CALCULATION OF THE GEOGRAPHICAL COORDINATES ON THE INTERNATIONAL ELLIPSOID OF REFERENCE.

by

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The fundamental problem of the calculation of the geographical coordinates of the angles of geodetic triangles has been treated by numerous authors and solved by means of various formulae which cannot all be mentioned in this article. The greatest geodesists have attacked this problem but it will suffice to cite only such names as A. M. LEGENDRE, PUISSANT, C. F. GAUSS, F. W. BESSEL, HOSSARD, A. R. CLARKE, W. JORDAN, F. R. HELMERT, G. C. ANDRAE, L. KRÜGER. Each Hydrographic Office usually favours the formulae it customarily employs and finds no reason to change, provided that it is satisfied that these formulae give the required degree of approximation.

I. — *ACCURACY.*

Attention is directed to a remarkable study of this question published by M. E. FICHOT, Hydrographe Général and present Director of the French Hydrographic Service, in the "*Annales Hydrographiques*" of 1907, pp. 47 - 136.

M. E. FICHOT has just supplemented this study with a note which appears in the "*Comptes Rendus des Séances de l'Académie des Sciences*", Vol. 188, p. 122.

The following notation will be employed :—

a Major axis of the ellipsoid of reference ;
b Minor axis of the ellipsoid of reference ;

$$e = \frac{\sqrt{a^2 - b^2}}{a} \quad \text{primary excentricity ;}$$

$$e' = \frac{\sqrt{a^2 - b^2}}{b} = \frac{e}{\sqrt{1 - e^2}} \quad \text{secondary excentricity ;}$$

φ Latitude of origin of the geodetic line ;
L Longitude of origin of the geodetic line ;
Z Azimuth at origin of the geodetic line counted clockwise from North ;

φ', L', Z' Latitude, longitude and azimuth at the extremity of the geodetic line ;
s Length of a geodetic arc ,
N The major normal :— radius of curvature in normal section ; perpendicular to plane of the meridian ;
ρ Radius of curvature of the normal meridian section ;
μ Modulus of the common logarithms.

The angles are expressed in radians and must be multiplied by $\frac{1}{\sin 1''}$ in order to convert them into seconds of arc.

The principal points which M. FICHOT establishes or cites are the following :—

a) The quantity $\frac{s}{N}$ is taken as infinitely small, of the 1st order ; e^2 is of the same order. In fact $e^2 = \frac{67}{10^4}$ and $\frac{s}{N}$ is equal to it for $s = 43$ kilometres (approximately), which is the average dimension of one side of a geodetic triangle.

The sides would have to be longer than 100 kilometres in order that e and not e^2 could be assumed to be of the same order as $\frac{s}{N}$.

$$e'^2 \text{ is of the same order as } e^2 : \quad \left(e'^2 - e^2 = \frac{0.455}{10^4}\right)$$

b) The formulae are based on the application of the LEGENDRE theorem which was extended by GAUSS to spheroidal triangles provided that they be considered as composed of geodetic lines. This theorem may then be stated as follows :—

If, with the lengths *a*, *b*, *c*, of the geodetic lines comprising a spheroidal triangle $A\ B\ C$, a plane triangle $A_p\ B_p\ C_p$ be constructed then :—

$$A - A_p = B - B_p = C - C_p = \frac{S}{3N\rho} = \frac{\varepsilon}{3}$$

the angular quantity neglected being of the *fourth order* of magnitude. (S is equally the area of the spheroidal triangle and that of the plane triangle, and $\frac{1}{N_p}$ the total curvature at any point in the interior of the spheroidal triangle).

Conversely, if, starting with the side *a* assumed to be known, the length *b* of a side of the geodetic triangle be calculated by means of the LEGENDRE theorem, *i. e.* by the formula :

$$b = a\,\frac{\sin\left(B - \frac{\varepsilon}{3}\right)}{\sin\left(A - \frac{\varepsilon}{3}\right)}$$

Thus a value of $\frac{b}{N}$ will be obtained in which the terms of the order $\frac{a^5}{N^5}$ are neglected, *i. e.* the terms of the 5th order.

c) The difference in length between the geodetic line and the corresponding normal section are always negligible in problems dealing with geodetic triangles (infinitely small quantities of the 7th degree).

d) The initial angles of divergence between the geodetic line and the corresponding normal section, are quantities of the 3rd order, the equation of which, neglecting terms of the 4th order, is :—

$$\frac{1}{12}\frac{e^2 s^2}{N^2} \sin 2Z \cos^2 \varphi$$

In order to use the LEGENDRE theorem to the limit of its applicability it is necessary to correct the observed angles by this amount. But this does not exceed 0''.014 for a side of 50 kilometres; it is smaller than the errors of observation and usually is not taken into consideration.

II. — *FORMULAE.*

The methods of calculation which are based on the LEGENDRE theorem could not be expected to give an accuracy greater than that of the 4th order for lengths and of the 3rd order for angles. This is the accuracy obtained with the formulae of CLARKE and of ANDRAE. M. FICHOT has shown that the same accuracy can be obtained by the method called that of the "Ingénieurs Géographes" which is mentioned by PUISSANT in the "*Mémorial du Dépôt Général de la Guerre*", Vol. VII, Paris - Picquet, 1832, and was employed for the calculation of the triangulation of France undertaken in 1817. It is based on the examination of a sphere tangent to the ellipsoid throughout the entire length of the parallel of origin. The formulae which have been deduced therefrom for the calculation of latitude, longitude, and convergence of meridians were carried at first to the second degree only. HOSSARD prepared tables of correction to take the terms of the 3rd degree into consideration: ("*Mémorial du Dépôt Général de la Guerre*", Vol. IX. pp. 519-531). M. FICHOT continues the development of the formulae to include the terms of the 4th degree. These terms were not unknown and are to be found particularly in the "*Handbuch der Vermessungskunde*" by JORDAN-EGGERT, 3rd Vol. p. 442. It is interesting however to see them deduced directly by such a simple method as that of the Ingénieurs Hydrographes. For the rest, to increase, the accuracy M. FICHOT employs only the infinitely small quantities $\frac{s}{N}$ and e^2, and refers them to the data at the point of origin of the geodeti arc.

Below $\varphi' - \varphi$ are given in a somewhat different form which lends citself more easily to calculation; for this, the secondary excentricity e' and the radius of curvature ρ are used.

$$(1)\quad \varphi'-\varphi = \frac{s}{\rho}\cos Z - \frac{s^2}{2N\rho}\left(tg\,\varphi \sin^2 Z + \frac{3}{2}e'^2 \sin 2\varphi \cos^2 Z\right)$$
$$-\frac{s^3}{6N^2\rho}\cos Z\left\{(1+3\,tg^2\varphi)\sin^2 Z + e^2\left[3(1-2\sin^2\varphi) - 2\sin^2 Z\,(1+2\sin^2\varphi)\right]\right\}$$
$$-\frac{s^4}{24N^3\rho}\sin^2 Z\,\frac{tang\,\varphi}{\cos^2\varphi}\left[4(2+\sin^2\varphi) - 3\sin^2 Z\,(3+2\sin^2\varphi)\right]$$

$$(2)\quad L-L' = \frac{s}{N}\frac{\sin Z}{\cos\varphi} + \frac{s^2}{2N^2}\sin 2Z\,\frac{tang\,\varphi}{\cos\varphi} - \frac{1}{3}\frac{s^3}{N^3}\sin Z\left(\frac{\sin^2\varphi - \cos^2 Z\,(1+3\sin^2\varphi)}{\cos^3\varphi} - e^2\cos\varphi\cos^2 Z\right)$$
$$-\frac{1}{6}\frac{s^4}{N^4}\sin 2Z\,\frac{\sin\varphi}{\cos^4\varphi}\left[1+2\sin^2\varphi + 3(1+\sin^2\varphi)\cos^2 Z\right]$$

$$(3)\quad Z'-Z = \pm 180 + \frac{s}{N}\sin Z\,tg\,\varphi + \frac{s^2}{4N^2}\sin 2Z\,(1+2\,tang^2\varphi + e^2\cos^2\varphi)$$
$$+\frac{1}{6}\frac{s^3}{N^3}\sin Z\,tang\,\varphi\left[2(3+4\,tang^2\varphi)\cos^2 Z - (1+2\,tang^2\varphi)\right]$$

All of the elements necessary for the calculation of these expressions on the system of the "*International Ellipsoid of Reference*" are to be found in the tables in "*Special Publication N° 2*" of the International Geodetic and Geophysic Union. These tables give the values of *log N*, *log* ρ and *log* $\dfrac{1}{2\,N\,\rho\,\sin 1''}$ for each minute.

(*See* "*Hydrographic Review*" of May, 1927, pp. 226-228).

Table IV of "*Special Publication N° 21*" of the International Hydrographic Bureau gives the values of *log N sin I''* and of *log* ρ *sin I''* on the same ellipsoid for each 10'.

But, calculation by these formulae is very long and complicated and such great accuracy is unnecessary for the sides of geodetic triangles of ordinary dimensions. The "*Bulletin Géodésique N° 12*" of 1926 (J. HERMANN, 6, rue de la Sorbonne, Paris), gives the formulae of Lieutenant Colonel E. BENOIT which have been adopted in France by the Geographic Service of the Army, as well as Tables which are calculated for every 10' for the "*International Ellipsoid of Reference*", and permit the formulae to be used with great rapidity (*). The formulae of Lieutenant Colonel BENOIT are accurate to the 3rd order inclusive and are therefore adequate for the calculation of the convergence of the meridians. If, in these calculations, it is desired to take the terms of the 4th order into consideration in computing the latitudes and the longitudes, it is necessary to add a term which will be indicated later. The notations previously employed will be retained indicating by an index *m* where a quantity is to be taken as the latitude equal to the mean of the latitudes of the two extremities of the geodetic line under consideration, and indicating by the prime when a quantity is to be taken as the latitude of the extremity of the geodetic line.

The introduction of these values greatly simplifies the equations; it

(*) It should be noted that in the formulae and in the BENOIT Tables the Latitude is indicated by the letter *L*, the Longitude by the letter *M*, length of geodetic arc by *K*.

rarely leads to the necessity of making successive approximations and the calculation may always be done rapidly.

Table I, which is included in the *"Bulletin Géodésique No 12"* of 1926, gives the values of *log P* to eight decimals, *log Q* to five decimals and *log R* to eight decimals for each 10' of latitude.

The equations of *P, Q, R* are the following :— (*)

$$P = \frac{1}{\rho \sin 1''}$$
$$Q = \frac{\operatorname{tang} \varphi}{2 N \rho \sin 1''}$$
$$R = \frac{1}{N \sin 1''}$$

III. — *LATITUDE.*

The latitude is first calculated by the approximate formula :—

$$(4)\quad \varphi' - \varphi = \frac{s}{\rho_m} \cos Z - \frac{s^2}{2 N_m \rho_m} \sin^2 Z \operatorname{tang} \varphi_m$$

which might be written with the notations given above.

$$(4\ bis)\quad \varphi' - \varphi = s P_m \cos Z - s^2 Q_m \sin^2 Z$$

After having calculated the logarithm of the second term

$$s^2 Q_m \sin^2 Z,$$

a correction $\mu\alpha$ expressed in units of the 5th decimal of the logarithm is added to the logarithm. The logarithm of $\mu\alpha$ is obtained by adding the logarithm or λ_m, obtained from Table II with 4 decimals, to the logarithm of $(\varphi' - \varphi)$; then the number is taken in order to obtain the correction $\mu\alpha$ which has the opposite sign to $\varphi' - \varphi$ when φ is included between — 30° and + 30°, otherwise it has the same sign as $\varphi' - \varphi$.

The equation for λ is :—

$$\mu \frac{\sin 1'' \; 3 \operatorname{tang}^2 \varphi - 1}{6 \operatorname{tang} \varphi}$$

(*) The quantities *P* and *R* are the inverses of the quantities called *B* and *A* in Table IV of Special Publication No 21 of the I. H. B.

Example :—

$\varphi = 47°4'\ 57''{,}212 \qquad s = 64.203',15 \qquad Z = 206°\ 20'\ 53''{,}7$

φ_m appro. $46°49',4$

$\log s$	$4{,}807\ 556\ 34$	$2 \log s$	$9{,}615\ 11$
$\log P_m$	$\bar{2}{,}510\ 310\ 52$	$\log Q_m$	$\bar{9}{,}431\ 45$
$\log \cos Z$	$\bar{1}{,}952\ 362\ 77$	$2 \log \sin Z$	$\bar{9}{,}294\ 43$
$\log 1^{\text{er}}$ terme	$3\ 270\ 229\ 63$		$0{,}340\ 99$
		$\mu\alpha$	$-1\ 48$
$\log \lambda_m$	$\bar{2}{,}8991$	$\log 2^{\text{e}}$ terme	$0{,}339\ 51$
$\log \mu\alpha$	$2{,}1693$		

φ	$47°\ 4'\ 57'',212$
1^{er} terme	$-\ 31'\ 3'',072$
2^{e} terme	$-\ 2'',185$
φ'	$46°\ 33'\ 51''\ 955$

In order to take the terms of the 4th order into consideration, it is necessary to add to this value the following quantity, for which the equation has been calculated. It has a value of several hundred thousandths of a second only for the ordinary lengths of s in latitude 45°.

$$\frac{s^2}{2N^2}\Big[e^4 \sin^2 Z \sin\varphi\cos^3\varphi + \frac{se^2}{4N}\cos Z\,(2\sin^2\varphi - \cos^2 Z) - \frac{s^2}{24N^2}\sin^2 Z \tan\varphi\,(1+3\tan^2\varphi)\,(1+\sin^2 Z - \tfrac{1}{3}\cos^2 Z \cot^2\varphi)\Big]$$

LONGITUDE.

This is calculated by the formula

$$(5)\quad L' - L = s\,\frac{R'\sin Z}{\cos\varphi'}$$

and a correction $\mu\beta$ is added to the logarithm of this term

$$\mu\beta = -\mu\beta_1 + \mu\beta_2$$

The term $\mu\,\beta_1$, and $\mu\,\beta_2$ are given in units of the eighth decimal of the logarithms of Table III which is entered, for the former, with the argument *log s R'* and, for the second, with the argument *log* $(L' - L)$.

The equation for $\mu\,\beta$ is :—

$$\mu\beta = -\frac{1}{6}\mu \sin^2 1'' \left[s^2 R'^2 - (L'-L)^2\right]$$

Example :—

φ' approx. $46^\circ\ 33'9$

log s	4,807 556 34		
log R'	$\bar{2}$,508 943 07	$-\mu\beta_1$	-729
log sR'	3,316 499 41		
log sin Z	9,647 212 73		
colog cos φ'	0,162 703 03	$+\mu\beta_2$	$+304$
log $L'-L$	3,126 415 17	$\mu\beta$	-425
$\mu\beta$	$-4\ 25$		
	3,126 410 92	$L'-L$	$22'\ 17'',861$
		L	$3'\ 44'',752$
		L'	$26'\ 2'',613$

If the terms of the 4th degree are to be taken into consideration, it is necessary to deduct from the value of $L' - L$ thus obtained, the following quantity which, in latitude 45° and for the usual lengths of s, does not exceed one ten thousandth of a second.

$$\frac{s^3}{12 N^3} \sin 2Z \left[e^2 \cos Z \cos\varphi + \frac{s}{N_m} \frac{\tang \varphi}{\cos^3\varphi}\left(4 \sin^2\varphi \cos^2 Z - \sin^2 Z\right)\right]$$

AZIMUTH.

This is calculated from the formula :

$$(6)\quad Z' - Z = \pm 180 + (L' - L)\sin\varphi_m$$

The correction $\mu\,\beta'$ for which the equation is

$$\mu\beta' = \frac{1}{6}\mu \sin^2 1'' \left[\frac{s^2 R'^2}{2} + \frac{(\varphi' - \varphi)^2}{4}\right] = \frac{1}{2}\mu\beta_1 + \frac{1}{4}\mu\beta_3$$

must be added to the logarithm of $(L' - L)\ sin\ \varphi_m$. It is computed by means of the same table as $\mu\,\beta$. The first part is none other than $\frac{\mu\,\beta_1}{2}$; therefore, in order to obtain its value it is but necessary to take half of the value of the correction $\mu\,\beta_1$ which was obtained when calculating the longitude. To compute the second part, the value of $\mu\,\beta_3$ is found in Table III in units of the eighth decimal of the logarithm opposite the argument $log\ (\varphi' - \varphi)$. It suffices to take one-fourth of this value.

Since the calculation involves nothing out of the ordinary, no example is given.

IV. — *CASE OF EXTREME LATITUDES. — POLAR REGIONS.*

The various expressions which are given above are valid only if the quantities $tan\ \varphi$ or $\frac{1}{cos\ \varphi}$ do not attain great magnitudes; in other words, if not used for high latitudes. It is evident that if the quantities $\varphi' - \varphi$ and $L' - L$ are of the same order of magnitude as the quantity $\frac{\pi}{2} - \varphi$, spheroidal triangle which has one angle at the pole and the others at the two extremities of the side of the geodetic triangle, cannot be solved by the development of the equation as a series. It will be necessary therefore to employ the ordinary formulae of spherical trigonometry on a sphere of radius N, then to add to the values thus obtained for $\varphi' - \varphi$, $L' - L$, $Z' - Z$, the correction factors of e^2 which are included in the formulae given by M. Fichot or, if the expression given at the beginning for $\varphi' - \varphi$ be taken, the terms in e'^2, after having substituted $\frac{1}{N}(1 + e'^2\ cos^2\ \varphi)$ to $\frac{1}{\rho}$

The corrections due to compression always retain the same degree of importance; they do not comprise the factors $tan\ \varphi$ or $\frac{1}{cos\ \varphi}$.

The Benoit formulae cannot be used in this case; but it should be noted that this condition rarely occurs. In latitude 70° the error due to the employment of these formulae does not exceed a thousandth of a second for geodetic triangles of ordinary dimensions.

EQUATORIAL REGIONS.

At a distance of less than 5° from the Equator, calculation of latitude cannot be done by the Benoit formula, since the correction $\mu\,\alpha$ then becomes infinity and consequently cannot be given in Table II. However, another expression may be obtained which is valid for the infinitely small factors of the 3rd degree by omitting all corrections to the formula (4 a) for latitude, provided that the coefficient Q be taken for the latitude $\frac{\varphi' + 2\varphi}{3}$ instead of for $\frac{\varphi' + \varphi}{2}$.

In order that the formula may be complete for the terms of the 4th degree inclusive, it is necessary to substract the expression

$$\frac{s^3}{3N^2\rho} \cos Z \sin^2 Z \tan^2 \varphi + \frac{s^3}{8N^2\rho} e'^2 \cos Z \left(\cos^2 Z - 2 \sin^2 \varphi\right) + \frac{s^4}{72N^3\rho} \sin^2 Z \tan \varphi \left(20 - 17 \sin^2 Z + 32 \tan^2 \varphi - 35 \sin^2 Z \tan^2 \varphi\right)$$

In the average latitudes the 1st term is of the 3rd degree.

But if $\varphi = \pm 5^\circ$, $tan^2\ \varphi$ is equal to $\frac{76}{10^4}$ and it may be considered as infinitely small of the 1st degree. Consequently, the first term of the above expression becomes of the 4th degree and the terms of the 4th degree which are omitted are limited to the following :

$$\frac{s^3}{3N^2\rho} \cos Z \sin^2 Z \tan^2 \varphi + \frac{s^3 e'^2}{8N^2\rho} \cos^3 Z + \frac{s^4}{72N^3\rho} \sin^2 Z \cdot \tan \varphi \left(20 - 17 \sin^2 Z\right)^{(*)}$$

V. — *CONCLUSION.*

The BENOIT formulae and Tables have the accuracy desirable in practice for the calculation of geographical positions in all normal cases. The Tables are computed for the International Ellipsoid of Reference and make the calculation extremely simple and rapid. (**)

Note. — Tables II and III, by means of which the corrections $\mu\,\alpha$ and $\mu\,\beta$ are calculated, are independent of the ellipsoid employed. Therefore, the BENOIT tables may be used with any ellipsoid, provided that Table I is replaced by a table giving the same elements corresponding to the ellipsoid actually employed.

VI. — *INVERSE PROBLEMS.*

The BENOIT formulae lend themselves readily to the solution of the inverse of the preceding problems within the limits of their applicability, as shown above.

1° Given the latitude of two points and, on the horizon of the first

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(*) In the terms of the 4th degree of all formulae mentioned in this article, the following terms may be used interchangeably N for ρ and e'^2 for e^2.

(**) An interesting study on this question may be found in the "*Geographical Journal*" of April, 1929, page 376.

point, the azimuth Z of the geodetic line joining them, to find the length of this geodetic line.

Equation (4) supplemented by the correction terms obtained from Table II may then be expressed :

$$\varphi' - \varphi = s P_m \cos Z - s^2 Q_m (1+\alpha) \sin^2 Z$$

Develop to the third degree inclusive, then

$$s = \frac{\varphi' - \varphi}{P_m \cos Z}\left[1 + (\varphi' - \varphi)\frac{Q_m \operatorname{tg}^2 Z (1+\alpha)}{P_m^2} + 2(\varphi' - \varphi)^2 \frac{Q_m^2 \operatorname{tg}^4 Z (1+\alpha)^2}{P_m^4}\right.$$

or, taking

$$s_1 = \frac{\varphi' - \varphi}{P_m \cos Z} \qquad p = R_m \operatorname{tg} \varphi_m \sin Z \operatorname{tg} Z \sin 1'' (1+\alpha)$$

then

$$s = s_1 \left[1 + \frac{p s_1 + p^2 s_1^2}{2}\right]$$

is obtained.

Example :

$\varphi' = 46° \; 33' \; 51''.955$ $\qquad\qquad Z = 206° \; 20' \; 53''.7$

$\varphi = 47° \; 4' \; 57''.212$

$\varphi' - \varphi = 31' \; 5''.257$

$\log \varphi' - \varphi$	3.270 738 68	$\log P_m$	8,510 310 50	$\log R_m$	8.508 94
$\log P_m \cos Z$	8.462 673 27	$\log \cos Z$	9.952 362 77	$\log \operatorname{tg} \varphi_m$	0,027 66
$\log s_1$	4.808 065 41		8,462 673 27	$\log \sin Z$	9,647 21
$\log\left(1 + \frac{p s_1}{2} + \frac{p^2 s_1^2}{2}\right)$	9.999 490 89	$p s_1 =$	− 0,002 348 67	$\log \operatorname{tg} Z$	9,694 85
$\log s$	4,807 556 30	$p^2 s_1^2 =$	+ 0,000 005 52	$\log \sin 1''$	4,685 57
s_m	64 203^m.15	$p s_1 + p^2 s_1^2 =$	− 0,002 343 15		2,564 23
		$\frac{p s_1 + p^2 s_1^2}{2} =$	− 0,001 171 575	$\mu\alpha$	− 148
		$1 + \frac{p s_1 + p^2 s_1^2}{2} =$	0.998 828 425	$\log p$	2.562 75
				$\log s_1$	4,808 07
				$\log p s_1$	7 370 82
				$\log p^2 s_1^2$	4.741 64

Once s is calculated, it will be a very simple matter to obtain the difference of longitude between the points by formula (5) completed by means of $\mu\beta$.

This method of longitude determination is not recommended however, when (as in the example above) the geodetic arc intersects the parallels of latitude at a small angle. Thanks to the facilities now available for the determination of longitude by means of W/T, it would appear to be more advantageous in certain cases to compute the difference in latitude between two points of known longitude, as well as the latitude of one of them and the azimuth on its horizon of the geodetic line joining them.

The formulae (7) and (8) given below, furnish another simple solution of the problem, but in this case it is necessary to make use of successive approximations, as it is not possible to determine the exact values of P_m, Q_m and R' accurately except in those cases in which the latitudes of the two points are known.

2° Knowing the latitudes and the longitudes of two points, find the azimuth and the length of the geodetic line joining the two points (in the case in which the data permit the application of the LEGENDRE theorem).

Formulae (4) and (5) completed may be written:

$$(7) \quad \varphi' - \varphi = s\, P_m \cos Z - s^2\, Q_m (1+\alpha) \sin^2 Z$$

$$(8) \quad L' - L = s\, \frac{R' \sin Z}{\cos \varphi'} (1+\beta)$$

eliminating s from these two equations,

$$\cot Z = \frac{\varphi' - \varphi}{L' - L} \, \frac{R'}{P_m \cos \varphi'} (1+\beta) + \frac{L' - L}{2} \, \frac{R_m}{R'} \cos \varphi' \operatorname{tg} \varphi_m \sin 1'' \, \frac{1+\alpha}{1+\beta}$$

is obtained.

This equation permits the calculation of Z up to the 3rd order inclusive. First an approximate value Z_1, is calculated by the formula

$$\cot Z_1 = \frac{\varphi' - \varphi}{L' - L} \, \frac{R'}{P_m \cos \varphi'} (1+\beta)$$

or rather, *log. cotan* Z_2 is calculated without taking the factor $1 + \beta$ into consideration. The value Z_2 thus obtained then permits an approximate value $s\,R'$ to be computed by the formulae

$$s R' = \frac{(L' - L) \cos \varphi'}{\sin Z_2} = \frac{\varphi' - \varphi}{\cos Z_2} \, \frac{R'}{P_m}$$

Having thus obtained an approximate value of *log* $s\,R'$, Table III then gives $\mu\beta_1$, $\mu\beta_2$ and consequently $\mu\beta$ expressed in units of the eighth decimal of the logarithm; adding these to the logarithm obtained for *cotan* Z_2 gives *log cotan* Z_1.

Assume :—

$$q = \frac{L'-L}{2} \frac{R_m}{R'} \cos \varphi' \operatorname{tg} \varphi_m \sin 1'' \frac{1+\alpha}{1+\beta}$$

The logarithm of q is calculated to 5 decimals without taking the factor $\frac{1+\alpha}{1+\beta}$ into consideration, then the quantity $\mu\alpha$ which is expressed in units of the fifth decimal of the logarithm is added to it and $\frac{\mu\beta}{1000}$ is then deducted which is thus expressed in units of the same decimal ($\frac{\mu\beta}{1000}$ generally becomes negligible).

Then :

$$Z = Z_1 - \frac{q \sin^2 Z_1}{\sin 1''} \left[1 - q \sin Z_1 \cos Z_1 + q^2 \sin^2 Z_1 \left(1 - \frac{4}{3} \sin^2 Z_1\right)\right]$$

is obtained.

Instead of developing the expression of Z, it will ordinarily be quicker to make the calculation by changing the expression of *cotan* Z to the following form

$$\operatorname{cotang} Z = \frac{\varphi'-\varphi}{L'-L} \frac{R'(1+\beta)}{R_m \cos \varphi'} \left(1 + \frac{(L'-L)^2 Q_m \cos^2 \varphi' (1+\alpha)}{(\varphi'-\varphi) R'^2 (1+\beta)^2}\right)$$

which may be expressed :—

$$\operatorname{cotang} Z = \operatorname{cotang} Z_1 (1 + \varepsilon)$$

Compute the *log.* of *cotan* Z_1, to 8 decimals, and that of ε to 5 decimals, then ε and $1 + \varepsilon$ are deduced and finally *log.* $(1 + \varepsilon)$ to 8 decimals.

In the case where *cotan* Z_1 is too large in absolute magnitude, compute

$$\log \operatorname{tang} Z = \log \operatorname{tang} Z_1 - \log (1 + \varepsilon)$$

Example :

$\varphi' = 46^\circ\ 33'\ 51'',955$ $\qquad$ $L' = 0^\circ\ 26'\ 2'',613$

$\varphi = 47^\circ\ 4'\ 57'',212$ $\qquad$ $L = 0^\circ\ 3'\ 44'',752$

$\varphi' - \varphi = 31'\ 5'',257$ $\qquad$ $L' - L = 22'\ 17'',861$

$\log \varphi'-\varphi$	3,270 738 68	$\log L'-L$	3,126 411 00	$\log \cos Z_2$	9.9525	$\log \sin Z_2$	9.6468
$\log R'$	8,508 943 09	$\log P_m$	8,510 310 50	$\log P_m$	8,5103	$\log(L'-L)\cos\varphi'$	2,9637
$\log R'(\varphi'-\varphi)$	1,779 681 77	$\log \cos \varphi'$	9.837 296 97	$\log P_m \cos Z_2$	8,4628	$\log s R'$	3,3169
	1,474 018 47	$\log(L'-L)P_m\cos\varphi'$	1,474 018 47	$\log R'(\varphi'-\varphi)$	1,7797	$\mu\beta_1 =$	731
$\log \cot Z_2$	0,305 663 30			$\log s R'$	3,3169	$\mu\beta_2 =$	305
	− 426					$\mu\beta =$	− 426
$\log \cot Z_1$	0,305 659 04	$\log(L'-L)^2$	6,25282	$\log(\varphi'-\varphi)$	3,27074		7,07023
$\log(1+\varepsilon)$	9.999 490 88	$\log \cos^2\varphi'$	2,67459	$\log R'^2$	7,01789	$\mu\alpha$	− 148
$\log \cot Z$	0.305 149 92	$\log Q_m$	1,43145		0,28863	$-2\mu\beta$	+ 1
			7,35886		7,35886	$\log \varepsilon$	7,06876
$Z = 206^\circ\ 20'\ 53'',72$					7,07023	$\varepsilon =$	− 0,0011716
						$1+\varepsilon =$	0,9988284

Remarks. — If *cotan* Z_1 is greater than 1, it is better to calculate *tan* Z_1 by the formula :—

$$\tan Z_2 = \frac{L'-L}{\varphi'-\varphi}\,\frac{P_m \cos\varphi'}{R'}$$

Then deduct $\mu\,\beta$ from *log tan* Z_2 to obtain *log tan* Z_1 and then Z_1.

Knowing Z, s may be computed by one of the formulae (7) or (8). Formula (8) determines s immediately and should be preferred when *tan* Z is smaller than I in absolute magnitude, otherwise the value of s obtained by formula (8) can only be considered as an approximation for computing s^2 and for substituting s^2 in formula (7) from which s is obtained.