THE POSITION AT SEA
BY RADIOGONIOMETRIC BEARINGS TAKEN ON BOARD.

by

INGÉNIEUR HYDROGRAPHE GÉNÉRAL P. DE VANSSAY DE BLAVOUS, DIRECTOR.

(1) *INFLUENCE OF THE COMPRESSION OF THE TERRESTRIAL ELLIPSOID.*

The problem of position-finding by radiogoniometric bearings taken on board differs notably from that of position-finding by means of altitudes of heavenly bodies. In the latter case, the observation of an altitude provides a geometric locus of the observer's position which can be represented by a small circle of the celestial sphere. The observation of a second body provides a second locus the intersection of which with the first gives a point on the celestial sphere whose geographical position is the same as that of the observer on the geoid. One has only to consider the shape of this surface when plotting on it the results of the calculations which have been made with the strictest accuracy on the sphere.

A radiogoniometric bearing gives us a geometric locus of the observer's position on the geoid, called the "curve of equal azimuth"; but the latter has not the simplicity of the circle of altitude. The angle given by the bearing is based on the one hand on the direction of the meridian, originating in principle from an astronomical observation the result of which is preserved on board by the magnetic compass, or from the direction furnished by the gyro-compass; on the other hand on the direction of optimum reception of the electric wave. The path followed by the latter after leaving the transmitting station is not very well known. If it behaved like a wave of light, its direction would be that of the normal section at the place of observation passing through the transmitting station — a well-known problem in geodesy. It would also be possible, if the wave were propagated through the land or sea, for it to come by the shortest route, i.e. the geodesic passing through the transmitting and the receiving stations. It would rather seem from experiments that the wave is propagated through the air, being reflected once or several times in the upper layers. To what extent do these reflections divert the wave from the plane of the normal section ? It is hard to say, but certain anomalies observed in the direction of reception can be explained in this way.

Further, on account of the metal of the ship, the wave will undergo another deflection, which certain appliances can enable us to minimise and control and which we know how to take into account.

Finally, on board ship the vertical around which the two directions defining the azimuthal angle are measured is often falsified by the motion of the ship and the imperfection of the mounting ; for there is no means of defining the vertical with the same excellent accuracy as is furnished by the sea horizon in measuring the altitude of heavenly bodies. This can cause an error in the azimuth which may attain some ten minutes.

For these various reasons, let alone the inaccuracy of the direction of maximum or minimum audibility which the electrical technicians are trying to reduce, wireless direction-finding cannot claim a degree of accuracy comparable to that of geodetic or astronomical measurement. But without a doubt this accuracy will improve, and it is not wihout interest, at least from the theoretical point of view, to find out what errors are introduced by the method of calculation used for the solution of the problem of radiogoniometric bearings ; more particularly as this error is the same in the case of a shore station, taking bearings of a transmitting ship under the best conditions of accuracy.

The problem cannot be treated like the one presented in geodesy in the study of the spheroidal triangle, because in the present case the triangle pole-transmitter-receiver is of far greater size then even the largest geodetic triangles. In the same way the use of a conformal projection of the ellipsoid on the sphere cannot help us, since for large values of the sides the angular deformation is no longer negligible. It is therefore customary to work out the calculations on the celestial sphere, as for an astronomical position, plotting the transmitting and receiving stations by their geographical latitudes and longitudes. This means the adoption for the azimuthal direction of that of a plane parallel to the verticals through these stations. This direction differs from that of the normal section and also from that of the geodesic joining the two stations. If Z is the azimuth of the geodesic, to calculate on the celestial sphere we must adopt an angle $Z + \varepsilon$.

The value ε does not become zero if the two stations are very close to one another ; in such a case it tends towards a limit which, for $Z = 45°$ and $L_0 = 0$, corresponds to about 11.5'. This could be foreseen, for the transformation carried out when passing from the ellipsoid to the sphere is not conformal.

Let P be the pole, Q the transmitting station at a latitude L, M the receiving station at a latitude L_0, G the difference of their longitudes, e the eccentricity of the terrestrial ellipsoid (see Fig. 1 *a*).

The term independent of the distance of the transmitting station can immediately be obtained by considering TISSOT's indicatrix. Let us describe on the ellipsoid an infinitely small circle with centre M and radius ds. Calling its radius of curvature ρ, we will have on the meridian

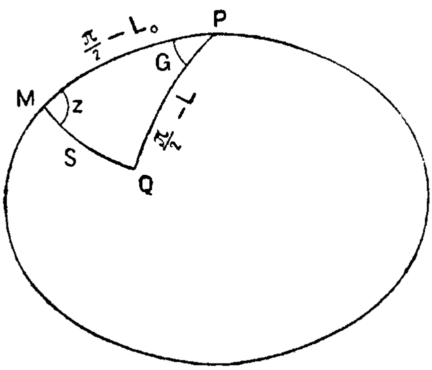
$$ds = \rho \, dL_0,$$

and on the parallel, calling the major normal ρ',

$$ds = \rho' \cos L_0 \, dG.$$

On the celestial sphere, this small circle will become an ellipse, the major axis of which will lie on the meridian and have the value

$$dL_0 = \frac{ds}{\rho} \, ;$$



Ellipsoid — FIG. 1*a*.

whilst the minor axis will lie on the parallel and have the value

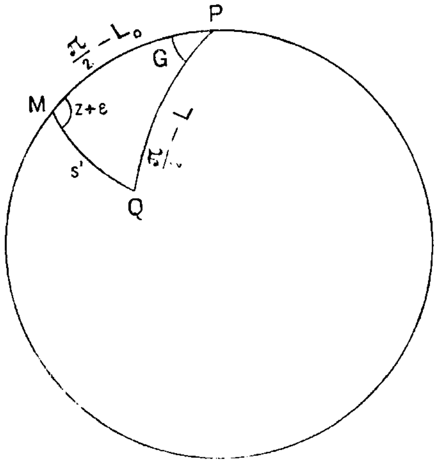
$$\cos L_0 \, dG = \frac{ds}{\rho'}.$$

We shall thus have

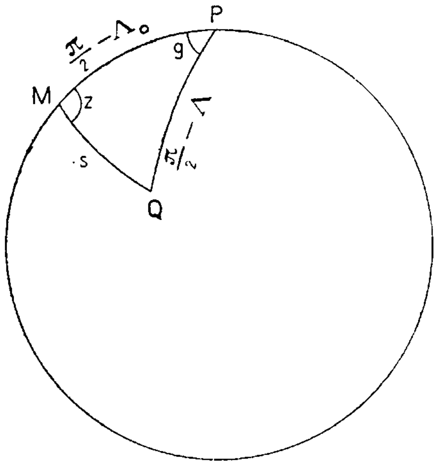
$$\tan (Z + \varepsilon) = \frac{\rho}{\rho'} \tan Z = \frac{1 - e^2}{1 - e^2 \sin^2 L_0} \tan Z.$$

Whence

$$\tan \varepsilon = - e^2 \frac{\cos^2 L_0 \sin Z \cos Z}{1 - e^2 (1 - \cos^2 Z \cos^2 L_0)}$$



Celestial sphere — FIG. 1 *b*



Jacobi's sphere — FIG. 1 *c*

To complete the expression of *tan* ε when the distance of the transmitting station is not negligible, we shall have recourse to JACOBI's auxiliary shpere relative to the point M (see Fig. 1*c*). We know that on this sphere the latitudes L_0 and L of the points M and Q are replaced by the reduced latitudes Λ_0 and Λ, which values are linked to L_0 and L by the relations

$$\tan \Lambda_0 = \sqrt{1 - e^2} \tan L_0, \qquad \tan \Lambda = \sqrt{1 - e^2} \tan L.$$

The geodesic MQ, of length S, is transformed into a great circle arc of length s, making the same angle Z with the meridian (we will suppose the major axis of the terrestrial ellipsoid and the radii of the spheres to be equal to unity).

The triangle MPQ of the ellipsoid has Z and G as its angles at M and P; that of the celestial sphere (see Fig. 1*b*) $Z+\varepsilon$ and G; that of JACOBI's sphere Z and g.

We can put $G = g - \eta$, and take for η the expression

$$\eta = \frac{e^2}{2} s (1 + \varphi) \sin Z \cos L_0 ,$$

φ being a quantity containing $\frac{e^2}{2}$ as a factor. Its expression has been given by HELMERT and completed by FICHOT (*Annales Hydrographiques*, Paris, 1921, p. 115). The same applies to the expression of S as a function of s (p. 113).

On JACOBI's sphere, the ordinary formula of the triangle gives

$$\sqrt{\frac{1 - e^2 sin^2 L_0}{1 - e^2}} \, cot\, Z \; sin\; g = tan\, L\, cos\, L_0 - sin\, L_0\, cos\, g.$$

On the celestial sphere we shall have

$$tan\, L\, cos\, L_0 = cot\,(Z + \varepsilon)\, sin\,(g - \eta) + sin\, L_0\, cos\,(g - \eta).$$

Adding these two equations member by member, we get

$$\sqrt{\frac{1 - e^2 sin^2 L_0}{1 - e^2}}\, cot Z\, sin\, g = cot\,(Z + \varepsilon)\,(sin\, g\, cos \eta - sin \eta\, cos\, g) + sin\, L_0 \left[cos\, g\,(cos \eta - 1) + sin\, g\, sin \eta \right].$$

Further, on JACOBI's sphere we have

$$cot\, g = \frac{cot\, s\, cos\, L_0 - \sqrt{1 - e^2}\, cos\, Z\, sin\, L_0}{\sqrt{1 - e^2 sin^2 L_0}\, sin\, Z}.$$

Applying this value to the previous equation, there emerges

$$\sqrt{\frac{1 - e^2 sin^2 L_0}{1 - e^2}}\; cot\; Z = cot\;(Z + \varepsilon)\left(1 - 2\, sin^2 \frac{\eta}{2} - sin\, \eta\, \frac{cot\, s\, cos\, L_0 - \sqrt{1 - e^2}\, cos Z\, sin\, L_0}{\sqrt{1 - e^2 sin^2 L_0}\, sin\, Z}\right)$$

$$+ \, sin\, \eta\, sin\, L_0 - 2\, sin^2 \frac{\eta}{2}\, sin\, L_0\, \frac{cot\, s\, cos\, L_0 - \sqrt{1 - e^2}\, cos\, Z\, sin\, L_0}{\sqrt{1 - e^2 sin^2 L_0}\, sin\, Z}.$$

This equation enables us to calculate ε accurately, after computing s in the triangle of JACOBI's sphere.

We still have to replace φ by its development in terms of the powers of e^2; the following is the expression of its first term :

$$\varphi = \frac{e^2}{8}\left[1 + cos^2 L_0\, sin^2 Z - \frac{sin\, s}{s}\,(sin^2 L_0 - cos^2 Z\, cos^2 L_0) - 2\, \frac{sin^2\, s}{s}\, cos\, Z\, sin\, L_0\, cos\, L_0\right].$$

By confining ourselves to terms of e^2 we shall have a simple expression of $tan\, \varepsilon$:

$$tan\, \varepsilon = - e^2 cos^2 L_0\, sin Z\, cos Z + \frac{e^2}{2}\left(1 - \frac{s}{tan\, s}\right) cos^2 L_0\, sin\, Z\, cos\, Z + \frac{e^2}{2}\, s\, sin\, Z\, sin\, L_0\, cos\, L_0,$$

in which we can use indifferently the value s of JACOBI's sphere, or the true value S (true distance divided by a mean value of the radius), or the value s' calculated on the celestial sphere.

We have confined ourselves hitherto to considering the observed azimuth Z as being that of the geodesic joining the observing and transmitting stations. If this azimuth were that of the normal section, the value which we have just indicated for ε would have to be diminished by the quantity δ, the expression of which, again neglecting terms of e^4 and further neglecting the terms of $e^2\, s^3$, may be written

$$\delta = \frac{e^2}{2}\, \frac{s^2}{3}\, cos^2 L_0\, sin\, Z\, cos\, Z.$$

It will be seen that it is possible in this case to suppress the second term of our expression of *tan* ε, without the error exceeding about 1', if s does not exceed 30°, and to adopt the formula

$$tan\ \varepsilon = \frac{e^2}{2} \sin Z \cos L_0 (s \sin L_0 - 2 \cos L_0 \cos Z).$$

The following procedure makes it possible to obtain this angle ε, relative to the azimuth Z of the normal section, more directly :—

The vertical plane, of azimuth $Z + \varepsilon$, which at the point M is parallel to the normal to the ellipsoid at Q, is defined by the relation

$$\cot (Z + \varepsilon) = \frac{\sin L \cos L_0 - \cos L \sin L_0 \cos G}{\cos L \sin G}.$$

The plane normal to the ellipsoid at M and passing through the point Q is defined by

$$\cot Z = \frac{\sqrt{1-e^2\sin^2 L}}{\cos L \sin G}\left[(1-e^2)\cos L_0\left(\frac{\sin L}{\sqrt{1-e^2\sin^2 L}}-\frac{\sin L_0}{\sqrt{1-e^2\sin^2 L_0}}\right)-\sin L_0\left(\frac{\cos L \cos G}{\sqrt{1-e^2\sin^2 L}}-\frac{\cos L_0}{\sqrt{1-e^2\sin^2 L_0}}\right)\right].$$

Subtracting these two equations member by member, we get

$$\frac{\tan \varepsilon}{\tan Z + \tan \varepsilon} = - e^2 \sin Z \cos Z \frac{\cos L_0}{\cos L \sin G}\left(\sin L - \sqrt{\frac{1 - e^2 \sin^2 L}{1 - e^2 \sin^2 L_0}} \sin L_0\right).$$

tan ε being a quantity of the order of e^2, we shall have, if we neglect the terms of e^4,

$$\tan \varepsilon = - e^2 \sin^2 Z \frac{\cos L_0}{\cos L} \frac{\sin L - \sin L_0}{\sin G},$$

a very simple formula to calculate, which will furnish ε if the exact position of the point M is known. But it is necessary to express *tan* ε as a function merely of three of the data of the spheroidal triangle. To do this, it is sufficient to eliminate G between the equations which give $\cot (Z + \varepsilon)$ and $\cot Z$. Confining ourselves to the terms of e^2, we get

$$\tan \varepsilon = - e^2 \sin Z \frac{\cos L_0}{\sin L + \sin L_0}\left(\cos Z \sin L \cos L_0 \pm \sin L_0 \sqrt{\cos^2 L - \cos^2 L_0 \sin^2 Z}\right).$$

The normal section from M in the direction Z is intersected by the parallel of latitude L at two points Q_1 and Q_2. The positive sign of the above formula corresponds to the point Q_1, the nearer to M, and the negative sign to the point Q_2, the further from M. These two points merge if the parallel L is tangent to the normal section at its vertex; the radical of the expression of *tan* ε is then zero. If the latitudes L and L_0 of the points M and Q are equal, without these points coming together, the negative sign must be employed; the value of ε is strictly zero. But if the points M and Q are very close to one another, the positive sign must be employed and we find ourselves back at the expression already given for this particular case.

If $Z = 0$, the correction ε is always zero.

In the Northern hemisphere, if we imagine the point Q starting at M and receding along a normal section of azimuth $Z < 90°$, ε starts with a negative value, then diminishes in absolute value to zero, which is reached when the point Q, having passed the vertex, has reached the latitude L_0 again. Afterwards ε becomes > 0, increases slowly, then, when approaching the nadir of point M, increases rapidly and, at this point, reaches the value $180 - Z$.

If $Z = 90°$ the equation becomes

$$tan\ \varepsilon = - e^2 \frac{cos\ L_0}{cos\ L} \frac{sin\ L - sin\ L_0}{sin\ G} = - e^2\ sin\ L_0\ cos\ L_0 \sqrt{\frac{sin\ L_0 - sin\ L}{sin\ L_0 + sin\ L}}.$$

ε starts from the value zero, then assumes a value > 0 and reaches the value $90°$.

If $Z > 90°$, ε starts with a positive value, taking the positive sign before the radical as far as the lower vertex when the radical becomes zero ; afterwards the negative sign is used ; ε continues to increase up to the value $180 - Z$ at the nadir of the point M. In the neighbourhood of this point, i.e. when the angle G, the difference of longitude between M and Q, approaches $180°$, it is no longer permissible to neglect the terms of e^4, for the denominator $sin\ L + sin\ L_0$ becomes of the order of e^2.

In the case when $L_0 = 0$, i.e. if the point M is on the equator, the two equations which give $cot\ (Z + \varepsilon)$ and $cot\ Z$ become

$$cotg\ (Z + \varepsilon) = \frac{tan\ L}{sin\ G}, \qquad cotg\ Z = (1 - e^2) \frac{tan\ L}{sin\ G}.$$

The value of ε will be the same over the whole extent of the normal section ; it now depends merely on Z :

$$tan\ \varepsilon = - e^2 \frac{sin\ Z\ cos\ Z}{1 - e^2\ sin^2\ Z}.$$

The point on the celestial sphere corresponding to Q describes another normal section of azimuth $Z + \varepsilon$.

In the equations which give $cot\ (Z + \varepsilon)$ and $cot\ Z$, Z may be eliminated ; we then have the value of ε as a function of the latitudes and the difference of longitude (complete expression) : $tan\ \varepsilon =$

$$e^2 sin\ G\ cos\ L\ cos\ L_0 \frac{sin\ L_0 \sqrt{\frac{1 - e^2 sin^2 L}{1 - e^2\ sin^2\ L_0}} - sin\ L}{1 - (sin\ L\ sin\ L_0 + cos\ L\ cos\ L_0 cos\ G)^2 + e^2 cos\ L_0\ (sin\ L_0 \sqrt{\frac{1 - e^2 sin^2 L}{1 - e^2 sin^2 L_0}} - sin\ L)(sin\ L\ cos\ L_0 - sin\ L_0\ cos\ L\ cos\ G)}$$

or, denoting by s' the arc MQ of the celestial sphere, $tan\ \varepsilon =$

$$e^2 sin\ G\ cos\ L\ cos\ L_0 \frac{sin\ L_0 \sqrt{\frac{1 - e^2\ sin^2\ L}{1 - e^2\ sin^2\ L_0}} - sin\ L}{sin^2\ s' + e^2\ cos\ L_0 (sin\ L_0 \sqrt{\frac{1 - e^2\ sin^2\ L}{1 - e^2\ sin^2\ L_0}} - sin\ L)\ (sin\ L\ cos\ L_0 - sin\ L_0\ cos\ L\ cos\ G)}$$

and neglecting the terms of e^4,

$$\tan \varepsilon = - e^2 \sin G \cos L \cos L_0 \frac{\sin L - \sin L_0}{\sin^2 s'}.$$

(2) *CALCULATION OF THE POSITION.*

We shall thus make our calculations on the sphere, locating the positions on it by their geographical situations, and if necessary replacing the observed azimuth Z by the value $Z + \varepsilon$ which we have just indicated ; but from now on, for the sake of simplicity in the formulae, we shall use Z to denote the azimuth in the form in which we use it on the sphere.

The locus of the points of equal azimuth on the sphere is a curve of rather complicated form, which we will discuss later. Applying the same principles as for the astronomical calculation of the position at sea, we shall replace it by one of its tangents in the neighbourhood of the estimated position, and we shall calculate the direction of this tangent and its distance from the estimated position.

We shall further take it that the estimated position is near enough to the true position for all the small quantities of the second order to be neglected. If it were otherwise, it would be much simpler to begin the work afresh, after obtaining a prior rectification of the position (which can incidentally often be done graphically), than to apply somewhat complicated corrections. If this consideration is fulfilled, the curve of equal azimuth can be treated without error as its tangent, and the graphic construction of this tangent by means of the results of the calculation can be carried out on the assumption that the earth is plane in the neighbourhood of the estimated position, without having recourse to any particular type of projection.

We shall denote the angle PQM by ω and the arc MQ calculated on the sphere by s (see Fig. 2).

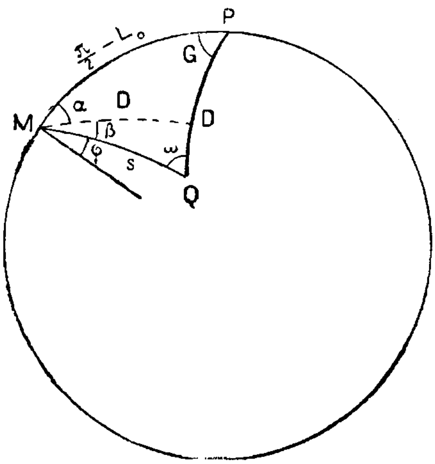


FIG. 2

It is easily shown that at the point M the tangent to the curve of equal azimuth makes an angle φ with the great circle arc MQ, which can be calculated by one of the two important formulae :

$$(1) \quad \begin{cases} \tan \varphi = \tan G \sin L_0 \\ \tan (Z + \varphi) = - \tan \omega \cos s. \end{cases}$$

If the estimated position answers the conditions we have mentioned, there is no inconvenience in using the values of G, L_0, ω and s in these formulae which correspond to the estimated position, for which the azimuth will have a value which we shall call Z'. The quantities ω and s are easily calculated in the spherical triangle MPQ ; but it is an advantage in this case to resolve it into two right-angled triangles by a great circle MD, perpendicular to PQ. Let α and β be the angles which it makes with MP and MQ ; we shall have

$$(2)\quad \begin{cases} \cot \alpha = \tan G \sin L_0 = \tan \varphi \\ \varphi = 90 - \alpha. \end{cases}$$

We shall then calculate the arc $MD = D$ and the arc PD by the formulae

$$(3)\quad \begin{cases} \sin D = \cos L_0 \sin G \\ \tan PD = \cot L_0 \cos G \\ DQ = 90 - (L + PD) \end{cases}$$

and finally,

$$(4)\quad \begin{cases} \cos s = \cos D \sin (L + PD) \\ \cot \omega = \cot D \cos (L + PD) \\ \cot \beta = \tan (L + PD) \sin D = \tan \omega \cos s = - \tan (Z' + \varphi) \\ Z' + \varphi = 90 + \beta \qquad Z' = \alpha + \beta. \end{cases}$$

All these formulae are very simple to calculate, but numerous tables exist which give the solutions without calculations.

Z' being given by formulae (4), we shall have the difference dZ between the observed azimuth Z and the calculated azimuth at the estimated position Z',

$$dZ = Z - Z'.$$

The distance δ of the estimated position from the tangent of azimuth $Z + \varphi$ will be

$$\delta = \frac{\tan s \cos \beta}{\sin Z} dz = \frac{\tan D}{\sin Z} dz,$$

this distance being laid off in the direction $Z + \varphi - 90 = \beta$, which is the bearing symmetrical with MD about the bisectrix of the angle PMQ. (1)

The tangent can also be laid off by its point of intersection with the meridian to a distance from the estimated position equal to

$$d\,L_0 = \frac{\tan s}{\sin Z} dz,$$

or again by the point where it meets the great circle MQ to a distance from the estimated position equal to

$$\frac{\cot L_0}{\sin Z} dz.$$

The accuracy of the geometric locus depends on the accuracy with which the angle Z is measured; an error of 1° in Z displaces the tangent by a quantity, expressed in degrees, equal to the arc

$$\frac{\tan D}{\sin Z}.$$

(3) *USE OF A MERCATOR PROJECTION.*

The curves of equal azimuth in the projection most commonly used for charts, that of MERCATOR, are transcendent curves whose systems one cannot hope to lay off on the chart when the latter includes several transmitters; but it will often be possible to provide a given chart with a special tracing which will enable these curves to be very easily determined by points

(1) A very elegant geometrical demonstration of this property, and also of that discussed in para. 5, will be found in the article : *"Le Segment capable sphérique"* by Mons. G. LECOQ, Annales Hydrographiques, Paris 1933.

in the neighbourhood of the estimated position. It is well-known, in fact, that if the transformations in the system of MERCATOR'S projection of a series of great circles are plotted on a tracing, it is sufficient to displace this tracing perpendicularly to the meridians, keeping the latitudes constant, to obtain on the chart the representation of every possible great circle. During this displacement, at the same point of a transformation, the great circle which it represents always makes the same angle with the meridian, and by joining on the tracing the points of the different transformations where these make the same angle with the meridians, one obtains the curves corresponding to the equal azimuths. If one has taken a bearing of the station Q of azimuth Z, one will, by a suitable movement of the tracing, cause one of the transformations to pass through the station and through the neighbourhood of the estimated point, and will prick on the chart the point of the transformation corresponding to the azimuth Z. The same operation will be carried out for one or two others of the neighbouring transformations, making them pass each time through the station Q. Joining the points thus pricked will give an element of the desired curve of equal azimuth.

The tracing must contain all the portions of the transformations which extend between the extreme latitudes of the chart. Thanks to the possibility of reversing it, it will be sufficient for this purpose that it should cover 90° of longitude. The use of this method is thus limited to route charts of which the longitude scale allows of a tracing of acceptable dimensions (2 cm. for 1° of longitude as a maximum).

A less precise, but very simple, method has been devised in the case where the distance of the ship from the transmitting station is not too great; i.e. when the arc s can be considered as an infinitesimal of the first order, and the second order can be neglected. In this case the quantities G and $L - L_0$ will also be of the first order, and the angle φ, given by formula (2), can be merged with the convergence of the meridians. The latter is, besides, twice the correction, known as the GIVRY correction, which is the difference between the spherical and loxodromic azimuths.

In these conditions, it can be said that the tangent at the point M to the curve of equal azimuth, formed by the loxodrome passing from M to Q, makes an angle equal and opposite to that made by this loxodrome with the great circle arc MQ.

At the ship's position, the azimuth of the loxodrome passing through Q is then $Z + \frac{\varphi}{2}$; we shall draw it through the point Q on our chart and it will meet the meridian of the estimated position at a point on the curve of equal azimuth. Through this point we shall draw a straight line of azimuth $Z + \varphi$ which is the desired tangent.

The construction of the radiogoniometric straight line on the chart thus only requires the use of the GIVRY correction, which is very easy to work out and is also furnished by many tables or by diagrams.

This procedure, the simplest and most convenient of all, cannot be used at any distance from the transmitting station. If we develop as a series the expression $A - Z$, being the difference between the loxodromic and the orthodromic azimuths, we find

$$A - Z = \frac{1}{1} G \sin L + \frac{1}{12} G^2 \left(1 + 3 \cos^2 L\right) \cot \omega.$$

Substituting $\frac{L + L_0}{2}$ for L, as is usually done in calculating the convergence of the meridians and GIVRY'S correction, the formula becomes

$$(5) \qquad A - Z = \frac{1}{2} G \sin \frac{L + L_0}{2} + \frac{1}{12} G^2 \cot \omega.$$

In these expressions the angles are expressed as parts of the radius, and powers of G greater than the second power have been omitted (see final note).

Applying the GIVRY correction to the angle Z, an error is thus committed which is in the neighbourhood of $\frac{1}{12} G^2 \cot \omega$ and also of $- \frac{1}{12} G^2 \cot Z$ or of $- \frac{1}{12} G^2 \cot A$.

The manner in which this error varies will be realised by noticing that it is proportional to the surface of the right-angled triangle whose hypotenuse is the loxodrome joining the ship to the transmitting station and whose other sides are the parallel and the meridian of these stations (see Fig. 3).

The following table gives the value in degrees of the error committed for a difference of longitude of 1° when the GIVRY correction is used, for different values of Z (or of ω) :

$$A - Z = \frac{1}{2} G \sin \frac{L + L_0}{2}.$$

Multiplying the numbers of the table by the *square* of the difference of longitude, expressed in degrees, we obtain the error of Z corresponding to the position of the ship. We can thus ascertain whether the second term of the expression (5) can be neglected. On the other hand, by taking this term into account, if the need arises, the application of this very simple method can be carried further.

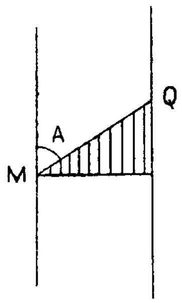


FIG. 3

Z°	*Error in degrees for 1° difference of long.*
10°	0,0082
20	0,0040
30	0,0025
40	0,0017
50	0,0012
60	0,0008
70	0,0005
80	0,0003

The term $\frac{1}{12} G^2 \cot \omega$ can also be expressed as a function of the distance s of the ship from the transmitting station, in which case it can be replaced by

$$\frac{1}{24} s^2 \frac{\sin 2\omega}{\cos^2 L}.$$

But this expression lends itself less well to presentation in the form of a table.

The Coast and Geodetic Survey of the United States of America in 1921 published three sheets of diagrams, n^os *R1*, *R2* and *R3*, and a special publication N° 75, by Oscar S. ADAMS, which enable $A - Z$ to be obtained more accurately than by formula (5). Their use necessitates a little more time, but guarantees a degree of accuracy which will always suffice for radiogoniometric measurements.

NOTE. — When the expression $\tan (A - Z)$ is developed by means of the MACLAURIN formula in terms of the successive powers of G for a single value of the angle ω, a development is obtained which falls into one of the five following forms, which are expressed as far as the terms of the third degree (inclusive) of G:

(a) $$\tan (A - Z) = \frac{G}{2} \sin L + \frac{G^2}{12} \cot\omega (1 + 3\cos^2 L) - \frac{G^3}{24} \sin L \left[3\cos^2 L + \cot^2\omega (1 + 6\cos^2 L)\right]$$

(b) $$\tan (A - Z) = \frac{G}{2} \sin \frac{L + L_0}{2} + \frac{G^2}{12} \cot \omega + \frac{G^3}{48} \sin L \cot^2 \omega (1 - 3 \sin^2 L)$$

(c) $$\tan (A - Z) = \frac{G}{2} \frac{\sin \frac{L + L_0}{2}}{\cos \frac{L - L_0}{2}} + \frac{G^2}{12} \cot \omega - \frac{G^3}{24} \sin L \cot^2 \omega$$

(d) $$A - Z = \frac{G}{2} \sin \frac{L + L_0}{2} + \frac{G^2}{12} \cot \omega + \frac{G^3}{48} \sin L \left[\cot^2\omega (1 - 3\sin^2 L) - 2\sin^2 L\right]$$

(e) $$A - Z = \frac{G}{2} \frac{\sin \frac{L + L_0}{2}}{\cos \frac{L - L_0}{2}} + \frac{G^2}{12} \cot \omega - \frac{G^3}{24} \sin L (\cot^2 \omega + \sin^2 L)$$

This development is applicable only if the successive powers of G (expressed as parts of the radius) are decreasing and if the omitted term can really be neglected. G should never exceed about 50 degrees at the most.

If the first term only be taken, any one of the four last formulae, which have the same term of G^2 smaller than that in formula (*a*), may be used.

If a larger number of terms is to be taken into account, one would try to avoid the calculation of the angle ω; this could be done by means of the equation

$$\cot \omega = - \cot Z + G \frac{\sin L}{\sin^2 Z}.$$

The preceding equations then become:

(a') $$\tan (A - Z) = \frac{G}{2} \sin L - \frac{G^2}{12} \cot Z (1 + 3\cos^2 L) + \frac{G^3}{24} \sin L (2 + \cot^2 Z + 3 \cos^2 L)$$

(b') $$\tan (A - Z) = \frac{G}{2} \sin \frac{L + L_0}{2} - \frac{G^2}{12} \cot Z + \frac{G^3}{24} \sin L (2 + \cot^2 Z + \frac{3}{2} \cos^2 L \cot^2 Z)$$

(c') $$\tan (A - Z) = \frac{G}{2} \frac{\sin \frac{L + L_0}{2}}{\cos \frac{L - L_0}{2}} - \frac{G^2}{12} \cot Z + \frac{G^3}{24} \sin L (2 + \cot^2 Z)$$

(d') $$A - Z = \frac{G}{2} \sin \frac{L + L_0}{2} - \frac{G^2}{12} \cot Z + \frac{G^3}{24} \sin L \left(2 + \cot^2 Z + \frac{3}{2} \cos^2 L \cot^2 Z - \sin^2 L\right)$$

(e') $$A - Z = \frac{G}{2} \frac{\sin \frac{L + L_0}{2}}{\cos \frac{L - L_0}{2}} - \frac{G^2}{12} \cot Z + \frac{G^3}{24} \sin L \left(2 + \cot^2 Z - \sin^2 L\right)$$

Formula (*e'*) is that with the smallest term of G^3 and thus would be the most accurate if this term be neglected.

(4) *LITTROW'S PROJECTION.*

This system of projection is defined by the formulae

$$x = \frac{\sin G}{\cos L_0}, \qquad y = \cos G \tan L_0.$$

The meridians are hyperbolas whose equation is

$$\frac{x^2}{\sin^2 G} - \frac{y^2}{\cos^2 G} = 1,$$

and the parallels are ellipses :

$$x^2 \cos^2 L_0 + y^2 \cot^2 L_0 = 1.$$

The meridian of origin is the y axis, the equator the x axis. The equation of the curve of equal azimuth will be

$$x \cot Z + y = \tan L.$$

It is thus a straight line which passes through the station Q, the meridian of which will have been taken as origin of the longitudes and as y axis. We shall have placed the station on the parallel corresponding to the latitude L.

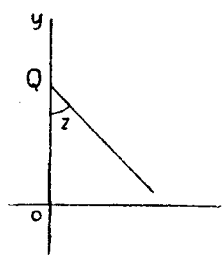


FIG. 4

This straight line makes an angle Z with the negative y axis (Fig. 4).

LITTROW'S projection thus enables us to represent a curve of equal azimuth by a straight line. This property endows it with great interest for the problem with which we are occupied.

The projection is conformal. The poles being at an infinite distance it cannot be utilised for polar regions. Further, the shape of its meridians and the very great distortion which it causes prevents us from using it conveniently as a projection for navigation. But it can render great services in the form of a diagram. It is already used by the British Navy (Capt. WEIR'S Azimuth Diagram) and by the German Navy (Azimut Messkarte N° 1982) for determining azimuths and for resolving various problems in nautical astronomy. It might be of interest to increase its scale and divide it into several sheets ; its extent in longitude can be limited to some thirty degrees for radiogoniometric purposes.

The method of using it is as follows :—

When a bearing has been taken of a station Q of azimuth Z, the station is placed on the y axis by its latitude and through this point is traced a straight line making an angle Z with the negative y's ; the ship's position is on this straight line.

If at the same time a bearing is taken of another station B, the straight line corresponding to its bearing is drawn in the same way. Then one chooses, somewhere near the latitude of the estimated position, two or three points on the straight line referring to the easternmost of the two stations, and moves them to the right, without changing their latitudes, while increasing their longitudes by a quantity equal to the difference of longitude of the two stations. The element of the curve thus obtained will cut the other straight line at the point sought for, which it is only necessary to plot on the chart by its latitude and longitude.

It does not seem that any procedure could provide a result in a quicker or simpler manner.

(We have supposed that the estimated position was to the eastward of the stations ; if it were to the westward it would merely be necessary to plot the longitudes in the reverse direction).

(5) *USE OF AN OBLIQUE PERSPECTIVE PROJECTION.*

While the previous system of projection turned the curve of equal azimuth into a straight line, this one transforms it into a circumference. We shall define the projection by the formulae

$$(6) \quad \begin{cases} x = \sin G \cos L_0 \\ -y = \cos G \cos L_0 + (1 - \sin L_0) \tan\left(\dfrac{\pi}{4} - \dfrac{L}{2}\right). \end{cases}$$

This is equivalent to projecting the point M of the sphere on the plane tangent to the pole P in a sense parallel to the direction $P'Q$ joining the opposite pole P' to the transmitting station (see Fig. 5).

To simplify the writing of the formulae, let us put

$$\frac{\pi}{2} - L = 2a\,;$$

a will be the angle $PP'Q$.

The equation of the curves of equal azimuth Z on the sphere is

$$(7) \qquad \cot 2a \cos L_0 = \cot Z \sin G + \sin L_0 \cos G.$$

Eliminating G between equations (6) and (7) we shall have

$$(8) \qquad \sin^2 L_0 - (y + \tan a) \sin L_0 \sin 2a + x \cot Z \sin 2a - \cos 2a = 0.$$

The elimination of G from the two equations (6) gives us, on the other hand,

$$(9) \qquad \sin^2 L_0 - (y + \tan a) \sin L_0 \sin 2a + (x^2 + y^2) \cos^2 a + y \sin 2a - \cos 2a = 0.$$

Equations (8) and (9) having to be satisfied at the same time, we must have

$$(10) \qquad x^2 + y^2 = 2\,(x \cot Z - y) \tan a.$$

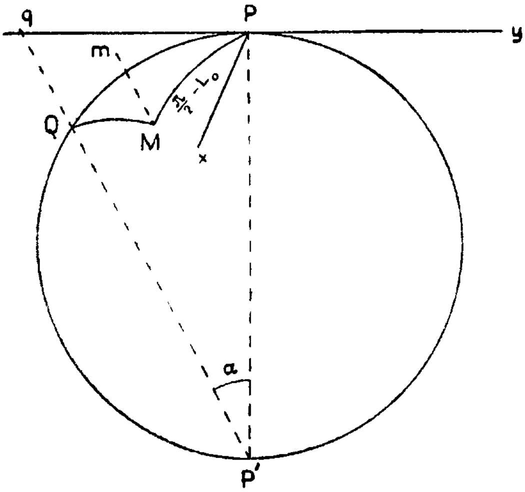


FIG. 5

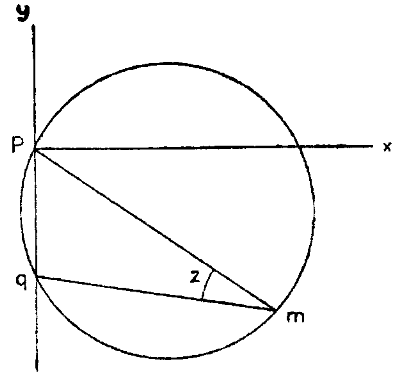


FIG. 6

This equation is that of the curves which are the projections of the curves of equal azimuth of the sphere. It represents a circle passing through the pole P and through the point q which is at the same time the projection of the pole P' and of the transmitting station Q. Further, between these two points P and q, the circle subtends the angle Z (see Fig. 6). The spherical segment containing an angle on the sphere is thus replaced by a plane segment subtending the same angle. The projection is, however, not conformal; the straight lines mP and mq are not the projections of the great circles MP and MQ of the sphere, which make an angle Z between them.

In this system of projection the parallels of the sphere are evidently circles equal to the parallels themselves, having their centres between P and q.

Their equation is no other than equation (9) which may be written

$$(9b) \qquad x^2 + \left[y + (1 - \sin L_0) \tan a\right]^2 = \cos^2 L_0.$$

The meridians are ellipses, whose equation is

$$\frac{x^2}{\sin^2 G} + (x \cot G + y)^2 \cot^2 a + 2\,(x \cot G + y) \cot a = 0.$$

They pass through the points P and q, have their centres in the middle of Pq, and make an angle G with it at P and q; the angles which define the longitudes are thus preserved at these points.

The two hemispheres separated by the great circle whose plane is perpendicular to the direction $P'Q$ have their projections superimposed and contained within the ellipse, the projection of this great circle, whose equation is

$$x^2 + \frac{(y + \tan a)^2}{1 + \tan^2 a} = 1.$$

When Z is less than $2a$, a portion of the segment containing the angle (10) is outside this ellipse and does not correspond to real points of the spherical segment.

This system of projection, which does not appear to have been mentioned up to the present (1) would enable the problem of the radiogoniometric bearing with respect to a determined station to be resolved quite simply; but it has the drawback of being different for each station considered; it would necessitate a special diagram for each one of them, and plotting the elements of the segments in the neighbourhood of the estimated position on the navigation chart, point by point, by means of latitudes and longitudes. We shall indicate a process which will avoid this inconvenience.

Besides, this system of projection is not the only one which turns a curve of equal azimuth into the segment of a circle subtending the observed angle; in Vol. IX, N° 1, of the *Hydrographic Review* of May 1932, p. 251, we described a system of inverse projection by LITTROW, proposed by Professor W. IMMLER for navigation in polar regions, which gives the same result and is independent of the position of the transmitting station; but its meridians and parallels are curves of too complicated a nature to enable this projection to be used other than as a diagram.

(6) *USE OF A POLAR STEREOGRAPHIC PROJECTION.*

Stereographic projection on the plane tangential to the pole presents great advantages and can be used easily for navigation, thanks to its properties of being conformal, of representing the meridians by converging straight lines, parallels by concentric circles, and every circle on the sphere by a circle. We will show, further, that the properties of the oblique perspective projection which we have just described can very easily be used on this projection.

In paragraph (7) will be found the study of the curves of equal azimuth; these curves are relatively simple on a polar stereographic projection, but one can avoid having to draw them.

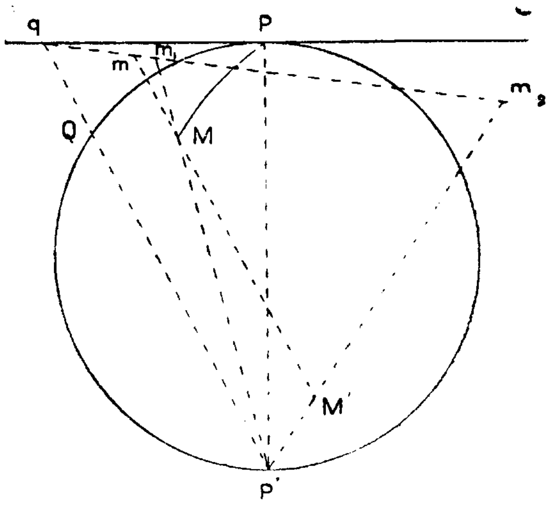


FIG. 7

Let us show at m_1 (Fig. 7) the stereographic projection of the point M of the sphere on the plane tangent to the pole. The oblique perspective pro-

(1) We were not aware of Mons. LECOQ's article, mentioned above, which reached us after going to press.

jection, peculiar to the station Q, of the same point M on the same plane, will be at m.

The straight line Mm, parallel to $P'Q$, is in the plane $qP'M$; the point m is thus on the straight line qm_1 ; and we have the relations

$$(11) \qquad \frac{m_1 m}{mq} = \frac{m_1 M}{MP'} = \frac{m_1 P'}{mP'} - 1 = tan^2 \left(\frac{\pi}{4} - \frac{L_0}{2}\right) = \frac{1 - sin\, L_0}{1 + sin\, L_0},$$

$$(12) \qquad \frac{m_1 m}{m_1\, q} = \frac{1 - sin\, L_0}{2}.$$

We can thus very easily pass from the polar stereographic projection of a point to its oblique perspective projection, and *vice versa*, by using a ratio which depends simply on the latitude of the point.

It should be noted that m is the oblique perspective projection of a second point M' of the sphere, which is also on the curve of equal azimuth Z ; the stereographic projection m_2 of this point M' will also be on qm produced ; it will generally be outside the projection and without interest in practice ; but all the same we would have (giving the correct sign to its latitude L'_0)

$$\frac{m_2\, m}{m_2\, q} = \frac{1 - sin\, L'_0}{2}.$$

The usage of this polar stereographic projection can then be as follows :—

It must bear a sufficiently close network of meridians and parallels. (Except in the polar regions, the pole will be outside the projection which may be constructed on any scale).

The meridian of each transmitting station will be drawn and must bear an indication of its points of intersection with the parallels of latitude of the system of oblique perspective projection peculiar to this station. This graduation is easy to establish by calculating the distance of each point of intersection from the parallel corresponding to the stereographic projection, a distance which is given by the formula

$$(1 - sin\, L_0)\left(\frac{cos\, L_0}{1 + sin\, L_0} - tan\, a\right),$$

in which L_0 is the latitude of the parallel and a the difference between 45° and half the latitude of the station.

The projection will be provided with a tracing of the same dimensions as the chart, bearing, for the same value of the earth's radius, a series of concentric circles of radius $cos\, L_0$. On each of these circles will have been inscribed, besides the latitude to which it corresponds, the value of the ratio $\frac{1 - sin\, L_0}{1 + sin\, L_0}$. The tracing will also bear the trace of one of its radii over a sufficient part of its length for its direction to be well determined, but the common centre of the circles will generally be outside the tracing.

When this radius is placed on the meridian of a transmitting station in such a way that one of the particular divisions L_0 inscribed on this meridian comes on the circle of the tracing that bears the same indication L_0, this circle will occupy the position of the parallel L_0 in the oblique perspective projection peculiar to the transmitting station. (But the other circles will not be in place).

Having taken a bearing of the transmitting station Q of azimuth Z (reckoned from North towards East), one will construct the arc of a circle, by points on the stereographic chart, being a segment subtending the angle Z between the pole and the point Q. This can be done without the pole being on the chart, either with a protractor with arms or with a station-pointer, marking the points of intersection of the segment with the meridians drawn; or by drawing, through the point Q, straight lines making angles $\pi - (Z + \alpha)$ with the meridian at the point Q for different values of α, and taking their intersections with the meridian making the angle α with the meridian of the station Q. This arc of a circle will be the curve of equal azimuth Z in the oblique perspective projection peculiar to Q; it will only be drawn in the vicinity of the estimated position. To find it, the estimated position must be transferred from the stereographic projection, where it is known, to the projection peculiar to Q. If L_0 is the latitude of the estimated position, the tracing will be applied as already described so as to have the parallel L_0 in this special projection (interpolating if necessary); and the intersection of this parallel with the straight line joining Q to the estimated position will be the position of the estimated position in the special projection.

It only remains to bring into the stereographic projection two or three of the points of the capable segment which we have just constructed in the special projection. Their positions on the circles of the tracing (placed each time in the position corresponding to them) will be selected, which will give their latitudes, and we shall take the intersection of the line joining them to the station Q with the parallel of their latitude in the stereographic projection.

If the intersection takes place at too small an angle for it to be determined with precision, we calculate the displacement to be applied to the point on the line joining it to the station, by multiplying the length which separates it from this station on the chart by the ratio inscribed on the tracing for the latitude of the position.

The same operation can be carried out for any other station contained in the chart, using the same tracing. The elements of the curves of equal azimuth, which have been constructed each time on a projection peculiar to the transmitting station, and transferred to the same stereographic projection, will cut at the point sought for.

We have thus been able accurately to construct, on as large a scale as desired, the useful portions of curves of equal azimuth corresponding to the transmitting stations of the chart, on a chart on a polar stereographic projection.

If the chart on the polar stereographic projection were on a small enough scale, especially in the polar regions, another procedure might have been employed, analogous to that which we have described for MERCATOR'S projection.

All the great circles of the sphere are represented by circles of radius $\frac{2}{\cos \varphi}$, φ being the latitude of their vertex. It is sufficient to draw these circles on a tracing contained in an angle of 90° between the extreme latitudes of the chart, and to make them turn round the pole, i.e. keeping the latitude of their points constant, to obtain the representation of every possible great circle (it is not necessary for the pole to be in the chart, and the tracing will be reversed as requisite). To ensure this displacement the tracing must show the parallels of the chart. The points where the great circles make the same angle with the meridians will be joined by a curve on the tracing.

When a bearing is taken of a station Q of azimuth Z, one seeks, by a suitable displacement of the tracing, for a great circle passing through the station and the neighbourhood of the estimated position, and marks on the chart the point corresponding to the angle Z. The same will be done for one or two other great circles, and, by joining the points thus obtained, we shall have the desired element of the curve of equal azimuth (1).

(7) *CURVES OF EQUAL AZIMUTH ON A POLAR STEREOGRAPHIC PROJECTION.*

If we use the same axes as above, and denote the length Pm_1 by ρ, the formulae of transformation will be :

$$x = \rho \sin G, \quad y = \rho \cos G, \quad \rho = 2 \tan \left(\frac{\pi}{4} - \frac{L_0}{2}\right) = \frac{2 \cos L_0}{1 + \sin L_0},$$

$$\sin L_0 = \frac{4 - \rho^2}{4 + \rho^2}, \quad \cos L_0 = \frac{4 \rho}{4 + \rho^2},$$

Introducing these values into equation (7) which gives the curves of equal azimuth on the sphere, we shall have the equation of the curves of equal azimuth of the stereographic projection,

$$(13) \quad \rho^2 (\sin G \cot Z - \cos G) - 4 \rho \cot 2 a + 4 (\sin G \cot Z + \cos G) = 0,$$

or, in cartesian co-ordinates,

$$(14) \quad (x^2 + y^2) (x \cot Z + y - 4 \cot 2 a) + 4 (x \cot Z - y) = 0.$$

This curve of the third degree is a circular cubic, referred to its centre P; it has as asymptote the straight line whose equation is

$$x \cot Z + y - 4 \cot 2 a = 0.$$

The curve meets the y axis at the origin P, and makes an angle Z with it. It meets it besides at two other points — the station q and its antipode t; at these two points it makes an angle Z with the y axis :

$$y_q = - 2 \tan a, \qquad y_t = + 2 \cot a.$$

These two points depend only on the value of a, and are independent of the value of the angle Z.

The asymptote also meets the y axis at a point, $y = 4 \cot 2a$, independent of the angle Z, and makes the angle Z with this axis. It meets the curve at

(1) The same procedure may be followed with a conic conformal projection of LAMBERT.

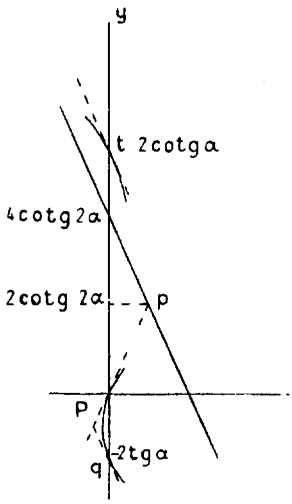


Fig. 8

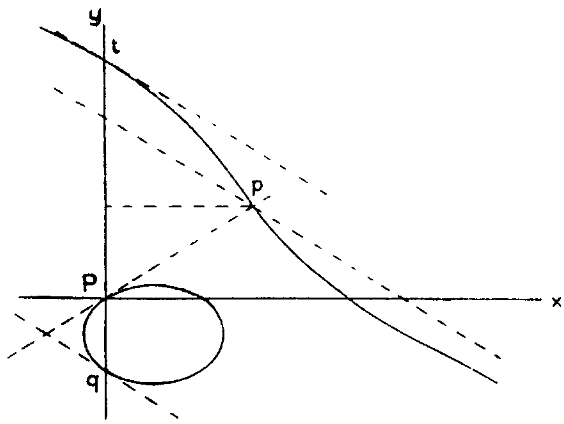


Fig. 9

a point p whose co-ordinates are

$$x = 2 \cot 2a \tan Z, \qquad y = 2 \cot 2a.$$

Equation (13) shows that the values of ρ are only real if

$$\sin G \leqslant \frac{\sin Z}{\sin 2a}.$$

They will thus always be real if $Z > 2a$. The curve will have a shape analogous to that of Fig. 9.

If $Z < 2a$, G can only vary between 0 and α and between $\pi - \alpha$ and π; α being given by

$$\sin \alpha = \frac{\sin Z}{\sin 2a}.$$

The vectors originating at the point P are thus tangents to the curve at the points where the latter meets the middle line,

$$x \cot Z + y - 2 \cot 2a = 0,$$

on which we have

$$\sin G = \frac{\sin Z}{\sin 2a}.$$

We shall find a shape analogous to that of Fig. 10.

If $Z = 2a$, the curve has a double point on the x axis, at the point whose co-ordinate is $x = 2$, corresponding to $L = 0$, $G = 90^0$. The two tangents are perpendicular to each other.

The tracing of the curves will be still further facilitated if we consider the parallels to which they are tangential. For this it is necessary that equation (13), in which G would be the unknown quantity, should have a double root We find the condition

$$\sin Z \cos L_0 = \pm \sin 2a.$$

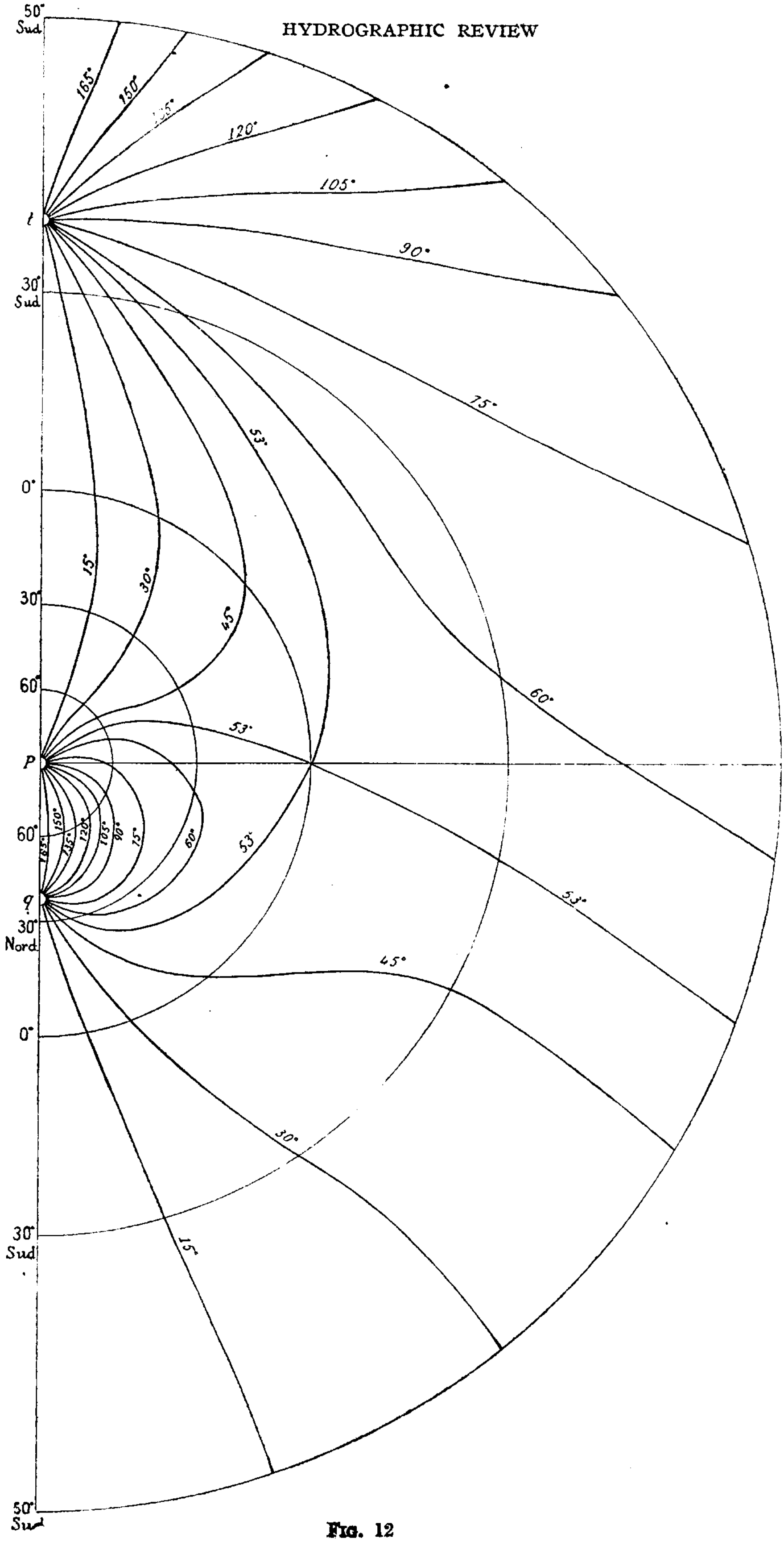

50° Sud
165°
150°
135°
120°
105°
90°
t
30° Sud
75°
53°
0°
15°
30°
45°
30°
60°
60°
53°
P
60°
150°
135°
120°
105°
90°
75°
60°
53°
165°
q
30° Nord
53°
45°
0°
30°
15°
30° Sud
50° Sud

FIG. 12

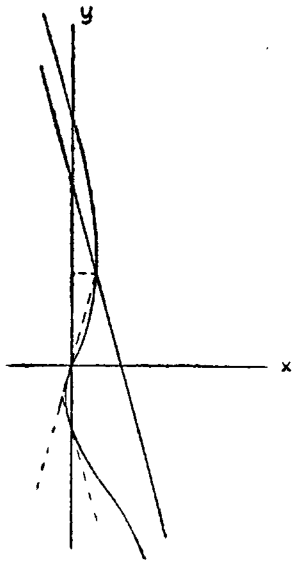


FIG. 10

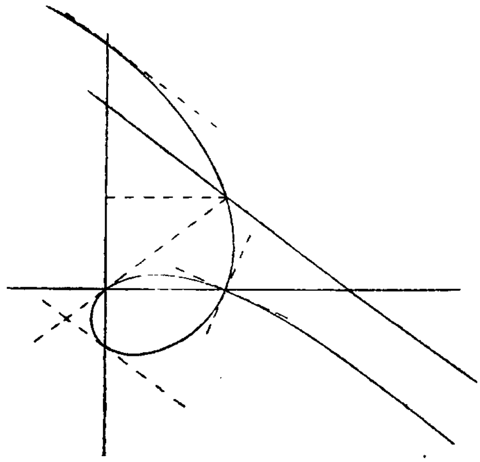


FIG. 11

We then have

$$\cos G = \tan L_0 \tan 2a, \quad \text{or} \quad \sin G = \frac{\cos Z}{\cos 2a}.$$

This relation shows that the spherical triangle is right-angled at Q when the curve of equal azimuth is perpendicular to the meridian of the observing station; which was also shown by equation (1).

The locus of the points of contact with the parallels is a circle whose centre is

$$x = 0, \qquad y = 2 \cot 2a$$

and which passes through the points t and q. This circle is the stereographic projection of the great circle perpendicular to the plane PQP' passing through Q and its antipode.

Opposite will be found the curves of equal azimuth drawn every 15° on a polar stereographic projection, for a transmitting station at a latitude of 37° ($2a = 53°$).

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