) v and u are obtained :-

$$
\begin{array}{rl}
\log \cos\psi = & 9.85850726 \\
\log \sin\lambda = & \underline{8.93983007\ (-)} \\
\log \sin \frac{v}{R} = & 8.79833733\ (-) \\
\frac{v}{R} = & -3^\circ\ 36'\text{-}13''.2516 \\
\text{Table VI} & -216'.36356 \\
\log = & 2.33518412\ (-) \\
\log R = & 6.80451673 \\
\log \sin 1' = & \underline{6.46372611} \\
\log y = & 5.60342696\ (-) \\
y = & -401\ 261.^{m}007
\end{array}
\qquad
\begin{array}{rl}
\log \text{cotang}\ \psi = & 0.01843569 \\
\log \cos\lambda = & \underline{9.99834779} \\
\log \text{cotang}\left(\frac{u}{R}+\psi_0\right) = & 0.01678348 \\
\frac{u}{R}+\psi_0 = & 43^\circ\text{-}53'\text{-}35''.3990 \\
\psi_0 = & \underline{41-36-46.9387} \\
\frac{u}{R} = & 2-16-48.4603 \\
\log \frac{u}{R \sin 1''} = & 3.91426170 \\
\log R = & 6.80451673 \\
\log \sin 1'' = & \underline{4.68557487} \\
\log u = & 5.40435330 \\
u = x = & +253\ 719.^{m}182
\end{array}
$$

To calculate X and Y a slightly different form will be given to the terms to be added to x and y to that of the equations (9).

Let : $\quad \frac{B}{A} = \text{tang}\,\beta \qquad \frac{y}{x} = \text{tang}\,\gamma$

Then :

$$X = x + \frac{\sqrt{A^2+B^2}}{3R^2}(x^2+y^2)^{\frac{3}{2}}\cos(\beta+3\gamma)$$

$$Y = y + \frac{\sqrt{A^2+B^2}}{3R^2}(x^2+y^2)^{\frac{3}{2}}\sin(\beta+3\gamma)$$

In the case where the limiting section is a system of two straight lines, $A^2+B^2 = \frac{A}{2}$ and $\beta = 90^\circ - \theta$ consequently :-

$$
\begin{array}{llll}
\log y = 5.60342696(-) & 5.60342696(-) & \log A = 9.42009442 & \log R^4 = 27.21806692 \\
\log x = \underline{5.40435330} & \log \sin\gamma = \underline{9.92696560} & \log 18 R^4 = \underline{28.47333943} & \log 18 = \underline{1.25527251} \\
\log \text{tang}\,\gamma = 0.19907366(-) & \log\sqrt{x^2+y^2} = 5.67646136 & \log \frac{A^2+B^2}{9R^4} = 0.94675499(-30) & \log 18 R^4 = 28.47333943 \\
\gamma = -57^\circ 41' 40''.719 & \log(x^2+y^2)^{\frac{3}{2}} = 17.02938408 & & \\
3\gamma = -173^\circ\text{-}5'\text{-}2''.157 & \log\frac{\sqrt{A^2+B^2}}{3R^2} = \underline{0.47337749\ (-15)} & & \\
\beta = \underline{316-30} & 2.50276157 & & 2.50276157 \\
\beta+3\gamma = 143-24-57.843 & \log\cos(\beta+3\gamma) = \underline{9.90470724\ (-)} & \log\sin(\beta+3\gamma) = & \underline{9.77524605} \\
& 2.40746881\ (-) & & 2.27800762 \\
& -255.546 & & +189.674 \\
& x = \underline{+253\ 719.182} & & y = \underline{-401\ 261.007} \\
\text{Monte Grammondo} & X = +253\ 463.636 & & Y = -401\ 071.333
\end{array}
$$

This calculation is much facilitated by the use of the Tables of Meridional Parts to five decimal places for the ellipsoid and for a sphere. Where the whole triangulation, even that of the first order, is calculated on a plane, it is but very rarely necessary to make it. But when, as is more frequent, the hydrographic survey is based on the geographical positions of a triangulation of the first order, it need be made for stations of the first order only. However, as the meridians and parallels have to be drawn on the construction sheet, a similar calculation has to be made for the whole number values of latitude and longitude. It will be considerably simplified as accuracy to the nearest metre will be sufficient for the purposes of graphic construction.

The linear modulus and the angular correction will now be calculated for the same point, Monte Grammondo, by formula (17) :-

$$
\begin{array}{lll}
\log X = 5.40391566 & \log Y = 5.60322162\ (-) & \\
\log \sin\theta = \underline{9.86056221} & \log\cos\theta = \underline{9.83781220\ (-)} & \\
\qquad 5.26447787 & \qquad 5.44103382 & \\
\quad 183\,856.03 & \log \Delta = 4.96484043 & \\
\quad \underline{276\,079.28} & \log \Delta^2 = 9.93968085 & \log \sin 1'' = 4.68557487 \\
\Delta = 92\,223.25 & \log R^2 = \underline{13.60903346} & \log R^2 = 13.60903346 \\
 & \log \frac{\Delta^2}{R^2} = 6.33064739 & \log 2 = 0.30103000 \\
 & 1 + \frac{\Delta^2}{2R^2} = 1.000107058 & \log 1 + \frac{\Delta^2}{2R^2} = \underline{0.00004649} \\
 & \qquad \underline{-\ 0.000050005} & \qquad 8.59568482 \\
 & m_a = 1.000057053 & \log \Delta = \underline{4.96484043} \\
\frac{\Delta}{2R^2(1+\mu)\sin 1''} = & 0''.000233968 & \log \frac{\Delta}{2R^2(1+\mu)\sin 1''} = 6.36915561
\end{array}
$$

The linear modulus at Monte Grammondo shows an augmentation in the scale of 0.057 m. par kilometre. The maximum angular correction to a bearing will be 0.23'' for an observed point at 1 kilometre (when X_3 and Y_3 coincide with Monte Grammondo) ; 4.68'' for a point at 20 kilometres. It has been shown that this number is near the maximum correction which is to be applied to the angle of a triangle.

CALCULATION OF THE GEOGRAPHICAL POSITION. The meridians and parallels having been drawn on the construction sheet as described above, it will scarcely ever be necessary to obtain the geographic coordinates of the coordinates X and Y by calculation. Amply sufficient accuracy for cartographic purposes will be obtained generally by direct measurement of the geographic coordinates, on the construction sheet itself. Nevertheless, calculation will be necessary if *geodetic positions* and *astronomical positions* are to be compared. Besides, it is advantageous to obtain accurate geographical positions of first order triangulation stations and of a few remarkable points independently of any system of projection.

The following formulae and methods of calculation are therefore suggested for the transformation of coordinates X and Y into geographical positions :-

Let :- $$\frac{B}{A} = \operatorname{tang}\beta \qquad \frac{Y}{X} = \operatorname{tang}\gamma \qquad P_n = \left(\frac{A^2+B^2}{9R^4}\right)^{\frac{n}{2}} \left(X^2+Y^2\right)^{n+\frac{1}{2}}$$

$$\alpha_n = n\beta + (2n+1)\gamma$$

Then :- $$x = X - P_1\cos\alpha_1 + 3P_2\cos\alpha_2 - 12P_3\cos\alpha_3 + 55P_4\cos\alpha_4 - 273P_5\cos\alpha_5 + 1428P_6\cos\alpha_6 - \dots$$

$$y = Y - P_1\sin\alpha_1 + 3P_2\sin\alpha_2 - 12P_3\sin\alpha_3 + 55P_4\sin\alpha_4 - 273P_5\sin\alpha_5 + 1428P_6\sin\alpha_6 - \dots$$

It will be but rarely necessary to calculate more than 2 or 3 terms of these developments.

Here is the calculation for Monte Grammondo :-

$$
\begin{array}{lll@{\qquad}lll}
\log Y & = & 5.60322162\ (-) & & & 5.60322162\ (-) \\
\log X & = & \underline{5.40391566} & \log\sin\gamma & = & \underline{9.92703192} \\
\log \operatorname{tang}\gamma & = & 0.19930596\ (-) & \log\sqrt{X^2+Y^2} & = & 5.67618970 \\
\gamma & = & -57^\circ\text{-}42'\text{-}30''549 & \beta & = & 316^\circ\text{-}30'\text{-}0 \\
n=1\quad \log\left(\frac{A^2+B^2}{9R^4}\right)^{\frac{1}{2}} & = & 0.47337749\ (-15) & \alpha_1 = \beta + 3\gamma & = & 143^\circ\text{-}22'\text{-}28''353 \\
\log(X^2+Y^2)^{\frac{3}{2}} & = & \underline{17.02856910} & & & \\
\log P_1 & = & 2.50194659 & & & 2.50194659 \\
\log\cos\alpha_1 & = & \underline{9.90447344\ (-)} & \log\sin\alpha_1 & = & \underline{9.77566980} \\
 & & 2.40642003\ (-) & & & 2.27761638 \\
\text{correction} & & +254.9295 & & & -189.5031
\end{array}
$$

$$
\begin{array}{lll@{\qquad}lll}
n=2\quad \log\frac{A^2+B^2}{9R^4} & = & 0.94675498\ (-30) & 2\beta & = & 633^\circ\text{-}00\text{-}0 \\
\log(X^2+Y^2)^{\frac{5}{2}} & = & \underline{28.38094850} & 5\gamma & = & \underline{-288\text{-}32\text{-}32.745} \\
\log P_2 & = & 9.32770348 & \alpha_2 & = & 344\text{-}27\text{-}27.255 \\
 & & & & & 9.32770348 \\
\log\cos\alpha_2 & = & 9.98382120 & \log\sin\alpha_2 & = & 9.42705684\ (-) \\
\log 3 & = & \underline{0.47712125} & & & \underline{0.47712125} \\
 & & 9.78864593 & & & 9.23188157\ (-) \\
\text{correction} & & +0.6147 & & & -0.1706
\end{array}
$$

$$
\begin{array}{lll@{\qquad}lll}
n=3\quad \log\left(\frac{A^2+B^2}{9R^4}\right)^{\frac{3}{2}} & = & 1.4201\ (-45) & 3\beta & = & 589^\circ\text{-}30 \\
\log(X^2+Y^2)^{\frac{7}{2}} & = & \underline{39.7333} & 7\gamma & = & \underline{-403\text{-}57\text{-}33.843} \\
\log P_3 & = & 6.1535\ (-10) & \alpha_3 & = & 185\text{-}42\text{-}26.157 \\
 & & & & & 6.1535 \\
\log\cos\alpha_3 & = & 9.9978\ (-) & \log\sin\alpha_3 & = & 8.9976\ (-) \\
\log 12 & = & \underline{1.0792} & & & \underline{1.0792} \\
 & & 7.2305\ (-) & & & 6.2303\ (-) \\
\text{correction} & & +0.0017 & & & +0.0002 \\
 & & +0.6147 & & & -0.1706 \\
 & & +254.9295 & & & -189.5031 \\
 & & \underline{+253\ 463.636} & & & \underline{-401\ 071.333} \\
x & = & +253\ 719.182 & y & = & -401\ 261.007
\end{array}
$$

By means of formula (8) and Table N° VI of Special Publication N° 21 the coordinates x and y are converted into the quantities u and v.

$$
\begin{array}{lll@{\qquad}lll@{\qquad}lll}
\log R & = & 6.80451673 & & & 6.80451673 & \frac{y}{R} & = & -216'.36356 \\
\log\sin 1'' & = & \underline{4.68557487} & \log\sin 1' & = & \underline{6.46372611} & \frac{v}{R} & = & -3^\circ\text{-}36'\text{-}13''252\ (\text{Table VI}) \\
\log R\sin 1'' & = & 1.49009160 & \log R\sin 1' & = & 3.26824284 & \frac{x}{R} & = & 2^\circ\text{-}16'\text{-}48''460 \\
\log x & = & \underline{5.40435330} & \log y & = & \underline{5.60342696}\ (-) & \psi_0 & = & \underline{41\text{-}36\text{-}46.939} \\
\log\frac{x}{R\sin 1''} & = & 3.91426170 & \log\frac{y}{R\sin 1'} & = & 2.33518412\ (-) & \frac{x}{R}+\psi_0 & = & 43\text{-}53\text{-}35.399
\end{array}
$$

The quantities ψ and λ are given by the formulae :-

$$\sin \psi = \cos \frac{v}{R} \sin \left(\frac{x}{R} + \psi_0 \right)$$

$$\operatorname{cotang} \lambda = \operatorname{cotang} \frac{v}{R} \cos \left(\frac{x}{R} + \psi_0 \right)$$

$\log \cos \frac{v}{R}$	$= 9.99914042$	$\log \operatorname{tang} \frac{v}{R}$	$= 8.79919692 \quad (-)$
$\log \sin \left(\frac{x}{R} + \psi_0 \right)$	$= 9.84093115$	$\log \cos \left(\frac{x}{R} + \psi_0 \right)$	$= 9.85771463$
$\log \sin \psi$	$= 9.84007157$	$\log \operatorname{tang} \lambda$	$= 8.94148229 \quad (-)$
ψ	$= 43^\circ\text{-}47'\text{-}3''.370$	λ	$= -4^\circ\text{-}59'\text{-}40''.650$

Finally formulae (5) and (6) will give φ and L.

(Table VI)	$2927'.85270$	$\log \lambda$	$= 4.25480539 \quad (-)$
constante	$8'.18817$	$\log \alpha$	$= 0.00045721$
	$2919'.66453$	$\log (L - L_0)$	$= 4.25434818 \quad (-)$
$\log =$	3.46533295	$L - L_0$	$= -4^\circ\text{-}59'\text{-}21''.730$
$\log \alpha =$	0.00045721	L_0	$= 12^\circ\text{-}30'$
	3.46487574	L	$= 7^\circ\text{-}30'\text{-}38''.27$
	$2916'.59241$		
(Table I)	$\varphi = 43^\circ\text{-}50'\text{-}29''.68$		

THE ROUSSILHE PROJECTIONS. — Ingénieur Hydrographe en Chef ROUSSILHE eported to the First General Conference of the Geodetic and Geophysical Union, which met at Rome in 1922, a method of definition and calculation of rectangular stereographic coordinates which will extend the use of calculations on the plane of triangulations (*See :* Work of the Geodetic Section, Vol. I, published by the Secretary, G. PERRIER, 1923).

At the Second General Conference, at Madrid, 1924, he again took the question up and generalized its application (*See :* Vol. 4, 1927).

The projections advocated are projections which are symmetrical with respect to the initial meridian. They are distinguished among themselves by the development which is adopted to represent this meridian. This is carried out as a function of the lengths s of the arc of the initial meridian :-

$$X_0 = s + t_3 s^3 + t_4 s^4 + \cdots$$

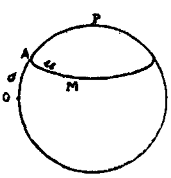

Fig. 9

The coefficients t are constants which characterize the projection. The position of a point M is defined by the lengths $OA = s$, measured on the meridian, and $AM = u$ measured on the parallel (*See :* Fig. 9).

A priori, the equation for the rectangular coordinates may be written in the following manner :-

$$X = X_0 + \lambda_2 u^2 + \lambda_4 u^4 + \lambda_6 u^6 + \cdots$$
$$Y = \lambda_1 u + \lambda_3 u^3 + \lambda_5 u^5 + \cdots$$

the coefficients λ as well as X_0 being functions of s developed in accordance with the powers of this variable.

These functions are determined by the condition that the projection should be conformal, *i. e.* that:

$$\frac{\delta X}{\delta u} + \frac{\delta Y}{\delta s} = u \frac{\tan\varphi}{N} \frac{\delta Y}{\delta u}$$
$$\frac{\delta Y}{\delta u} - \frac{\delta X}{\delta s} = - u \frac{\tan\varphi}{N} \frac{\delta X}{\delta u}$$

These two equations, in which $\frac{\tan\varphi}{N}$ also must be developed in accordance with the powers of s, give, by identification, the values of the coefficients of the functions λ.

The calculation, which is more lengthy than difficult, will not be developed here.

The projection thus defined is conformal to whatever infinitely small limits are desired. It will suffice to carry out the development to a sufficient extent, but in practice it is unnecessary to go beyond the 7th order.

Let :- $$\frac{\tan\varphi}{N} = \frac{\tan\varphi_0}{N_0} + k_1 s + k_2 s^2 + \cdots$$

when $$k_1 = \frac{1 + \tan^2\varphi_0 - e^2 \sin^2\varphi_0}{\rho_0 N_0} \qquad k_2 = \tan\varphi_0 \frac{1 + \tan^2\varphi_0}{\rho_0^2 N_0}$$

Then :- $$\lambda_1 = \frac{\delta X_0}{\delta s} = 1 + 3t_3 s^2 + 4t_4 s^4$$
$$\lambda_2 = \frac{\tan\varphi_0}{2N_0} + \left(\frac{k_1}{2} - 3t_3\right) s + \cdots$$
$$\lambda_3 = \frac{k_1}{6} - \frac{\tan^2\varphi_0}{3N_0} - t_3 + \cdots$$

Consequently, by keeping to the terms of the 3rd order, the expressions for X and Y become :-

$$X = s + \frac{\tan\varphi_0}{2N_0} u^2 + t_3 s^3 + \left(\frac{k_1}{2} - 3t_3\right) u^2 s + \cdots$$
$$Y = u + 3t_3 u s^2 + \left(\frac{k_1}{6} - \frac{\tan^2\varphi_0}{3N_0^2} - t_3\right) u^3 + \cdots$$

The first terms of the linear modulus developed as functions of X and Y become :-

$$m = 1 + 3t_3 X^2 + \left(\frac{k_1}{2} - \frac{\tan^2\varphi_0}{2N_0^2} - 3t_3\right) Y^2 + \cdots$$

The conic section of equal linear alteration is therefore represented by the equation :-

$$3t_3 X^2 + \left(\frac{K_1}{2} - \frac{\tan^2\varphi_0}{2N_0^2} - 3t_3\right) Y^2 = \mu$$

or

$$3t_3 X^2 + \left(\frac{1}{2\rho_0 N_0} - 3t_3\right) Y^2 = \mu$$

It is an ellipse of the axis which lies along the meridian. It would be advantageous to chose such a t_3 that μ will have the smallest value possible at the periphery of the area to be represented but the axes of the ellipse cannot be inclined as in the projections of TISSOT and LABORDE and thus obtain the minimum linear alteration possible.

A projection analogous to that of GAUSS (*b*) will be obtained if $t_3 = o$; a projection analogous to the conical projection of LAMBERT (*c*) if :

$$t_3 = \frac{1}{6\rho_0 N_0}$$

(neglecting the terms e^4)

We have a projection of the stereographic type if :

$$t_3 = \frac{1}{12\rho_0 N_0}$$

It is easy to diminish the value of μ by half at the periphery by employing the coefficient K_0 as for the projection of LABORDE.

If we continue the development of the expression of m as far as the terms of the 3rd order inclusive, these terms may be expressed :-

$$+ X\left[4t_4 X^2 + \left(K_2 - K_1\frac{\tan\varphi_0}{N_0} - 12t_4\right) Y^2\right]$$

In this expression :-

$$K_2 - K_1\frac{\tan\varphi_0}{N_0} = e^2 \tan\varphi_0 \, \frac{1 + \sin^2\varphi_0}{\rho_0^2 N_0}$$

It can be seen that, if a value of zero be given to t_4 the term of the 3rd order of the development of m is reduced at least to the 4th order because it contains e^2 as a factor.

Therefore there does not seem to be any advantage in continuing the development of X by the terms of s^5 or s^6. M. ROUSSILHE proposes that this be done to approach as closely as it is desired to the geometrical definition of the GAUSS projection, of the LAMBERT conical projection or the stereographic projection.

The terms of the 3rd order of the development of m might be still further reduced by determining t_4 so that the conic section

$$4t_4 X^2 + \left(e^2 \tan\varphi_0 \frac{1 + \sin^2\varphi_0}{\rho_0^2 N_0} - 12t_4\right) Y^2 = \text{constante}$$

may be similar to the conic sections of equal values of μ.

To do this it is necessary to give t_4 the value

$$t_4 = \frac{3}{2} e^2 \operatorname{tang} \varphi_0 \frac{1+\sin^2\varphi_0}{\rho_0^2 N_0} \cdot \frac{1}{12 + \frac{1}{t_3 \rho_0 N_0}}$$

the terms of the 3rd order may then be expressed :-

$$2 e^2 \operatorname{tang} \varphi_0 \frac{1+\sin^2\varphi_0}{\rho_0^2 N_0} \cdot \frac{X}{12 t_3 + \frac{1}{\rho_0 N_0}} \left[3 t_3 X^2 + \left(\frac{1}{2 \rho_0 N_0} - 3 t_3 \right) Y^2 \right]$$

It is reduced to the 4th or 5th order on account of the presence of the factor e^2, as in the case where t_4 is taken as 0 and in the case of the LABORDE projection.

The calculation of the angular corrections may be carried out exactly as was shown for the LABORDE projection.

This system of projection may be employed up to very great distances from the origin if care is taken to continue the development of $\frac{\operatorname{tang} L}{N}$ as far as required by the exigences of the triangulation to be calculated, taking into consideration the powers of e^2 when necessary, and employing the number of quantities $\lambda_1, \lambda_2, \lambda_3$ etc. which are necessary, each developed as far as is required to ensure the conformity of the projection to the accuracy desired.

If it were desired to employ it for the calculation of triangulations of the first order, this method will certainly lead to longer calculations than with the LABORDE method. It is more suitable for those triangulations in which the accuracy requires but few terms of the development and on the condition that the area to be represented is nearly symmetrical with respect to the meridian.

1 st December 1929.