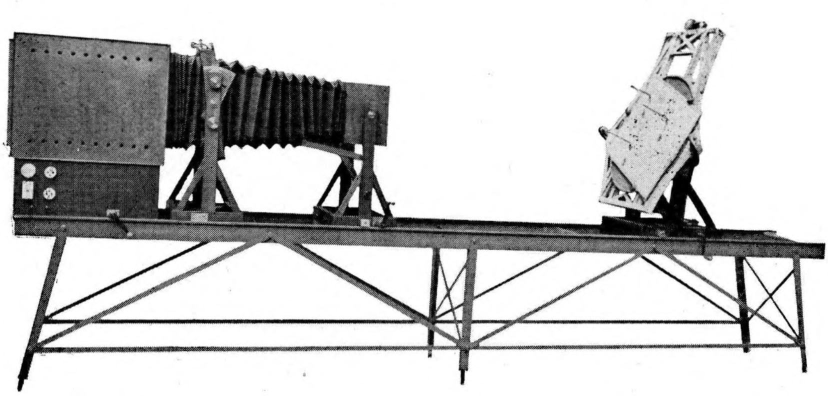

Roussilhe
Apparatus

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NOTE ON THE SOLUTION OF THE POTHENOT PROBLEM IN SPACE

(APPLICATION TO THE RESTITUTION OF AERIAL PHOTOGRAPHS)

by

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In practical geodesy, the French method, called the "approximate point method" (or "des Ingénieurs Hydrographes"), is characterized by the following procedure :—

1° First, with the aid of graphical construction and appropriate instruments, we determine an *approximate position* of the point to be fixed.

2° Then we compute, with relation to the approximate position, the determining factors (distance and direction) of the various *geometric loci* which serve for the determination of the point (intersecting sighting lines and tangents to the arc of the segments containing the angles).

3° Finally a *large scale diagram* is prepared giving the exact location of the geometric loci as determined by observations and a *mean point* is chosen by eye — or else by means of a graphical construction corresponding to the method of least squares (barycentre of perpendiculars to each locus), or finally, by calculation (a procedure which is rather long and which, generally, does not improve the result obtained by the graphic method).

I shall show that we can follow exactly the same procedure in fixing the point S from which an aerial photograph is taken : it is an analogy in space to the Pothenot problem (three-point problem) in a plane.

Geometrically, the locus of two control marks of *restitution* correspond to a torus, generated by the segment of the circle containing the angle determined by the focus F of the photographic objective and the two images of the control marks on the negative.

If three points of control are available it is necessary in fixing the point S to construct or calculate the points common to three toruses.

By a method analogous to that for determining a line of position in astronomical navigation, we determine first (to about the 1st order) an *approximate position* S_a in space ; then for each of the toruses, an *approximation point* located exactly on the surface ; for which point the plane tangent to the torus under consideration may easily be defined.

The three similar tangent planes therefore form a triangular pyramid of which the apex (with an approximation of about the 2nd order) is the point S which is sought.

In order to obtain the resection by means of an easy geometric construction, we intersect the pyramid with two parallel horizontal planes close to each other. The triangles are then homothetic and the *centre of homotheticity* is the point sought.

This graphic method is rapid and simple and moreover, it permits us to use more than threee toruses since generally there are at least four restitution control marks.

When the degree of approximation of the result is insufficient the data already calculated permit a second rapid approximation to be made.

In fact, in the problem of restitution of aerial photographs there are *three unknown quantities*, the plane coordinates x, y, and the altitude z of the optical centre S of the objective. This complication, however, is not new and existed previously in the well-known topographic problem : to *fix the point by bearings and orient the station.* In this case there are in fact three unknown quantities ; the coordinates of the point of observation and the error in the initial orientation.

However, the solution follows rapidly from a consideration of the two homothetic figures and the position of the containing segments. This similarity of the two questions has led me to believe for some time that the three-point problem in space might be solved in the same manner and that, in consequence, as far as the solution by calculation is concerned, it might be reduced to a simple problem in a plane.

Accordingly I shall analyse the special conditions for the determination of a *point of approximation,* the formulae for calculating the *geometric loci,* the construction of the *compensating diagram,* permitting the best employment of an abundant number of measurements of which the best may be selected and the *accuracy of the result* obtained.

I. *APPROXIMATE SOLUTION.*

Given an aerial photograph P, taken with an objective of focal length f, and assumed to give an exact geometric perspective of the conformal image of the plane of the ground (C on the scale E ; See Fig. 1), the restitution consists in determining a plane R_1 and the point of view S_1, such that the perspective of P on the plane R_1 shall be identical with the original figure C.

The problem admits of an infinite number of solutions ; the planes R_1 are various positions of the plane C when turned about the straight line D :— the intersection of P and C. The view point S_1 describes a circle with a centre h and radius hS :— h being the limiting straight line of plane P (*horizon* of S if the plane C itself is horizontal).

I have shown (*) how the problem may be solved by placing a photographic objective, suitably corrected to act as a thin lens at S_1. The essential conditions are :—

(*) Comptes-rendus de l'Académie des Sciences, Paris, 17th March, 1922.

a) The principal plane of the objective must pass through the straight line *D*, the principal optical axis is therefore *tilted* through the angle μ, with respect to the *axis of perspective* $\omega\ S_1\ O_1$ defined by the centre of the plate ω and the optical centre of the objective S_1.

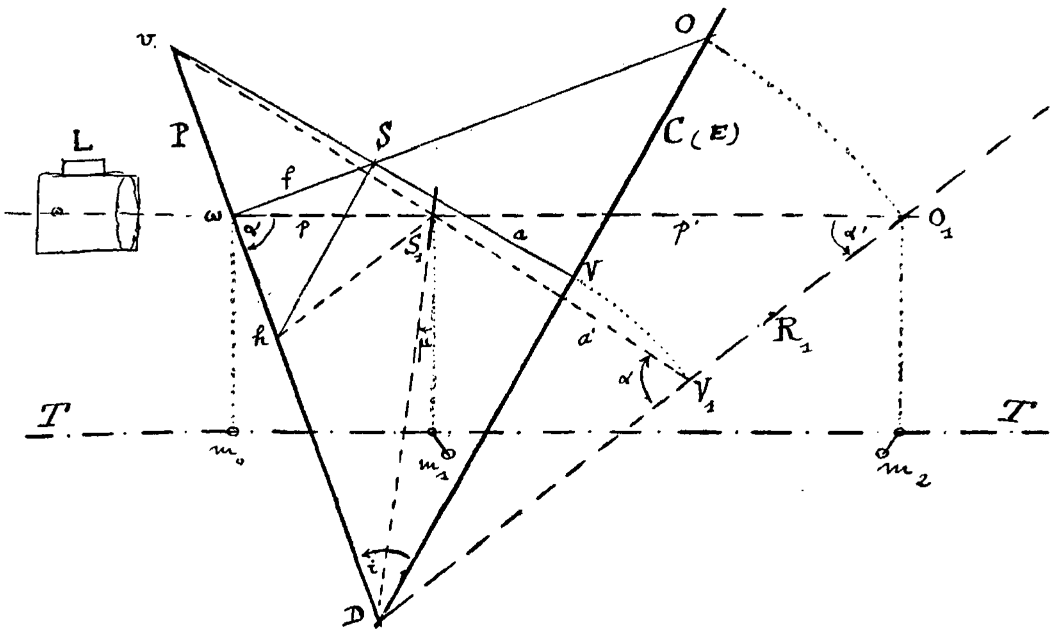


Fig. 1.

b) The useful *aperture* of the objective is reduced to half (about f/11 for the usual 0.50 m. apparatus) in such a manner as to increase the depth of the field without increasing the curvature (image absolutely sharp).

c) The objective gives *magnification* *K* defined by

$$1.5 < K < 3$$

in such a manner as to preserve the sharpness of the enlarged images ($K < 3$) and to obtain a constant value for the focal length φ which is given by $\frac{1}{p} + \frac{1}{p'} = \frac{1}{\varphi}$ and to be able to neglect the interval between the nodal points in view of the depth of the field ($K > 1.5$).

Under these conditions there are still an infinite number of optical solutions characterized this time by the focal length φ of the transforming objective.

An easy solution which permits the employment of series objectives and gives an adequate correction for the residual deformation by *distortion*, consists of taking $\varphi = f$, thus using for the transformation the same objective employed in taking the photograph.

Simple formulae may be employed for calculating all of the data for restitution, given :—

f = the focal length of the objective used for taking the view.
φ = the focal length of the objective used for restitution.
i = the inclination of the negative P to the plane C.
E = scale of construction.
A = altitude of the position S.
V = foot of the vertical from S.
γ = orientation of the horizontal lines of the negative.

In practice, however, we know only the focal lengths f and φ, and the centre of the plate.

The other conditions are unknown and *must be determined*: that constitutes actually the three-point problem in space.

These unknown conditions are replaced by three *controls of restitution* and the settings of the transformation apparatus as diagrammed in Fig. 1, are adjusted in such a manner that the perspective images of the controls on the negative coincide on the plane R_1 with the geodetic positions of the control points in this plane.

The angles α and α' of the planes P and R_1 with the axis of the perspective ωS_1 are conjugated as well as the lengths p and p'. Certain *diagrams* make it possible to obtain the correct position for establishing coincidence by *successive approximations*, the entire difficulty being the determination of the *orientation* of the negative in its plane. (The horizontal lines of the negative must always be perpendicular in space to the plane of Figure 1).

Finally, we readily obtain :—

a) a *transformation* of the negative — photographic proof on a solid back — identical with the initial figure C if the terrain is horizontal, *i. e.* if the images on the negative are not deformed as a result of *relief*.

b) the length OV defining the foot V of the perpendicular from S, the altitude of flight A, the inclination i of the negative and the orientation θ of the line OV; in other words all the initial conditions defining the position of the negative in space.

Special diagrams in connection with the apparatus permit the solving of the problem in less than half an hour with a degree of accuracy which will be discussed later.

II. *RESTITUTION OF A THREE DIMENSIONAL SPACE.*

The factors A and V are *indispensable* for the correct restitution of the original negative when the control points of restitution M are located at different altitudes and if, as is usually the case, the terrain is broken.

In fact, in order to obtain the correct perspective in the plane R_1, it is necessary to replace the geodetic positions M_0 of the control points in space by their conical perspective M_1 on the plane R_1; the point S_1 being the centre of the perspective. (*See* Figure 2).

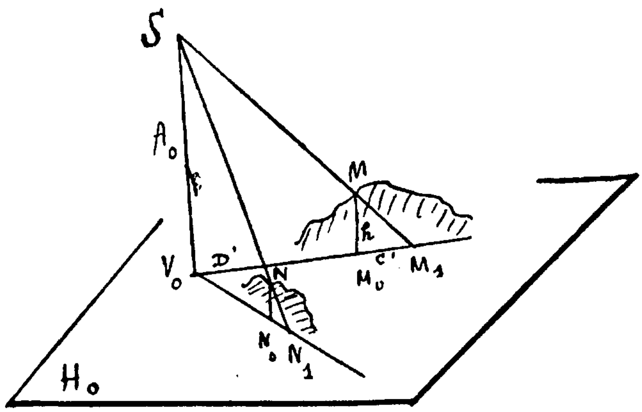


Fig. 2

Calling h the height of the point M above a given horizontal plane of altitude H_0 (that of the control at the lowest altitude).

A_0 the approximate altitude of S deduced from the primary tentative transformation;

D the distance $V_0 M_0$:—

the *inverse correction* for altitude $c' = \overline{M_0 M_1}$ is given by

$$c' = h \frac{D'}{A_0 - h}$$

This is laid off on the vector $V_0 M_0$ in the direction $V_0 M_0$ if $h > o$. Conversely, to every point N_1 of the perspective transformation corresponds a point N_0 — correctly restituted — obtained by applying the *direct correction for altitude* c to the point N_1 :

$$c = h' \frac{D}{A_0}$$

h' being the height of the point N, D being the distance $V_0 N_1$. This time the correction c is applied in the direction $N_1 V_0$ if $h > o$.

The proper transformation of the negative necessitates therefore the knowledge of the *coordinates* and the *altitudes* of the restitution controls, as well as the knowledge of the *coordinates and of the altitude of the point S*.

As for the *restitution* proper of the negative, two cases are possible :—

a) if the altitudes h of the detailed positions are known, the transformation is distorted by the application of the corrections c to the points of this transformation.

b) if the shape of the relief is unknown, we then combine, in at least two of the transformations covering the same zone, the vectors $V_0 N_1$, $V'_0 N'_1$, etc., which are none other than the azimuthal traces of the vertical planes SVN, $SV'N$, etc. Otherwise stated, and precisely as in ordinary topography, the position of *station* S is determined by the solution of the *3 point problem* (bearings to known points) — the *planimetry* of the terrain is constructed by the method of *intersections*; the *height* sbeing given by calculation :

$$h' = c \frac{A_0}{D} = c' \cdot \frac{A'_0}{D} \ldots\ldots$$

Finally, the complete restitution is effected in two stages :—

1° the *transformation* proper or, "automatic perspective".

2° the corrections for altitude altering the so-called transformation in the proper "restitution" (a work of calculation and draughting).

It is possible however to render the second stage of the restitution equally *automatic* in so far as pertains to the points of control and, consequently, to simplify the correct work of tranformation for broken terrain.

Let us return to diagram of Figure I.

The initial data of the negative being supposedly known to start with, the length and angles which correspond to the geometric transformation are, taking $a = AE$:—

$$\left\{ \begin{aligned} p &= \varphi \frac{a + f \cos i}{a} \\ p' &= \varphi \frac{a + f \cos i}{f \cos i} \\ \cos \alpha &= \frac{p^2 - f^2}{2 p f} \tan g\, i \\ \cos \alpha' &= \frac{p^2 + f^2}{2 p f} \sin i \end{aligned} \right.$$

Conversely, if we assume the horizontal lines of the negative to be conveniently oriented and if, after the controls and their images have been brought into coincidence, we measure, p, p', α and α'

$$\left\{ \begin{aligned} \cos i &= \frac{\sin \alpha}{\sin \alpha} \\ A &= \frac{p' - \varphi}{\varphi} \frac{f \cos i}{E} \\ O V &= \frac{p' - \varphi}{\varphi} \frac{f \sin i}{E} \end{aligned} \right.$$

These *inverse equations,* translated into diagrams, permit the rapid determination of the unknown quantities, i, A and OV, or at least their approximate values.

Solving the triangle $S_1 V_1 O_1$, we have :—

$$\left\{ \begin{aligned} V_1 &= \pi - \alpha \\ V_1 S_1 &= a' = a \frac{p}{f} \end{aligned} \right.$$

These *new properties* of the perspectives may then be translated in the following manner :—

"When the plane of the perspective R_1 revolves about the axis of homotheticity D the quadrilateral ω S_1 V_1 D is distorted in such a manner that the angles ω and V_1 remain unchanged and that the ratio $\frac{S_1 V_1}{S_1 \omega}$ remains constant".

Consequently, in the correct restitution of a three-dimensional space — defined as a plan in relief of which the altitudes are normally computed from the plane C of the initial chart — all occurs as though the restituted relief were both *canted* and *amplified.*

Assume then (*See* Figure 3) that N_0 N_1 be the transformation of the points M_0 M_1 and N be the conical perspective of N_1, such that N_0 N be parallel to V_1 S_1.

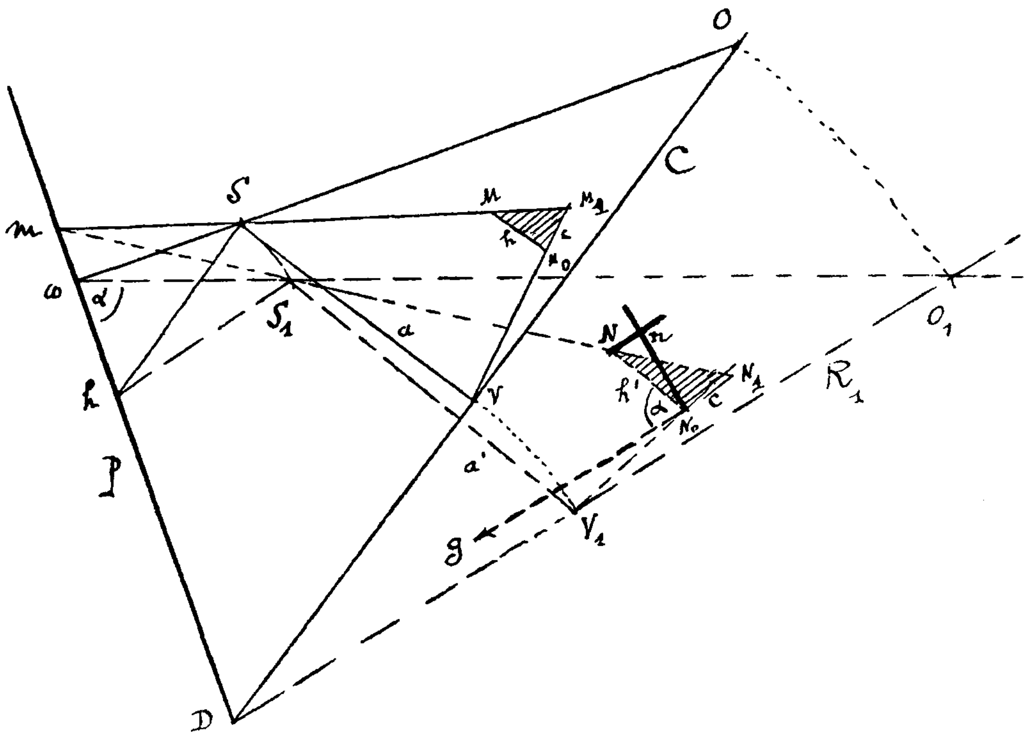


Fig. 3

The line N_0 N makes with the plane R_1 (therefore with its line of maximum inclination $N_0\,g$ parallel to $V_1\,D$) an angle equal to α, which is constant for all points of the same transformation.

The transformed altitude $N_0\,N = h'$ is given by

$$\frac{h'}{a'} = \frac{c}{V_1\,N_1} = \frac{c}{VM_0} = \frac{h}{a}$$

Therefore

$$h' = h\,\frac{a'}{a} = h\,\frac{p}{f}$$

The point N is the *correct representation* in the *restituted space* of the point M in the initial space.

It is easy actually to achieve this representation. In the plane $NN_0 g$, the normal and tangential components to the transformed altitude h' are

$$\begin{cases} x = n\,N = h \dfrac{p}{f} \cos \alpha \\ y = N_0\, n = h \dfrac{p}{f} \sin \alpha \end{cases}$$

These may be calculated with the aid of a simple diagram and with an accuracy which is adequate when the relative altitudes h are known and sufficiently close values of p and α : the latter may be obtained, for instance, by making a primary approximation to the transformation, neglecting the altitudes h at the start.

Let us conceive a *pylon* (See Fig. 3 and 4) comprising a *centre* located at the geodetic point N_0, one *arm* $N_0 n$ perpendicular to the plane R_1 and graduated in millimetres, a *tangential* nN also graduated in millimetres, placed at a distance y from the plane R_1, by means of a slide.

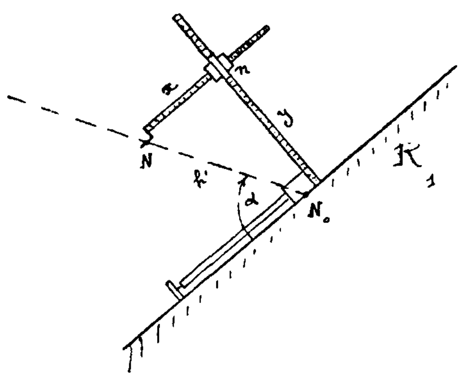


Fig. 4

At the extremity of the tangential arm an *index* N, placed with the aid of a second slide at a distance x from the perpendicular arm.

Finally a lever with a roller, fixed to the centre N_0, is weighted in such a manner that it is automatically maintained in the direction $N_0 g$ of the maximum inclination to the plane R_1 (the axis of perspective of the transformation apparatus ωS_1 is placed horizontally).

When making the methodical search for the correct position of transformation, the focus is first adjusted at the centre and the approximate values of p and p' are deduced, knowing f and φ on one hand and the approximate values of A and OV, on the other hand.

Then the planes P and R_1 are securely adjusted to their conjugate positions with the aid of the diagram

$$\cos i = \frac{\sin \alpha'}{\sin \alpha}$$

(the angle i being approximately known).

The subsequent setting and repeated trials are for the purpose of *methodically* determining the optimum orientation of the negative in its plane *i. e.* turning the negative in its own plane (by a device especially provided in the ROUSSILHE apparatus), and correlatively causing the system of the controls to be turned in the plane R_1 — then following the deformations of the image projected on the plane R_1.

If all of the restitution control marks (except that at the lowest level) are provided with *altitude pylons* — set according to the provisional values of ρ and α — the positions N', N''...... materialized by the indexes of these pylons will then automatically represent the restituted space, and it will suffice to bring into coincidence the optical images of the controls with the point M on the one hand and the indices N', N''...... on the other.

These settings are made in a methodical manner by varying independently the angle α' (and therefore α) and the orientation γ of the negative in its plane; and in each case bringing about exact coincidence of the image of the 2 controls with the two corresponding indices. We then obtain, for the image of the control M (without pylon) the *characteristic curves* between which it will suffice to interpolate in order to determine the manner in which the apparatus should be wedged up in its correct position.

With the aid of these contrivances we may obtain a very close approximation of the position S in about *one hour ;* after which a second approximation of the transformation is made either by readjusting the settings of the pylons, or by working without them and by making the inverse corrections to altitude on the three controls (by calculation and graphically).

Finally, we may obtain in less than *two hours* a *photographic proof* of the transformation and the *initial conditions* of the resection, to the following approximations (experimental results for the negatives of about $\frac{1}{4,800}$ taken at 2,400 metres with an objective of 0.50 metres and restituted at $\frac{1}{2,000}$) :

$$\left.\begin{array}{r} 15 \text{ to } 20' \text{ for the inclination } i \\ 3 \text{ to } 7^{m} \text{ for the position } V \\ 2 \text{ to } 3^{m} \text{ for the altitude } A \end{array}\right\} \text{variables dependent on inclination } i.$$

NOTE : *The graphic approximations :*

$$1.5 \text{ m/m to } 3.0 \text{ m/m for } V,$$
$$1.0 \text{ m/m to } 1.5 \text{ m/m for } A,$$

remain very nearly constant regardless of the scale of the negatives provided one works with the same coefficient of magnification ($K = 2.4$ about).

The interest displayed in the completed mechanism for the restitution in relief shows the utility of the contrivances employed in the construction of the transformation apparatus of the ROUSSILHE type. The axis of perspective ω S_1 O_1 is placed *horizontally* and there results, for negatives inclined to the horizontal up to 30° and more, values of α' comprised between 90° and 56°.

Under these conditions the altitude columns (pylons) function to the best advantage possible.

The restitution apparatus recently constructed in GERMANY on similar principles has the axis of perspective placed *vertically*. This arrangement does not allow the automatic restitution by means of the columns.

For the rest, effort has been made in the various restitution apparatus to keep the plane R_1 rigorously horizontal with the idea of producing a graphical mechanism for automatically tracing the plan. In that case it is necessary to make the combined *negative-objective* system movable (which involves complications) and to adjust the focal length of the objective to preserve the sharpness of the transformed image — the geometric position of the negative remaining exactly the same as that occupied in the initial space. It is not certain, in such cases, if the employment of special lenses to adjust the focal length might not result in a material alteration in the functioning of the entire optical system.

III. *ELIMINATION OF UNSUITABLE SOLUTIONS.*

The problem of the restitution on three control marks leads geometrically to a known problem, namely,

"to fit a triangle, the lengths of whose sides are given, on a given trihedral pyramid."

The solution is obtained directly by developping the trihedral *SABC* along one of its faces (*See* Fig. 5) and constructing a *curve of errors.*

Sa and *Sa'* being the developments of the edge *SA*, we lay off on these two vectors two lengths equal to *S7* and *S7'*, for example :—

With the point 7' as a centre, describe a circle of radius *CA*. We then obtain on *Sc* two points 7_1 and 7_2. With these two points as centres, describe two circles of radius *BC*. The circle with centre 7 (on *Sa*) intersects the two circles 7_1 and 7_2 in 4 points : *l, m. n, p.*

The curve thus drawn is a curve of the 8th degree, but it is also symmetrical with respect to *S* ; it intersects therefore the vector *Sb* in 4 points (1, 2, 3, 4) (which may be real as is the case in Fig. 5).

These are the solutions of the *direct transformation* ; the four symmetrical solutions with respect to *S* will give an *inverted* triangle *ABC*.

M. DE VANSSAY, Ingénieur Hydrographe Général (*) has shown that the problem may be solved algebraically by calculating the roots of the equation of the 4th degree.

Finally when we transform the negative on three control points only we are given the choice of 4 possible solutions.

If, however, we know a priori one of the edges *SA, SB, SC* (the greatest one for example), the ambiguity is then reduced to 2 *possible solutions,* defined by the classification of the lengths *SA* and *SB* :

$$SC > SA > SB$$
$$SC > SB > SA$$

(*) *Hydrographic Review*, Vol. IV, N° 2.

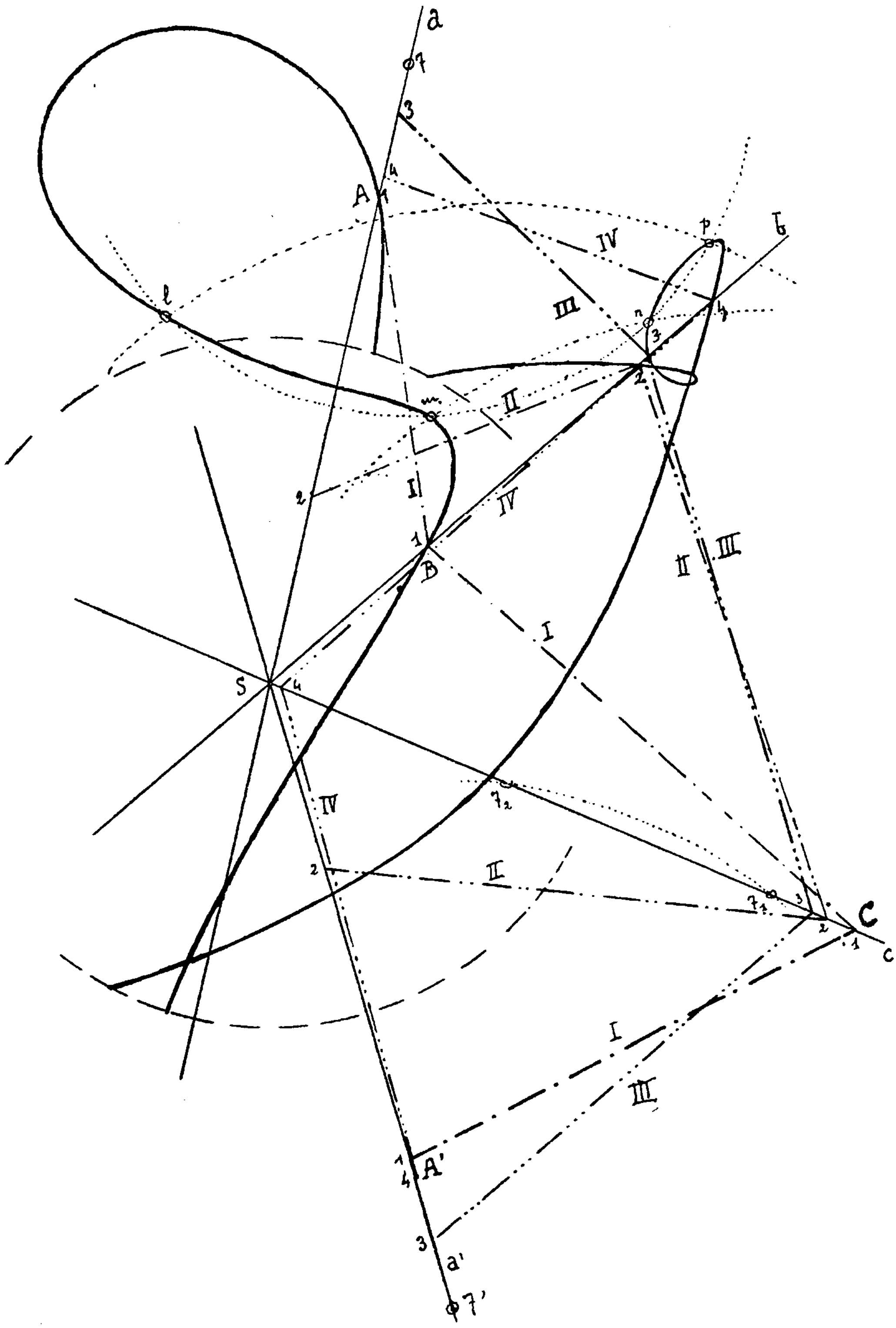


Fig. 5

(The greatest of these edges will be determined, for example, if we know approximately the relative positions of the controls ABC and of the point V, which is the foot of the perpendicular from S).

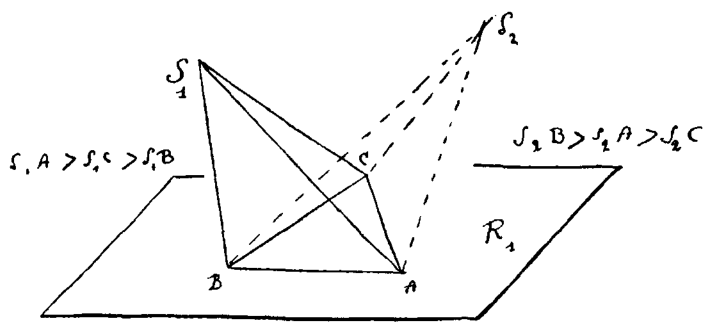


Fig. 6

Consequently, if we know, for example after a rapid transformation with the ROUSSILHE apparatus, the order of classification of the lengths SA, SB, SC, we may directly eliminate the three unsuitable solutions.

If we know definitely only the greatest (or the least) of the lengths SA, SB, SC, then two solutions are possible. The ambiguity is removed by using *a fourth point of control D* — the admissible solution then being the only one which will be common to the two transformations relating to the groups ABC, ABD, for example.

The transformation of the negative being effected for instance on the base ABC (SA being the longest side), of two possibilities, we have :—

1° The 4th control will be found in exact coincidence — in which case the solution is good.

2° The 4th coincidence is not realised : in which case the negative is displaced in its plane, and correlatively the plan in the plane R_1, in such a manner as to permute the order of magnitude of the sides SB, SC. We then recommence the methodical search for bringing about coincidence.

IV. *THE CALCULATION OF THE RESECTION* S *IN SPACE.*

I shall follow the identical procedure given at the beginning of this article.

a) OBSERVED ANGLE.

Knowing the coordinates x_i, y_i and x_k, y_k of two controls m_i and m_k of the negative, with relation to the two diametrical axes of the plate (*see* Figure 7), we calculate the angle S_{ik} defined by the optical centre S of the objective.

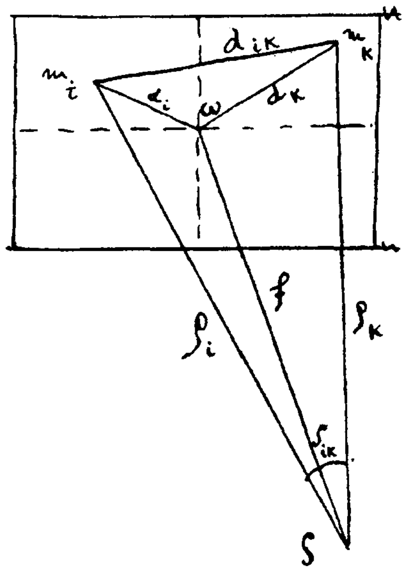


Fig. 7

$$\frac{x_i}{\cos \varphi_i} = \frac{y_i}{\sin \varphi_i} = d_i$$

$$\frac{f}{\cos \psi_i} = \frac{d_i}{\sin \psi_i} = \rho_i$$

$$\frac{x_k - x_i}{\cos v_{ik}} = \frac{y_k - y_i}{\sin v_{ik}} = d_{ik}$$

$$2p_{ik} = \rho_i + \rho_k + d_{ik}$$

$$\tan g \frac{1}{2} S_{ik} = \sqrt{\frac{(p_{ik} - \rho_i)\ (p_{ik} - \rho_k)}{p_{ik}\ (p_{ik} - d_{ik})}}$$

The calculation does not present any special difificulty; but it cannot be avoided without using a more or less complicated instrument for the direct measurement of the angles S_{ik}.

b) CALCULATED ANGLE (*See* Figure 8).

We may calculate equally the angle $M_i S_a M_k$, defined by the ground controls $M_i M_k$, and by the approximation S_a of the position S, the point of resection in space. This position is given, for example, by the transformation apparatus.

In order to facilitate the calculation, we shall first make the *corrections for altitude.*

Let XYh_i represent the coordinates of M_i
and $X_i Y_i O$ » » » of N_i the conical perspective of M_i seen from S_a and on the plane $z = o$.

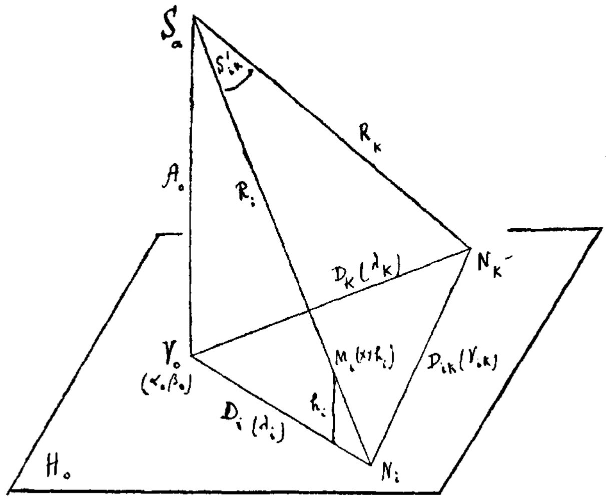


Fig. 8

The equations of the straight line $S_a M_i N_i$ give :—

$$X_i - X = (X - \alpha_o) \frac{h_i}{A_o - h_i}$$

$$Y_i - Y = (Y - \beta_o) \frac{h_i}{A_o - h_i}$$

This accomplished, we compute successively :—

$$\left\{ \begin{array}{l} \dfrac{X_i - \alpha_o}{\cos \lambda_i} = \dfrac{Y_i - \beta_o}{\sin \lambda_i} = D_i \\ \dfrac{D_i}{\cos \mu_i} = \dfrac{A_o}{\sin \mu_i} = R_i \\ \dfrac{X_k - X_i}{\cos V_{ik}} = \dfrac{Y_k - Y_i}{\sin V_{ik}} = D_{ik} \\ 2\ P_{ik} = R_i + R_k + D_{ik} \\ \tan g \dfrac{1}{2} S'_{ik} = \sqrt{\dfrac{(P_{ik} - R_i)(P_{ik} - R_k)}{P_{ik}(P_{ik} - D_{ik})}} \end{array} \right.$$

Whence the difference (with its sign and expressed in *seconds*)

$$d\ S_{ik} = S_{ik} - S'_{ik}$$

c) ARC OF SEGMENT CONTAINING THE ANGLE.

In the plane of the face $S_a N_i N_k$ (which is only approximately the face SN_iN_k of the photographic polyhedral), we know two points on the *true* arc of the segment containing the angle S_{ik} described about N_iN_k as a base: these are the intersections of the arc with the edges S_aN and S_aN_k.

In fact, if $d\,S_{ik}$ is small (*See* Figure 9) we have :—

$$\frac{S_a\ T_i}{d\,S_{ik}\,sin\ 1''} = \frac{R_k}{sin\ S_{ik}}$$

Let X, Y, Z be the coordinates of the point T_i. We have :—

$$\frac{X-\alpha_o}{X_i-\alpha_o} = \frac{Y-\beta_o}{Y_i-\beta_o} = \frac{Z-A_o}{-A_o} = \frac{S_a\ T_i}{R} = \frac{R_k}{R_i}\ \frac{d\,S_{ik}\,sin\ 1'}{sin\ S_{ik}}$$

From which we obtain for the coordinates of the point T :

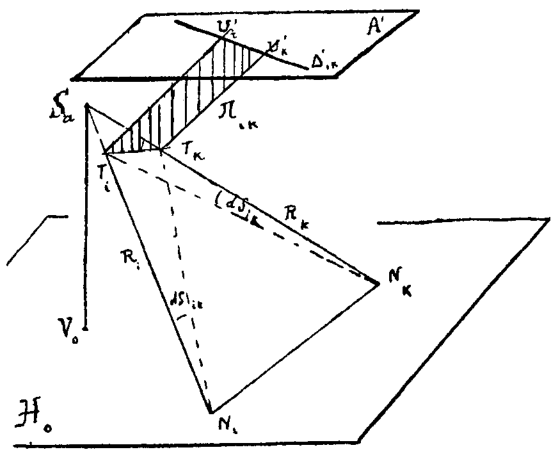


Fig. 9

$$(T_i)\quad \begin{cases} X-\alpha_o = (X_i-\alpha_o)\ \dfrac{R_k}{R_i}\ \dfrac{d\,S_{ik}\,sin\ 1''}{sin\ S_{ik}} \\ Y-\beta_o = (Y_i-\beta_o)\ \dfrac{R_k}{R_i}\ \dfrac{d\,S_{ik}\,sin\ 1''}{sin\ S_{ik}} \\ Z-A_o = \quad -A_o\ \dfrac{R_k}{R_i}\ \dfrac{d\,S_{ik}\,sin\ 1''}{sin\ S_{ik}} \end{cases}$$

We obtain immediately the coordinates of T by permuting i and k;

However, in space, the arc of the segment containing the angle S_{ik} generates a torus and, in the region adjacent to the point S_a, the plane tangent to this surface is defined (to about the 1st order) by the normals T_iT_k to the plane : $S_aN_iN_k$.

The equation of this plane is :

$$\begin{vmatrix} X-\alpha_o & Y-\beta_o & Z-A_o \\ X_i-\alpha_o & Y_i-\beta_o & -A_o \\ X_k-X_i & Y_k-Y_i & o \end{vmatrix} = o$$

That is :

$$l\,(X - \alpha_o) + m\,(Y - \beta_o) + n\,(Z - A_o) = o$$

with :—

$$\frac{l}{A_o\,(Y_k - Y_i)} = \frac{m}{-A_o\,(X_k - X_i)} = \frac{n}{(X_i - \alpha_o)\,(Y_k - Y_i) - (Y_i - \beta_o)\,(X_k - X_i)}$$

Returning to the notations above, we have then :

$$\frac{l}{A_o \sin V_{ik}} = \frac{m}{-A_o \cos V_{ik}} = \frac{n}{D_i \sin\,(V_{ik} - \lambda_i)}$$

The perpendiculars to the plane $S_a N_i N_k$ dropped from T_i, correspond to the equations ($X'Y'Z'$ being the coordinates)

$$\frac{X' - X}{A_o \sin V_{ik}} = \frac{Y' - Y}{-A_o \cos V_{ik}} = \frac{Z' - Z}{D_{ik} \sin\,(V_{ik} - \lambda_i)}$$

The two corresponding perpendiculars define the plane Π_{ik} tangent to the torus generated by the arc of the segment containing the angle S_{ik}.

d) LARGE SCALE GRAPH.

Let us pass a horizontal altitude plane

$$A'_o = A_o + dA'$$

The perpendicular from T_i is intersected at a point U'_i ($X'_i\,Y'_i\,Z'_i$) defined by :

$$\frac{X'_i - X}{A_o \sin V_{ik}} = \frac{Y'_i - Y}{-A_o \cos V_{ik}} = \frac{A_o + dA' - Z}{D_i \sin\,(V_{ik} - \lambda_i)}$$

Substituting for XYZ its values (equations T_i) we obtain :

$$X'_i - \alpha_o = \left[X_i - \alpha_o + \frac{A_o^2 \sin V_{ik}}{D_i \sin\,(V_{ik} - \lambda_i)}\right]\frac{R_k \sin 1''}{R_i \sin S_{ik}}\,d\,S_{ik} + \frac{A_o \sin V_{ik}}{D_i \sin\,(V_{ik} - \lambda_i)}\,d\,A'$$

$$Y'_i - \beta_o = \left[Y_i - \beta_o - \frac{A_o^2 \cos V_{ik}}{D_i \sin\,(V_{ik} - \lambda_i)}\right]\frac{R_k \sin 1''}{R_i \sin S_{ik}}\,d\,S_{ik} - \frac{A_o \cos V_{ik}}{D_i \sin\,(V_{ik} - \lambda_i)}\,d\,A'$$

Finally, the coordinates of the *characteristic points* U'_i, U'_k, defining the trace Δ'_{ik} on the altitude plane $A' = A_o + dA'$ of the plane Π_{ik} tangent to the torus defined by the controls *i* and *k*, are given by :—

$$U'_i \left\{ \begin{array}{l} X'_i - \alpha_o = L'_i\,dS_{ik} + M'_i\,dA' \\ Y'_i - \alpha_o = N'_i\,dS_{ik} + P'_i\,dA' \\ Z'_i - A_o = dA' \end{array} \right.$$

$$U'_k \left\{ \begin{array}{l} X'_k - \alpha_o = L'_k\,dS_{ik} + M'_k\,dA' \\ Y'_k - \beta_o = N'_k\,dS_{ik} + P'_k\,dA' \\ Z'_k - A_o = dA' \end{array} \right.$$

with the notations :

$$\begin{cases} F_i = X_i - \alpha_o + \dfrac{A_0^2 \sin V_{ik}}{D_i \sin (V_{ik} - \lambda_i)} \\ F_k = X_k - \alpha_o + \dfrac{A_0^2 \sin V_{ik}}{D_i \sin (V_{ik} - \lambda_i)} \\ G_i = Y_i - \beta_o + \dfrac{A_0^2 \cos V_{ik}}{D_i \sin (V_{ik} - \lambda_i)} \\ G_k = Y_k - \beta_o + \dfrac{A_0^2 \cos V_{ik}}{D_i \sin (V_{ik} - \lambda_i)} \end{cases}$$

(It is unnecessary to permute i and k in these last terms :—

$D_i \sin (V_{ik} - \lambda_i) = - D_k \sin (V_{ki} - \lambda_k)$ height of the triangle $V_a N_i N_k$).

$$\begin{cases} L'_i = F_i \dfrac{R_k \sin 1''}{R_i \sin S_{ik}} \\ L'_k = F_k \dfrac{R_i \sin 1''}{R_k \sin S_{ik}} \\ N'_i = G_i \dfrac{R_k \sin 1''}{R_i \sin S_{ik}} \\ N'_k = G_k \dfrac{R_i \sin 1''}{R_k \sin S_{ik}} \end{cases}$$

$$\begin{cases} M'_i = M'_k = \dfrac{A_o \sin V_{ik}}{D_i \sin (V_{ik} - \lambda_i)} \\ P'_i = P'_k = \dfrac{- A_o \cos V_{ik}}{D_i \sin (V_{ik} - \lambda_i)} \end{cases}$$

These coefficients may be calculated, or may be determined partly from the diagram of the restitution controls

REMARKS. If the angle dS_{ik} is small, the points U'_i and U'_k are very close togethers: by multiplying the difference $(X'_k - X'_i)$ and $(Y'_k - Y'_i)$ by a suitable coefficient, we may always trace the straight line Δ'_{ik} fixing one point, U'_i for example, and using the directional parameters.

For the second altitude, since

$$A'' = A_o + dA''$$

does not differ from A' except by a *round number of metres, m* for example, we obtain two other characteristic points U''_i, U''_k, of which the coordinates do not differ from the first except by the quantities :—

$$\begin{cases} mM'_i \\ mP'_i \\ m \end{cases}$$

the calculation of which is easy.

Finally we draw on the diagram the straight lines Δ' and Δ'' to a convenient scale, for example 1/200.

The two diagrams obtained are then homothetic: if the differences dS_{ik} are small and considered as being of the first order, the centre of homoteticity of the two diagrams is the true point *S*, to about the second order.

But we may consider, however, the true point as the intersection of the planes Π tangent to the toruses, or as the intersection of the straight lines common to those planes, in pairs; the horizontal projections of these straight lines are easy to construct.

Let us superpose these two diagrams on their common origin V_o (*See* Figure 10).

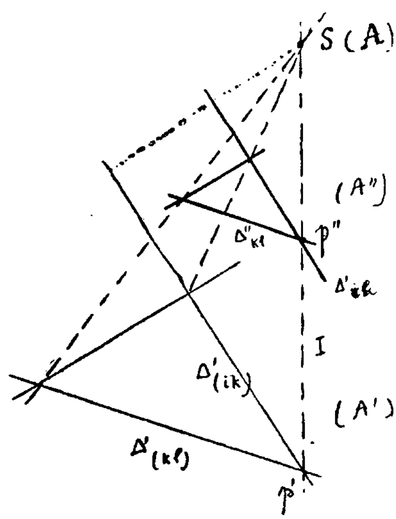


Fig. 10

The point *p'*, the intersection of the straight lines Δ'_{ik} and Δ'_{kl}, is the projection of a point in the straight line *I* common to the planes Π_{ik} and Π_{kl}. The point *p''* the intersection of the straight lines Δ''_{ik} and Δ''_{kl}, is equally on the straight line *I*.

It suffices therefore to join the homologous points in the two homothetic diagrams and to take the *mean point* of the straight lines *I* thus obtained.

If 4 *restitution controls* are avaiblable, we then actually measure in fact 6 *arcs subtending the angle.* We obtain then 15 *straight lines I* converging 3 and 3. (*ik, kl, il*) (an excellent check).

The point *V* being selected, we take from the diagram the distances λ' and λ'' to the two homologous straight lines Δ' Δ''.

$$\frac{A - A'}{\lambda'} = \frac{A'' - A}{\lambda''} = \frac{A'' - A'}{\lambda' + \lambda''}$$

From which :

$$A = A' + (A'' - A') \frac{\lambda'}{\lambda' + \lambda''}$$

We thus obtain as many values for the altitude as there are couples Δ' Δ'' (6 in the case of four controls).

e) THE PRACTICAL CALCULATION.

We shall determine *serially* (using logarithms to 6 decimal places and tables of trigonometric functions in degrees, or a calculating machine and table of natural values of trigonometric functions in grades) (circumference of circle divided into 100 parts) :

1) value of d, φ, ρ, ψ and S
2) The values of D, λ, R, μ and S', whence dS.
3) The coefficients $FGLMNP$
4) The coordinates of the characteristic points.

We then have to complete the diagram, calculating the mean error of the point selected, the altitude and its mean error.

The complete calculation for the 6 arcs of the circumscribed circle may then be effected in from 5 to 6 hours, especially if one employs a calculating machine.

NOTE : In principle, if the point of departure S_a is not sufficiently closely chosen, it is necessary to make a *second approximation by calculation.*

Practically, if we start from a point S_a determined by automatic transformation with the ROUSSILHE apparatus, a second calculation will always be *unnecessary.*

V. *ACCURACY OF THE RESULT.*

First case. In the present state of manufacture of the instantaneous, (snop-shot) ortho-chromatic plates employed for aerial photography, the coordinates of the restitution controls can hardly be determined on the negative closer than $\frac{1}{20}$th of a millimetre ; all contrivances for more accurate measurement, such as that used for terrestrial metro-photography for the slow exposure plates, give only an approximation which is illusory in this case.

Besides, the coordinates of the ground controls are not generally known closer than to about 0.1 metre ; further, this requires the establishment of their position by perfect triangulation and by accurate calculation, using a sufficiently conformal system of projection. In taking into consideration, among other things, the graphical errors of transfer on the transformation diagram, the small errors in establishing the coincidence of the images and the controls, and finally the errors made in calculating the inverse corrections for altitude from the approximate position S_a, we must allow for an error of 0.2 *metre* in the mean approximation of the points N.

To simplify the problem, let us suppose that the four controls M of the negative form a suqare of 160 $^m/_m$, on each side, centred in a plate of 18×24 $^c/_m$

The angles observed defined by the sides of the square (leaving the diagonals out of consideration) are given, for $f = 0.5$ metre, by :

$$tan \frac{1}{2} S = \frac{80}{500} \text{ nearly,}$$

from which we infer $S = 18°25'$ about.

The error at the base of the angle $\frac{S}{2}$ being $d\,b = \frac{1}{2} \frac{1^{mm}}{20} \sqrt{2}$,

the correlative error of the angle S is given by :

$$\frac{d\,S_1 \sin 1''}{2 \cos^2 \frac{S}{2}} = \frac{\sqrt{2}}{2 \times 20 \times 500}$$

We find $dS_1 = 28''$ *about.*

But we should not neglect the error on the focal length : taking into consideration the distortion of objectives of 0.5 m., the value of f is not actually known except to within about 0.2^{mm}.

The calculation of the angle S by the equation

$$cotan \frac{S}{2} = \frac{500}{80} \text{ gives :}$$

$$\frac{d\,S_2 \sin 1''}{2 \sin^2 \frac{S}{2}} = \frac{0,2}{80} = \frac{1}{400},$$

from which $dS_2 = 26''$ *about.*

As for the calculated angle S', by taking the average conditions of a negative approximately horizontal in view to a restitution of 1/2000 :

$A = 2400$ m.
$f = 0.5$ metre
controls spaced about 800 metres apart
$SN = 2{,}500$ m. about.

We find : —

$$tan \frac{1}{2} S' = \frac{400}{2500}$$

$$\frac{d\,S_3 \sin 1''}{2 \cos^2 \frac{S}{2}} = \frac{0{,}20 \sqrt{2}}{2 \times 2500}$$

and $dS_3 = 23''$ *about.*

By grouping these three possible errors—which represent rather the most favourable balanced conditions—the differences $dS = S - S'$ cannot be calculated with an average approximation closer than $26'' \sqrt{3}$, or say, about 45". Otherwise stated, the measuring apparatus consisting of the system of *plate and objective* may practically be likened *to an ordinary tacheometer which gives the angles to about the sexagesimal minute.*

With respect to the arc of the circumscribed circle, its differential departure for $dS = 45''$, is : $p = \dfrac{2500 \times 2500 \times 45 \sin 1''}{800} = 1.7\,\text{m}.$

A simple geometrical construction shows that with the negative inclined between 0° and 30°, the error in the plane varies

from : $MQ_1 = 3.9$ m.
to $M_2S_a = 10.1$ m.

Similarly, the error in altitude varies

from $MP = 0.7$ m.
to $M_2P_2 = 5.0$ m. (*See* Figure 11)

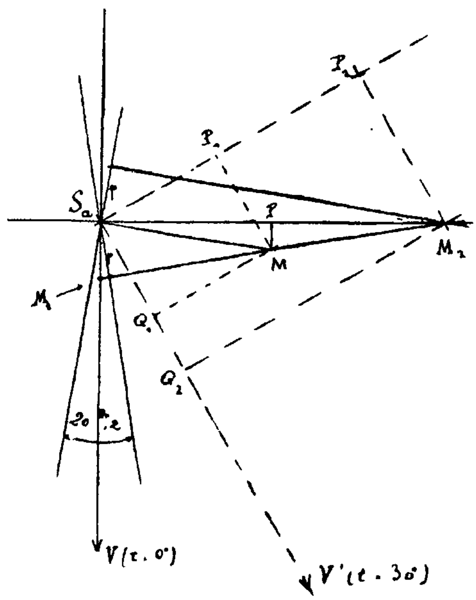


Fig. 11

Comparing these results with the approximations obtained when using the automatic transformation apparatus (*See* Section II) in the course of numerous operations, *i.e.* :—

from 3 to 7 metres for the position of the point V,
2 to 3 m. for the altitude A,

we see that the calculation does not materially improve the approximation to the position. THIS CALCULATION IS THEREFORE UNNECESSARY FROM A PRACTICAL STANDPOINT and under the present conditions of manufacture of the plates and objectives, the ROUSSILHE apparatus *completely and rapidly solves the 3 point problem in space, with the greatest possible accuracy.*

Second case. Let us suppose now (a solution not yet realized on which the constructors might work to advantage) that the photographic plates might be so tenuously made that one might measure to within 1/100th of a millimetre with a calibrated comparator.

Let us suppose also that the objectives used have been specially studied and constructed in such a manner that we may determine focal lengths to within about 0.05 $^m/_m$ that and the restitution controls are correct and brought into coincidence to within 0.1 m. (about) (then working to the normal scale of 1/1000) ; the difference S dS may then be calculated to within about 14" (in place of 45").

In such case, the *negative-objective comparator-system* will be the same as a repeating theodolite of the average type (with a mean error of 20") with regard to precision. Under these ideal conditions, the differential departure of the arcs of the circumscribed circle will be reduced to about 0.53 m., and the approximation of the *results will be tripled, at least.*

We may then make the following estimates of the possible limits of accuracy :

from 1.3 to 3.4 m. for the position of point V
0.2 to 1.7 m. for the altitude A

Finally, the restitution apparatus, with the diagrams and the altitude colums, operates like a protractor with n arms or, more exactly, like a *station pointer* (stigmograph).

Its employment, following the method of successive approximations which has been studied and standardized in all its details, permits the determinations of the coordinates of a station in space *in less than* 2 *hours* and of obtaining at the same time a correct transformation of the negative used.

The approximation to the position of the point of resection thus obtained is nearly as close as that obtained by calculation.

It is nearly *a third* of that which might be obtained by calculating the bearings in space once we have available photographic plates which permit readings to 1/100th of a millimetre and special objectives rigorously corrected — a condition which is far from the case at present.

The method of automatic transformation of large-scale aerial photographs — a method which I have already applied during the past ten years in the revision of 200 land-surveying charts — is susceptible moreover to fertile applications outside of the realm of topography.

In particular, it permits of the rapid and accurate setting of the negatives in the apparatus for accurate aerial stereo-topometry, now under consideration. For the rest, with the aid of radio installations of a type which may readily be conceived, it may be constituted into a powerful means of controlling aeronautical performances and the study of navigational instruments and flight apparatus.

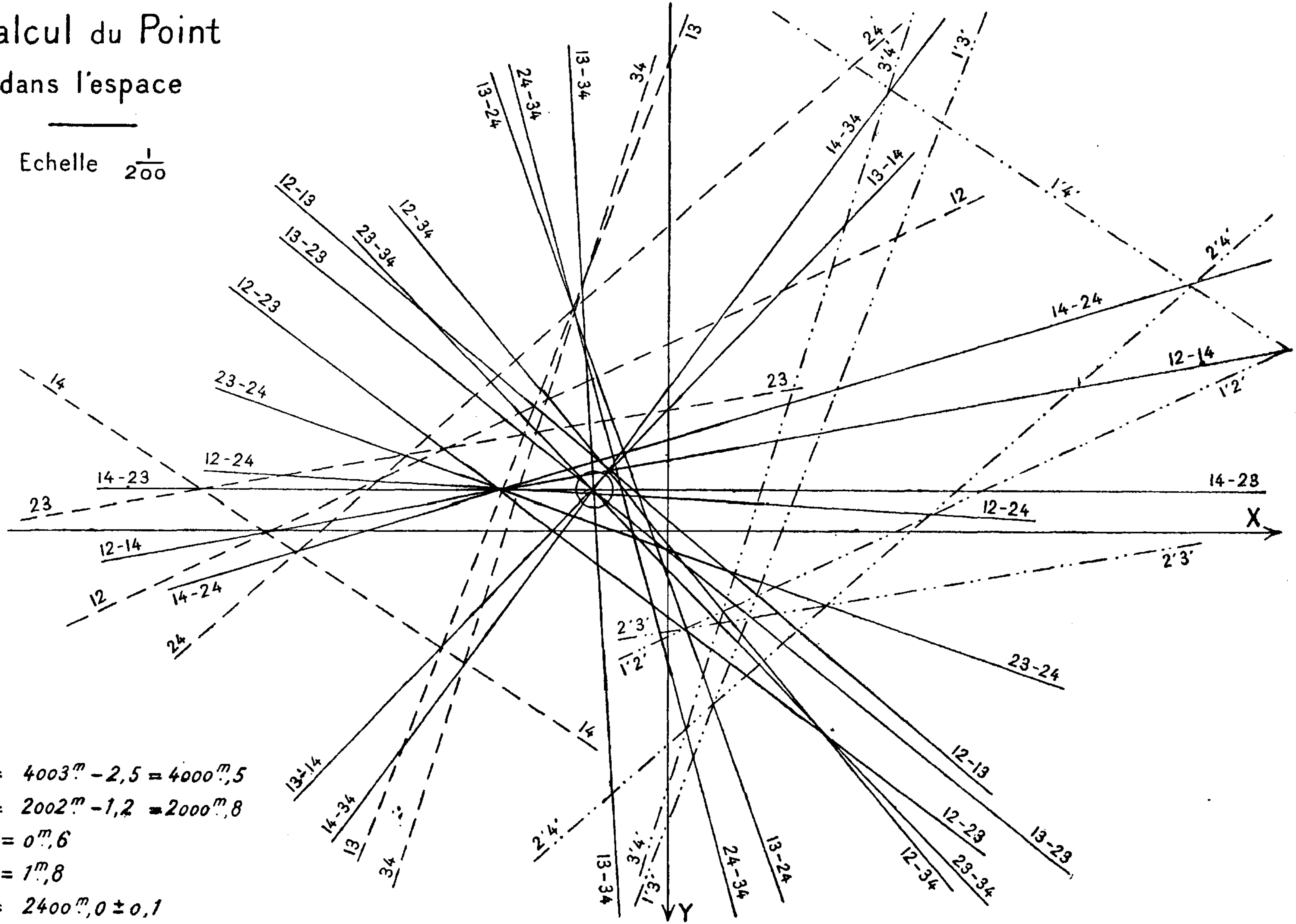
Paris, the 26*th of November* 1928.

NOTE. The transformation apparatus of the ROUSSILHE type is now manufactured by the Société Cinéma-Tirage L. MAURICE, 66, rue Saint-Denis, at *Gennevilliers* (Seine-France). The price of the large type (restitution scale 1/1000 to 1/15000) with 2 objectives of 0.3 and 0.5 m. is 35,000 francs, with all accessories.

A small model (1/5000 to 1/10000) is now under construction — approximate price 20,000 francs: with two objectives of 0.18 and 0.3 m.

Calcul du Point
dans l'espace

Echelle $\frac{1}{200}$



$$\begin{cases} X = 4003^m - 2,5 = 4000^m,5 \\ Y = 2002^m - 1,2 = 2000^m,8 \end{cases}$$

$$\begin{cases} e_m = 0^m,6 \\ e_M = 1^m,8 \end{cases}$$

$$A = 2400^m,0 \pm 0,1$$

CALCULATION OF POSITION IN SPACE — Scale: 1/200.

Calcul de X					Calcul de Y				
Points	1	2	3	4	Points	1	2	3	4
X'	3.807.39	4385,70	4.903.57	4.308.25	Y'	2059.40	2.769.68	2.324.55	1.834.31
α_0	4 003,00	4 003,00	4.003,00	4.003.00	β_0	2.002,00	2.002,00	2.002,00	2.002,00
$X'-\alpha_0$	-195,61	382,70	900.57	305.25	$Y'-\beta_0$	+57.40	767.68	322.55	-367.69
A_0		2.402,00	2.402,00	2.402,00					
h	0	36.0	90.0	24.0					
A_0-h		2.366.0	2.312.0	2378.0					
$\log(X'-\alpha_0)$		2,582858	2,954517	2,484656	$\log(Y'-\beta_0)$		2,885180	2,508597	2.565482
$\log h$		1.556303	1.954243	1.380211	$\log h$		1.556303	1.954243	1.380211
$\mathrm{colog}(A_0-h)$		6.625985	6.636012	6.623788	$\mathrm{colog}(A_0-h)$		6.625985	6.636012	6.623788
Σ		0.765146	1.544772	0.488655	Σ		1.067468	1.098852	0.569481
ΔX		+5,82	+35,06	+3.08	ΔY		+11,68	+12,56	-3.71
$X'-\alpha_0$		382,70	900,57	305,25	$Y'-\beta_0$		767.68	322,55	-367.69
$X-\alpha_0$	-195,61	+388,52	+935.63	+308,33	$Y-\beta_0$	+57.40	+779 36	335,11	-371.40

TABLE II. — CALCULATION OF THE DISTANCES $\rho_i = S_{mi}$ (plate).

Points	1	2	3	4
x_i	mm +100,00	mm -90,00	mm -90,00	mm +100,00
y_i	+70.00	+70,00	-70,00	-70,00
$\log y_i$	1.845 098	1.845 098	1.845 098 (-)	1.845 098 (-)
$\log x_i$	2.000 000	1.954 243 (-)	1.954 243 (-)	2.000 000
$\log tg\ \varphi_i$	9.845 098	9.890 855 (-)	9.890 855 (+)	9.845 098 (-)
φ_i	$38^c.8800$	$42^c.0832$	$42^c.0832$	$38^c.8800$
$\log y_i$	1.845 098	1.845 098	1.845 098	1.845 098
$\log \sin \varphi_i$	9.758 505	9.788 126	9.788 126	9.758 505
$\log d_i$	2.086 593	2.056 772	2.056 772	2.086 593
$\log x_i$	2.000 000	1.954 243	1.954 243	2.000 000
$\log \cos \varphi_i$	9.913 407	9.897 271	9.897 271	9.913 407
$\log d_i$	2 086 593	2.056 772	2.056 772	2 086 593
$\log d_i$	2.086 593	2.056 772	2.056 772	2.086 593
$\log f$	2.698 970	2.698 970	2.698 970	2 698 970
$\log tg\ \psi_i$	7.387 623 $15^c.2437$	7.358 002 $14^c.2731$	7.358 002 $14^c.2731$	7.387 623 $15^c.2437$
$\log f$	2.698 970	2.698 970	2.698 970	2.698 970
$\log \cos \psi_i$	9.987 429	9.988 992	9.988 992	9.987 429
$\log \rho_i$	2.711 541	2.709 978	2.709 978	2.711 541
ρ_i	mm 514.68	mm 512.85	mm 512.85	mm 514.68

TABLE III. — CALCULATION OF THE OBSERVED ANGLES S_{ik}

Angles	1-2	1-3	1-4	2-3	2-4	3-4
y_k	+ 70.00	− 70.00	− 70.00	− 70.00	− 70.00	− 70.00
y_i	+ 70.00	+ 70.00	+ 70.00	+ 70.00	+ 70.00	− 70.00
$\Delta y = y_k - y_i$	0	− 140.00	− 140.00	− 140.00	− 140.00	0
x_k	− 90.00	− 90.00	+ 100.00	− 90.00	+ 100.00	+ 100.00
x_i	+ 100.00	+ 100.00	+ 100.00	− 90.00	− 90.00	− 90.00
$\Delta x = x_k - x_i$	− 190.00	− 190.00	0	0	+ 190.00	+ 190.00
log Δy		2.146128			2.146128	
log Δx		2.278754			2.278754	
log tg v	0	9.867374	0	0	9.867374	0
v_{ik}	200.0000		300.0000	300.0000		0.0000
log Δy		2.146128			2.146128	
log sin v		9.773200			9.773200	
log d_{ik}		2.372928			2.372928	
log Δx		2.278754			2.278754	
log cos v		9.905826			9.905826	
log d_{ik}		2.372928			2.372928	
d_{ik}	190.00	236.01	145.00	145.00	236.01	190.00
ρ_i	514.68	514.68	514.68	512.85	512.85	512.85
ρ_k	512.85	512.85	514.68	512.85	514.68	514.68
$2p_{ik}$	1217.53	1263.54	1169.36	1165.70	1263.54	1217.53
p_{ik}	608.765	631.77	584.68	582.85	631.77	608.765
$p_{ik} - d_{ik}$	418.765	395.76	444.68	442.85	395.76	418.765
$p_{ik} - \rho_i$	94.085	117.09	70.00	70.00	118.92	95.915
$p_{ik} - \rho_k$	95.915	118.92	70.00	70.00	117.09	94.085
$\log(p - \rho_i)$	1.973520	2.068520	1.845098	1.845098	2.075255	1.981887
$\log(p - \rho_k)$	1.981887	2.075255	1.845098	1.845098	2.068520	1.973520
Σ	3.955407	4.143775	3.690196	3.690196	4.143775	3.955407
log p_{ik}	2.784450	2.800559	2.766918	2.765557	2.800559	2.784450
$\log(p_{ik} - d_{ik})$	2.621970	2.597432	2.648048	2.646257	2.597432	2.621970
Σ	5.406420	5.397991	5.414966	5.411814	5.397991	5.406420
$\log tg^2 \frac{1}{2} S$	8.548987	8.745784	8.275230	8.278382	8.745784	8.548987
$\log tg \frac{1}{2} S$	9.274493	9.372892	9.137615	9.139191	9.372892	9.274493
$\frac{1}{2} S_{ik}$	$11^{c}.8393$	14.7536	8.6854	8.7165.6	14.7536	11.8393
S_{ik}	$23^{c}.6786$	$29^{c}.5072$	$17^{c}.3708$	$17^{c}.4331$	$29^{c}.5072$	$23^{c}.6786$

TABLE IV.— CALCULATION OF THE DISTANCES $R_i = S_a M_i$

Points	1	2	3	4
$Y_i - \beta_0$	+ 57.40	+ 779.36	+ 335.11	− 371.40
$X_i - \alpha_0$	− 195.61	+ 388.52	+ 935.63	+ 308.33
$\log(Y_i - \beta_0)$	1.758912	2.891738	2.525187	2.569842 (−)
$\log(X_i - \alpha_0)$	2.291391 (−)	2.589413	2.971104	2.489016
log tg λ	9.467521	0.302325	9.554083	0.080826
λ	− 18.1710	70.5591	21.8953	55.8901
λ_i	181.8290	70.5591	21.8953	55.8901
$\log(Y_i - \beta_0)$	1.758912	2.891738	2.525187	2.569842
log sin λ	9.449586	9.951803	9.527874	9.886159
log D	2.309326	2.939935	2.997313	2.683683
$\log(X_i - \alpha_0)$	2.291391	2.589413	2.971104	2.489016
log cos λ	9.982063	9.649478	9.973791	9.805333
log D	2.309328	2.939935	2.997313	2.683683
log A_0	3.380573	3.380573	3.380573	3.380573
log D_i	2.309327	2.939935	2.997313	2.683683
log tg μ	1.071246	0.440638	0.383260	0.696890
μ_i	$94^{c}.6099$	$77^{c}.8579$	75.0251	87.3746
log A_0	3.380573	3.380573	3.380573	3.380573
log sin μ	9.998442	9.973185	9.965686	9.991403
log R_i	3.382131	3.407388	3.414887	3.387170
R_i	2410.63	2554.98	2599.48	2450.02

TABLE V. — CALCULATION OF THE DISTANCES S'_{ik} AND OF THE DIFFERENCES dS_{ik}

Angles	1-2	1-3	1-4	2-3	2-4	3-4
$Y_k - \beta_0$	+ 779.36	+ 335,11	371.40	+ 335,11	- 371.40	- 371.40
$Y_i - \beta_0$	+ 57.40	+ 57.40	+ 57.40	+ 779.36	+ 779.36	+ 335,11
$\Delta y = y_k - y_i$	+ 721.96	+ 277.71	- 428,80	- 444.25	- 1150.76	- 706.51
$X_k - \alpha_0$	+ 388.52	+ 935.63	+ 308.33	+ 935.63	+ 308.33	+ 308.33
$X_i - \alpha_0$	- 195.61	- 195.61	- 195.61	+ 388.52	+ 388.52	+ 935.63
$\Delta X = X_k - X_i$	+ 584.13	+ 1131.24	+ 503.94	+ 547.11	- 80.19	- 627.30
log ΔY	2.858513	2.443592	2.632255(-)	2.647627(-)	3.060985(-)	2.849118(-)
log ΔX	2.766510	3.053555	2.702379	2.738075	1.904120(-)	2.797475(-)
log tg V	0.092003	9.390037	9.929876(-)	9.909552(-)	1.156865(+)	0.051643(+)
V_{ik}	56.5934	15.3254	400^G - 44.8825	400^G - 43.4181	295.5709	253.7762
log Δy	2.858513	2.443592	2.632255	2.647627	3.060985	2.849118
log sin V	9.890650	9.377330	9.811604	9.799585	9.998948	9.873775
log D	2.967863	3.066262	2.820651	2.848042	3.062037	2.975343
log ΔX	2.766510	3.053555	2.702379	2.738075	1.904120	2.797475
log Cos V	9.778647	9.987293	9.881728	9.890033	8.842083	9.822132
log D_{ik}	2.967863	3.066262	2.820651	2.848042	3.062037	2.975343
D_{ik}	928.67	1164.83	661,68	704.76	1153,55	944.81
R_i	2410.63	2410,63	2410,63	2554.98	2554.98	2599.48
R_k	2554.98	2599.48	2450,02	2599.48	2450,02	2450,02
$2P_{ik}$	5.894 28	6.174 94	5.522.33	5.859.22	6.158.55	5.994,31
P_{ik}	2.947.14	3.087.47	2.761,165	2.929.61	3.079.275	2.997.155
$P_{ik} - D_{ik}$	2.018,47	1.922,64	2.099.485	2.224,85	1.925.725	2.052,345
$P_{ik} - R_i$	536,51	676.84	350.535	374,63	524.295	397.675
$P_{ik} - R_k$	392,16	487.99	311.145	330.13	629.255	547.135
log$(P_{ik} - R_i)$	2.729578	2.830486	2.544731	2.573603	2.719576	2.599528
log$(P_{ik} - R_k)$	2.593463	2.688411	2.492963	2.518685	2.798827	2.738094
Σ	5.323041	5.518897	5.037694	5.092288	5.518403	5.337622
log P_{ik}	3.469401	3.489603	3.441092	3.466809	3.488449	3.476709
log$(P_{ik} - D_{ik})$	3.305022	3.283898	3.322113	3.347301	3.284595	3.312250
Σ	6.774423	6.773501	6.763205	6.814110	6.773044	6.788959
log tg² ½S	8.548618	8.745396	8.274489	8.278178	8.745359	8.548663
log tg ½S'	9.274309	9.372698	9.137244.5	9.139089	9.372679.5	9.274331.5
½ S'	11.8344	14.7472.5	8.6780.7	8.7145.4	14.7466.5	11.8350
S'_{ik}	23.6688	29.4945	17.3561	17.4291	29.4933	23.6700
$dS_{ik} = S - S'$	+ 98"	+ 127"	+ 147"	+ 40"	+ 139"	+ 86"

TABLE VI. — CALCULATION OF THE COEFFICIENTS F AND G

Angles	1-2	1-3	1-4	2-3	2-4	3-4
log sin 1"	4.196120	4.196120	4.196120	4.196120	4.196120	4.196120
log sin S_{ik}	9.560415	9.650385	9.430537	9.432053	9.650385	9.550415
log sin"/sin S_{ik}	4.635705	4.545735	4.765583	4.764067	4.545735	4.635705
log R_k	3.407388	3.414887	3.389170	3.414887	3.389170	3.38917
log R_i	3.382130	3.382130	3.382130	3.407388	3.407388	3.41488
log R_k/R_i	0.025258	0.032757	0.007040	0.007499	9.981782	9.97428
log q_{ik}	4.660963	4.578492	4.772623	4.771566	4.527517	4.609988
log q_{ki}	4.610447	4.512978	4.758543	4.756568	4.563953	4.661422
V_{ik}	56.5934	15.3254	355.1175	356.5819	295.5709	253.7762
λ_i	181.8290	181.8290	181.8290	70.5591	70.5591	21.8953
$V_{ik} - \lambda_i$	274.7644	233,4964	173,2885	286.0228	225.0118	231.8809
log A_0	3.380573	3.380573	3.380573	3.380573	3.380573	3.380573
log sin V_{ik}	9.890650+	9.377330+	9.811604-	9.799585-	9.998948-	9.873775-
log Cos V_{ik}	9.798647+	9.987293+	9.881728+	9.890033-	8.842083-	9.822132-
log A_0 sin V_{ik}	3.271223+	2.757903+	3.192177-	3.180158-	3.379521-	3.254348-
log A_0 Cos V_{ik}	3.179220+	3.367866+	3.262301+	3.270606+	2.222656-	3.202705-
log D_i	2.309327	2.309327	2.309327	2.939935	2.939935	2.997313
log sin$(V_{ik} - \lambda_i)$	9.964946-	9.700891-	9.610000+	9.989448-	9.583034-	9.681344-
log h_{ik}	2.274273-	2.010218-	1.919327+	2.929383-	2.522969-	2.678657-
log r_{ik}	0.996950-	0.747685-	1.272850-	0.250775+	0.856552+	0.575691+
log s_{ik}	0.904947-	1.357648-	1.342974+	0.341223-	9.699687+	0.524048+
log $A_0 r_{ik}$	4.377523-	4.128258-	4.653423-	3.631348+	4.237125+	3.956264+
log $A_0 S_{ik}$	4.285520-	4.738221-	4.723547+	3.721796-	3.080260+	3.904621+
$A_0 r_{ik}$	- 23851,89	- 13.435,62	- 45021,80	+ 4.279,06	+ 17263,36	+ 9.041,98
$X_i - \alpha_0$	- 195.61	- 195.61	- 195.61	+ 388.52	+ 388.52	+ 935.63
F_i	- 24047.50	- 13631,23	- 45217,41	+ 4.667.58	+ 17.651.88	+ 9.977 61
$X_k - \alpha_0$	+ 388.52	+ 935.63	+ 308.33	+ 935.63	+ 308.33	+ 308.33
F_k	- 23463,37	- 12499.99	- 44713.47	+ 5214.69	+ 17571.69	+ 9350 31
$-A_0 S_{ik}$	+ 19.298.35	+ 54729.43	- 52911,12	+ 5269,82	- 1202.98	- 8.028.25
$Y_i - \beta_0$	+ 57.40	+ 57.40	+ 57.40	+ 779.36	+ 779.36	+ 335.11
G_i	+ 19355.75	+ 54786.83	- 52.853,72	+ 6049.18	- 423,62	- 7.693.14
$Y_k - \beta_0$	+ 779.36	+ 335.11	- 371.40	+ 335.11	- 371.40	- 371.40
G_k	+ 20077.71	+ 55064.54	- 53282.52	+ 5604.93	- 1574.38	- 8.399.65

TABLE VII. — CALCULATION OF THE COORDINATES OF THE CHARACTERISTIC POINTS OF THE SEGMENTS CONTAINING ANGLES

$A' = 2399''$ $A'' = 2402''$	1 - 2		1 - 3		1 - 4		2 - 3		2 - 4		3 - 4	
	U'(12)	U''(21)	U'(13)	U''(31)	U'(14)	U''(41)	U'(23)	U''(32)	U'(24)	U''(42)	U'(34)	U''(43)
$\log F_i$	4.381070 -	4.370391 -	4.134535 -	4.096910 -	4.655306 -	4.650437 -	3.669092 +	3.717228 +	4.246791 +	4.244813 +	3.999026 +	3.970806 +
$\log q_{ik}$	4.660963	4.610447	4.578492	4.512978	4.772623	4.758543	4.771566	4.756568	4.527517	4.563953	4.609988	4.661422
$\log L'_{ik}$	9.042033 -	8.980838 -	8.713027 -	8.609888 -	9.427929 -	9.408982 -	8.440658 +	8.473796 +	8.774308 +	8.808766 +	8.609014 +	8.632248 +
$\log dS_{ik}$	1.991226 +	1.991226 +	2.103804 +	2.103844 +	2.167317 +	2.167317 +	1.602060 +	1.602060 +	2.143015 +	2.143015 +	1.934498 +	1.934498 +
$\log L'_{ik} dS_{ik}$	1.033259 -	0.972064 -	0.816831 -	0.713692 -	1.595246 -	1.576299 -	0.042718 +	0.075856 +	0.917323 +	0.951781 +	0.543512 +	0.566746 +
$L'_{ik} dS_{ik}$	- 10.80	- 9.38	- 6.56	- 5.17	- 39.38	- 37.70	+ 11.03	+ 11.91	+ 8.27	+ 8.95	+ 3.50	+ 3.69
M'_{ik}	- 9.930		- 5.594		- 18.743		+ 1.781		+ 7.187		+ 3.764	
dA'	- 3		- 3		- 3		- 3		- 3		- 3	
$M'_{ik} dA'$	+ 29.79	+ 29.79	+ 16.78	+ 16.78	+ 56.23	+ 56.23	- 5.34	- 5.34	- 21.56	- 21.56	- 11.29	- 11.29
$X' - \alpha_0$	+ 18.99	+ 20.41	+ 10.22	+ 11.61	+ 16.85	+ 18.53	+ 5.69	+ 6.57	- 13.29	- 12.61	- 7.79	- 7.60
$\log G_i$	4.286810 +	4.302714 +	4.738677 +	4.740872 +	4.728076 -	4.726585 -	3.781697 +	3.748570 +	2.626976 -	3.197109 -	3.886103 -	3.924862 -
$\log q_{ik}$	4.660963	4.610447	4.578492	4.512978	4.772623	4.758543	4.771566	4.756568	4.527517	4.563953	4.609988	4.661422
$\log N'_{ik}$	8.947773 +	8.913161 +	9.317169 +	9.253850 +	9.495699 -	9.485128 -	8.553263 +	8.505138 +	7.154493 -	7.761062 -	8.496091 -	8.535686 -
$\log dS_{ik}$	1.991226 +	1.991226 +	2.103804 +	2.103804 +	2.167317 +	2.167317 +	1.602060 +	1.602060 +	2.143015 +	2.143015 +	1.934498 +	1.934498 +
$\log N'_{ik} dS_{ik}$	0.938999 +	0.904387 +	1.420973 +	1.357654 +	1.663016 -	1.652445 -	0.155323 +	0.107198 +	9.297508 -	9.904077 -	0.430589 -	0.520182 -
$N'_{ik} dS_{ik}$	+ 8.69	+ 8.02	+ 26.36	+ 22.79	- 46.03	- 44.92	+ 1.43	+ 1.28	- 0.20	- 0.80	- 2.70	- 3.31
P'_{ik}	+ 8.034		+ 22.785		- 22.028		+ 2.194		- 0.501		- 3.342	
dA'	- 3		- 8		- 3		- 3		- 1		- 3	
$P'_{ik} dA'$	- 24.10	- 24.10	- 68.35	- 68.35	+ 66.08	+ 66.08	- 6.58	- 6.58	+ 1.50	+ 1.50	+ 10.03	+ 10.03
$Y' - \beta_0$	- 15.41	- 16.08	- 41.99	- 45.56	+ 20.05	+ 21.16	- 5.15	- 5.30	+ 1.30	+ 0.70	+ 7.33	+ 6.72
dA'	0											
$M'_{ik} dA'$	0											
$X'' - \alpha_0$	- 10.80	- 9.38	- 6.56	- 5.17	- 39.38	- 37.70	+ 11.03	+ 11.91	+ 8.27	+ 8.95	+ 3.50	+ 3.69
$P'_{ik} dA'$	0											
$Y'' - \beta_0$	+ 8.69	+ 8.02	+ 26.36	+ 22.79	- 46.03	- 44.92	+ 1.43	+ 1.28	- 0.20	- 0.80	2.70	- 3.31