DEFLECTION OF THE PLUMB-LINE.

by

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1. *Geoid and the Reference Spheroid.*

The visible land surface of the earth is very irregular on account of its mountains and valleys. The ocean surface, however, if it be freed from the disturbing effects of the winds and tides is very regular, so much so that the mean sea level lends itself to accurate determination along the sea shore and forms the standard zero reference datum for all land surface heights. This ocean surface is held in equilibrium by the force of gravity attracting it towards the earth, and the rotational or centrifugal forces acting outwards; it is therefore an equipotential surface and is truly level. By which is meant that at every point the actual pull of gravity is in a direction perpendicular to the surface.

If this mean sea-level surface be supposed to be extended under the land surface by narrow sea-level canals, then this whole equipotential water surface would give a shape that is called the "geoid".

The direction of the pull exerted by the earth at any point is called the *Plumb-line direction* The geoid has the characteristic property that it is everywhere perpendicular to the Plumb-line. Owing to this characteristic this surface plays a very important role in all geodetic operations; for example, it is the datum surface from which heights of all points on the surface of the earth are reckoned.

In a geodetic survey, the first step is to measure a base on the ground, and then start a series of triangulation from it. To compute this triangulation, a knowledge of the form of the geoid is required, but this is not known at the time of survey. Even if we knew the local form of the geoid, it would be a very inconvenient surface for mathematical computations, as the irregularities produced in it by the attraction of irregular features of the earth, visible and invisible, would make the mathematical formulae very complicated.

Geodetic measurements show that the form of the geoid is approximately that of a spheroid of revolution about its minor axis, and so it has become customary now to base all the triangulation computations on an oblate spheroid of reference.

The geoid surface is elevated under the continents and depressed under the oceans.

2. *Geodetic Datum.*

The reference spheroid is fixed by 5 quantities:

$$a,\ 1/\varepsilon,\ \xi_0,\ \eta_0 \text{ and } h_0.$$

a is the length of its Semi-major axis,

ε is Ellipticity,

ξ_0, η_0 are N-S & E-W components of inclination of spheroid to the geoid at some point, chosen as the origin, h_0 is the vertical separation between the two surfaces at this origin.

The above quantities constitute a geodetic datum.

The last quantity h_0 is only important for heights, and not for latitudes and longitudes.

In Fig. 1, O is the origin, h_0 being taken equal to zero. Different countries have different geodetic datums.

In India, the spheroid of reference chosen is Everest spheroid, whose dimensions are given by

$$a = 20{,}922{,}931.80 \text{ ft.}$$

$$\varepsilon = \frac{1}{300.8}$$

At Kaliānpur origin $\quad \xi_0 = 0''.3$ S.,

$\eta_0 = 2''.9$ W.

In height the geoid and the spheroid coincide at Kaliānpur.

In United States, Canada and Mexico, the triangulation is computed on CLARKE's 1866 spheroid, whose dimensions are:

$$a = 6{,}378{,}258 \text{ metres,}$$

$$\varepsilon = \frac{1}{294.98}$$

The origin chosen is at Meades' Ranch in Kansas for the whole of North America.

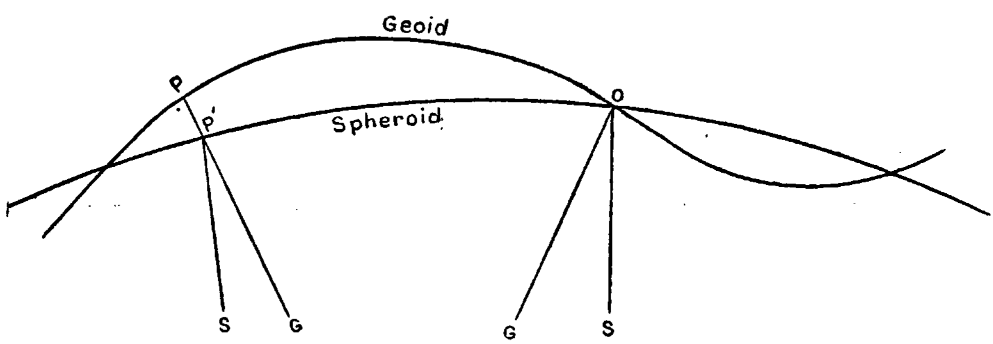

Fig. 1

3. *Plumb-line Deflections.*

Having chosen a Reference spheroid, triangulation can be computed on it by spheroidal geometry and the latitudes, longitudes and azimuths of the points so determined are called geodetic values. These quantities can be determined at a station quite independently by astronomical observations, and generally differ from the geodetic values by small amounts. We will denote the geodetic latitude, longitude and azimuth by λ_g, L_g and A_g respectively, and the corresponding astronomical quantities by the suffix *a*.

When a triangulator levels his theodolite, its vertical axis is perpendicular to the geoid, hence while the astronomical co-ordinates refer to the geoidal normal the geodetic co-ordinates refer to the spheroidal normal. Their differences $\lambda_a - \lambda_g$, $L_a - L_g$ and $A_a - A_g$ are a measure of the inclination of the two surfaces, which is called the Plumb-line Deflection.

It is usual to resolve this Plumb-line Deflection into two components, one along the meridian (denoted by ξ) and the other along the Prime Vertical (η).

In Figure 1, *OG,OS* and *PG,P'S* are normals to the geoid and the spheroid respectively at the origin *O*, and any other point *P*. *P'* is a point on the spheroid vertically below *P*. *GOS* represents the Plumb-line Deflection at *O*, and *SP'G* is the deflection at *P*.

It is a matter of simple geometry to show that

$$\begin{aligned} \lambda_a - \lambda_g &= \xi, \\ L_a - L_g &= \eta \sec \lambda, \\ A_a - A_g &= \eta \tan \lambda. \end{aligned}$$

If ξ, η are positive, then there is a Southerly and Westerly deflection, *i. e.* the Inward normal to the geoid lies to the South or West of the spheroidal normal.

It must be realised that the astronomical values are absolute quantities, while the geodetic values depend on the Reference spheroid selected. Hence, the magnitude of the Plumb-line deflections in an area depends on the Reference spheroid, which as explained above is chosen for convenience of computation.

The Plumb-line is attracted towards mountains and is deflected away from oceans on account of variation of mass. Ordinarily these deflections amount to only a few seconds of arc, but in some places at the foot of the Himalayas they are as much as one minute of arc with reference to the Everest spheroid.

4. *Isostasy.*

It has been explained above that deflections of the Plumb-line are determined from differences between astronomical and geodetic latitudes and longitudes. Now, the Plumb-line would be attracted by a big land mass, and it was first thought that these deflections were simply due to attraction of topographic features.

In India, the problem was of great interest on account of the presence of the mighty Himalayas, and the extensive Tibetan Plateau.

In 1852, Archdeacon PRATT (1) of Calcutta set about to calculate the direct effect of the Himalayas in deflecting the Plumb-line at three triangulation stations situated on the Great arc of India in longitude 77°30'. This produced the baffling result that the calculated deflections were found to be much greater than the observed ones. Thus at Kaliāna (in latitude 29°30'),

(1) Phil. Trans. of the Royal Society, 1854-59.

the northern terminus of the Indian arc, observed meridional Plumb-line deflection = 5".2, while PRATT calculated that the mass of the Himalayas would produce at this point a deflection of 27".8.

It was then that PRATT and AIRY put forward their theories. PRATT postulated that the earth was originally spheroidal, and mountains had risen by vertical expansion, so that the mountains had material of less density beneath them.

AIRY visualised the outer crust to be floating on a denser highly plastic material like closely packed icebergs at sea. The higher the mountains, the more deeply its roots penetrate in the heavier fluid.

Both these theories of compensated topography were applied to the three stations mentioned above, but failed to explain the observed deflections.

BURRARD (1) reconsidered the whole question in 1901. He had more data to work upon, namely 161 meridian deflections and 11 prime vertical deflections. He came to the conclusion that PRATT's theory of perfect compensation under the Himalayas could not by itself explain the observed deflections in India. He was led to infer the existence of an extensive hidden chain of excessive density in Central India. He found that the Plumb-lines on either side of this supposed chain were deflected towards each other.

In 1889, DUTTON read a paper on physical geology before the Philosophical Society of Washington, in which he introduced the term Isostasy to denote the condition of equilibrium of outer crust of the earth's crust. By this, he implied that all visible topographical features were compensated by deficiency of mass underneath them.

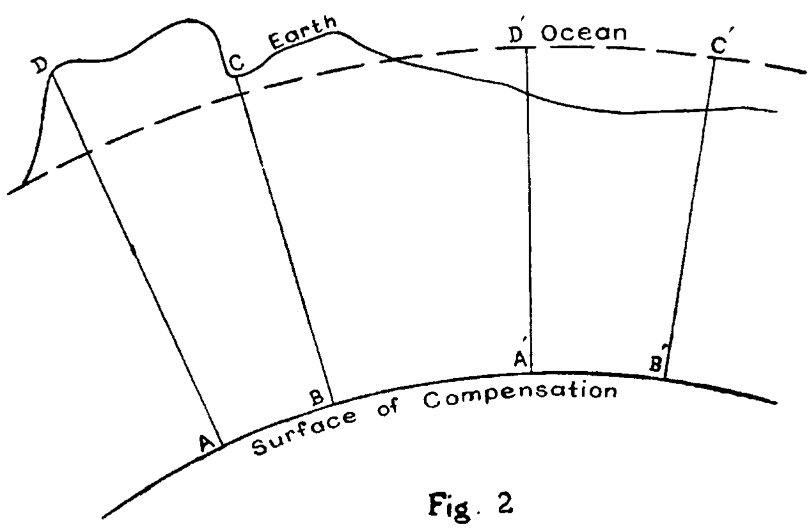

Fig. 2

HAYFORD, in 1909, developed formulae and tables to calculate the effect of the superficial masses and their compensation on Plumb-line deflections.

(1) "The attraction of the Himālaya mountains upon the Plumb-line in India". Survey of India Prof. Paper N° 5.

He assumed that on a certain surface, called the surface of compensation which is about 60 miles below sea-level, the earth's crust exerts a uniform pressure.

In Fig. 2, *ABCD, A'B'C'D'* represent two columns having unit area of cross-section and bounded by vertical planes. HAYFORD's theory of Isostasy is that the amount of mass in these columns is the same even though they are of different height, and they exert the same pressure on the compensation surface. It follows that the matter under the oceans is of higher density than that under the continents.

5. *Problem of the figure of the Earth.*

This problem consists in finding that spheroid of reference which gives a best fit to the geoid, and the latest solutions have made use of the theory of Isostasy. Earlier attempts at this problem consisted in utilising lengths of meridian arcs determined by triangulation with astronomical determinations of latitude. Thus, at the end of 18th century, LAPLACE (1) derived elements of the earth's figure from a discussion of 7 arcs of meridian.

CLARKE in 1866 determined elements of spheroid by using meridian arcs only, but later on, in 1880, he used arcs of parallel too, and recalculated the figure of the earth.

EVEREST (2) in 1830 used two meridian arcs, and determined the elements of the best-fitting spheroid.

The triangulation is computed on one of the spheroids so determined in various countries.

HAYFORD (3) in 1909 determined the figure of the earth from Plumb-line deflections in United States, reduced on the basis of Isostatic compensation. His method is as follows:-

The Plumb-line deflections can arise from four causes : (*a*) from the errors δ_a, $\delta\varepsilon$, $\delta\xi_0$, & $\delta\eta_0$ in the geodetic datum chosen, (*b*) from errors in astronomical observations, *i. e.* in λ_a, L_a and A_a, (*c*) from errors generated in the triangulation, *i. e.* accumulated error in geodetic azimuth and consequently in latitude and longitude, and (*d*) from the actual deflections of the vertical due to the effect of the whole topography.

Of these the most important are (*a*) and (*d*). (*d*) can be allowed for by assuming some sort of supplementary hypothesis like Isostasy.

HAYFORD worked out 5 least square solutions which he designated by *B,E,H,G,* and *A* to determine the most probable values of the constants representing the earth. Solution *A* is based upon the assumption that depth of compensation $T = 0$, solution *B* assumes no Isostatic compensation, *i. e.* $T = \infty$, solutions *E,H,G,* suppose compensation to be uniformly distributed throughout the depths 162.2, 120.9 and 113.7 kms. respectively.

He found that the residuals, *i. e.* those portions of the deflections of the vertical which remained unexplained after each solution, were least for solution *H*,

(1) Mécanique Céleste, Part I, Book III, Chapter V.
(2) G. T. Survey of India, Vol. II, p. 125.
(3) J. F. HAYFORD : "The Figure of the Earth and Isostasy from the Measurements in the United States".

which therefore was accepted as furnishing the most probable values of a and ε.

The values found were

$$a = 6{,}378{,}388 \text{ metres} \pm 0.018$$

$$\frac{1}{\varepsilon} = 297.0 \pm 0.5$$

HAYFORD based his work on deflections in the United States, but it was believed that the use of Isostasy had given a spheroid from the limited area of the United States that would fit the whole earth more closely than any other spheroid. The International Union of Geodesy and Geophysics at Madrid accepted in 1925 the HAYFORD figure as the International Spheroid of Reference. This adoption has not found general favour yet, as it is believed that although this figure fits the area of United States tolerably well, it does not fit Europe at all well (1).

6. *Form of Geoid.*

When an area has been fairly covered with stations at which the Plumb-line deflections have been observed, it becomes possible to mark out the geoid for that area. This has been done in United States (2) and in India (3).

Briefly put, the method consists of integrating the two components of deflection and finding the separation between the geoid and spheroid. Having found this separation at a number of points, contours can be drawn, showing the form of the irregular geoid with reference to the chosen spheroid.

The following simple example will serve as an illustration. Consider an arc of triangulation running along a meridian and suppose there are five stations of this triangulation, at which the meridional components of deflection have been observed. Columns 2 and 3 of the following table give the deflections and distances between the stations.

TABLE.

Station.	*Plumb-line deflection* (ξ)	*Distance in miles.*	*Rise of geoid ft.*
1	+ 36"		
		30	
2	+ 10"		— 17,2
		40	
3	— 2"		— 6,0
		50	
4	+ 4"		+ 3,7
		25	
5	+ 20"		+ 5,0

(1) The Geographical Journal, June, 1927 "A Graphical Discussion of the Figure of the Earth", by A. R. HINKS.

(2) "The Figure of the Earth and Isostasy from Measurements in the United States", by J. F. HAYFORD, 1909, p. 57.

(3) Survey of India, Geodetic Report, Vol. I, p. 8.

The rise of the geoid from station to station is given in the last column. It is based on the simple fact that if the deflection is 1 second, the geoid would differ from the spheroid by 1 foot at a distance of 40 miles. An essential assumption involved is that, between two stations, the deflection varies linearly; thus the average deflection between stations 1 and 2 is taken to be $\frac{36 + 10}{2} = 23''$. In plains, where the deflections are usually small and vary very slowly, this assumption is not a serious source of error. But in mountainous country, deflections vary irregularly, and if the stations are very far apart, interpolation may give quite a wrong picture of the geoid. In such a case deflection stations have to be put close to each other. An apt example is afforded by the three Sub-Himalayan stations Dehra Dūn, Rājpur and Mussoorie.

Observed meridian deflections at Dehra Dūn and Mussoorie which are about 15 miles apart are nearly the same, $-37''$. Rājpur lies midway between these stations and the deflection there is $-48''$. Obviously a simple interpolation is hopeless in this case.

7. *Effects of the Deflection of the Vertical.*

(i) From para. 4, it is obvious that the choice of deflections at the origin of the survey is a question of successive approximations. We can start provisionally with any spheroid and assume deflections at the origin to be nil. When a triangulation net of sufficient extent has been executed and computed we can then get deflections at the origin from the criterion that for the spheroid which best fits the geoid locally, mean deflection over the whole country should be zero.

An example is afforded by the deflections at the Kaliānpur (1) origin for India, which have been changed from time to time.

(ii) When a theodolite is levelled at a station, its horizontal circle is parallel to the tangent plane to the geoid at the station. The horizontal angles measured with a theodolite thus refer to the tangent planes to the geoid. Since the geoid and spheroid are generally inclined to each other, all observed horizontal angles should be corrected for the deflection of the vertical before they can be used for the computation of triangulation.

The Plumb-line deflections are generally unknown at the time of computation of triangulation, and so correction on this account has been ignored as a rule. Fortunately this correction is negligible in most cases. There are, however, cases where it is imperative to correct the horizontal angles for this dislevelment, which becomes particularly large when rays are steeply elevated, as is often the case in Base-line extension. This source of error is not likely to give large triangular errors.

There is one horizontal angle in Indian triangulation, namely the angle at Dehra Dūn Observatory between Banog H.S. and Sirkanda H.S., which requires a correction of $4\frac{1}{2}''$ on this account. This is, however, a very extreme case.

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(1) See Supplement to Geodetic Report, Survey of India, Volume VI, p. vi.

(iii) It has been mentioned in para. 2, that each country has its own Reference spheroid, on which triangulation is computed. These spheroids differ considerably from each other, but being derived from meagre data do not fit the respective areas very well.

Thus, the Everest spheroid which was selected about a hundred years ago deviates from the geoid in India by as much as 150 feet in Baluchistān and Northern India. In spite of this, it is likely to remain in use indefinitely, as a change to another spheroid involves republishing of all data, and would entail vast expense. Similarly, for geographical purposes, the Americans are likely to stick to the Clarke spheroid, as for general cartographical purposes any reference spheroid is good enough.

From a scientific point of view, however, it is essential to determine with ever-increasing precision the dimensions of the best fitting spheroid. For this purpose we have to choose a figure, which fully expresses our existing knowledge.

Having found a new spheroid, the Plumb-line deflections on the original spheroid have to be converted to this one. The method of doing this is explained in supplement to *Survey of India Geodetic Report* VI, p. x.

The most accurate spheroid would be obtained when a sufficient extent of globe has been surveyed, and when all the detached surveys of the individual continents are linked together and adjusted as a unit whole.

Having found a satisfactory spheroid, the undulations of the geoid with respect to it can be traced as in para. 5, and this gives rise to one of the major problems of geodesy, namely, to determine the irregularities of densities in the earth's crust which will explain these undulations.

In this connection, it may be mentioned that the measurement of the force of gravity by means of pendulums also assists greatly in the investigation of this problem.

8. *Summary.*

To put the matter briefly:

(*a*) The geoid is the true or equipotential surface of the earth, and the direction of the Plumb-line, or true vertical, is always perpendicular to this surface.

(*b*) For computation purposes a spheroid of reference has to be assumed for the figure of the earth, and the departure of the geoid from this spheroid can be calculated by the deviation of the Plumb-line, or vertical. This is obtained from astronomical observations combined with triangulation data.

(*c*) The shape of the geoid is largely influenced by the topographical irregularities of the earth's surface, but still more by the variations of density in the outer crust. Various theories of isostasy have been put forward to account for the undulations of the geoid, and the irregular deviations of the Plumb-line.