RADIO-ACOUSTIC POSITION LINE.

FIXING A POINT BY MEANS OF THE RATIOS OF THE TIME TAKEN BY SOUND TO TRAVEL FROM THE POINT TO THREE FIXED STATIONS. (*)

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I. — The methods of fixing by means of audible signals hitherto proposed have involved the use of the actual value of the velocity of sound in the medium. Thus, for example, the difference of the times at which a sound from an unknown point arrives at two fixed stations has been used to fix the direction of the point. Assuming the velocity of sound in the medium, the air or sea water, the observed difference of the times for the sound from the unknown point P to reach the observing stations gives the difference of the distances of P from A and B. It is then known that, within limits for which the curvature of the earth is not appreciable, P lies somewhere on a hyperbola of which A and B are the foci and whose asymptotes OT_1, OT_2

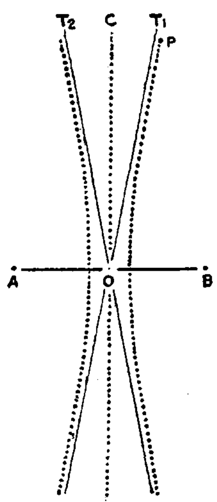

Figure 1

HYPERBOLA: LOCUS OF POINTS SUCH THAT THE DIFFERENCE OF DISTANCES FROM TWO FIXED POINTS A AND B IS CONSTANT.

(*) Forwarded by the *Hydrographer of the British Navy* on October 25th 1928.

are known directions through O, the mid-point of AB. Also if P is at a distance which is great with respect to AB the direction of P from O will not differ greatly from that of the asymptote OT_1, which is inclined to OB at the angle whose tangent is $\sqrt{\dfrac{a^2 - d^2}{d}}$ where $2a$ denotes the length of AB and $2d$ the difference between the distances PA and PB, *i. e.* the difference of the times multiplied by the known or assumed velocity of sound. (*See* fig. 1.)

Thus, knowing the difference between the distances PA and PB, an approximate bearing of P from O, the mid-point of AB, can be drawn, the direction of one or other of the asymptotes OT_1 or OT_2 according as the distance from A or B is the greater. The right hand branch of the hyperbola refers to points situated so that the distance from A is the greater, the left hand branch to those whose distance from A is the smaller and the conjugate axis OC to points equidistant from A and B.

By the above method, the actual value of the velocity of sound in the medium being known and constant in all directions, the time observations suffice to give an approximate fix for P, by means of the approximate bearings of P from the mid-points of the several pairs of observing stations.

If a simultaneous visible or wireless signal were made at P so that the actual times along PA and PB are found, and not merely the difference of the times, the fix becomes, with the assumption of the value of the velocity of sound, the ordinary case by distances from A and B.

2. — In the method described by Capt. L. TONTA in the pamphlet *A New Radio-Acoustic Position Line*, a wireless as well as audible signal is used but the actual value of the velocity of sound is not involved. The data used is the ratio of the times taken by sound to travel from P to two fixed stations A and B. By this means the velocity of sound is practically eliminated, except in so far as it is assumed that it is constant in the vicinity including the observing stations and the point whose position is sought. This latter may not be true, particularly in sea water, but possibly the effect of the variations in the velocity of sound along the directions of PA, PB, *etc.*, would be inappreciable.

The principle of the method is the geometrical fact that the locus of a point which moves so that its distances from two fixed points are in constant ratio is a circle. The truth of this proposition will be seen by consideration of fig. 2. The path of P is such that $\dfrac{PA}{PB}$ is constant and equal say to K. It is obvious that two points on the locus of P are M and N, the points which divide AB internally and externally in the ratio $K:1$. If P be joined to M and N it follows that, since $\dfrac{PA}{PB}$ is the same ratio as $\dfrac{AM}{MB}$ (*i. e.* $K:1$), the line PM bisects the angle APB and also since the ratios

$\frac{PA}{PB}$ and $\frac{AN}{NB}$ are equal the line PN bisects the exterior angle of the triangle APB *i. e.* the angle between AP produced and BP. Therefore MPN is a right angle and thus P lies on the circle on MN as diameter.

Since $\frac{AM}{MB} = K$, *i. e.* $\frac{AB - MB}{MB} = K$;

$\frac{AB}{MB} = K + 1$, and since $\frac{AN}{BN} = K$,

i. e. $\frac{AB + NB}{NB} = K$, $\frac{AB}{NB} = K - 1$

therefore $MB = \frac{AB}{K+1}$ and $NB = \frac{AB}{K-1}$

therefore $MN = AB \left[\frac{1}{K+1} + \frac{1}{K-1} \right] = \frac{2\,KAB}{K^2 - 1}$

Thus $MN = \frac{4\,aK}{K^2 - 1}$ where $2a$ denotes the length of AB

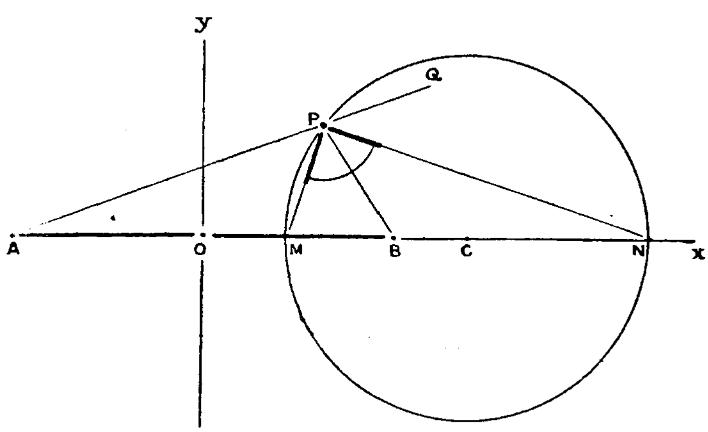

Figure 2

LOCUS OF P WHERE RATIO OF DISTANCES OF P FROM A AND B IS CONSTANT
PM BISECTS APB PN BISECTS BPQ

Thus the locus of P is the circle on MN as diameter, *i. e.* whose centre is midway between the points which divide AB internally and externally in the given ratio K and whose radius is therefore $\frac{K.\ AB}{K^2 - 1.}$

The locus of P may be easily found analytically. Take axes as shewn in fig. 2., origin at the mid-point of AB and axis of x along AB. Then A and B are the points $(-a,o)$ and (a,o). Thus if P is the point (x,y) :

$$PA^2 = (x + a)^2 + y^2, \qquad PB^2 = (x-a)^2 + y^2$$

therefore since $\dfrac{PA}{PB} = K, \qquad (x + a)^2 + y^2 = K^2 \left[(x-a)^2 + y^2) \right]$

i. e. $\qquad x^2 (1-K^2) + 2ax (1+K^2) + y^2 (1-K^2) + a^2 (1-K^2) = 0$

i. e. $\qquad (1)\ x^2 + y^2 - 2ax \dfrac{K^2+1}{K^2-1} + a^2 = 0 \qquad K > 1$

which shews that locus of P is a circle whose centre is at the point $a \dfrac{K^2+1}{K^2-1}$

and whose radius is $\dfrac{2Ka}{K^2-1}$. The points where the circle cuts axis of x

are given by the equation :

$$(2) \qquad x^2 - 2ax \frac{K^2+1}{K^2-1} + a^2 = 0$$

i. e. $$\left[x - a \frac{K-1}{K+1} \right] \left[x - a \frac{K+1}{K-1} \right] = 0$$

Therefore $x = a \dfrac{K-1}{K+1}$ and $x = a \dfrac{K+1}{K-1}$, these being the lengths of OM and ON.

This shews that the circle passes through the points which divide AB internally and externally in the ratio $K : 1$ as noted in the geometrical method.

If α denotes the angle whose tangent is K, the distance of the centre of the position circle from the mid-point of AB is $a \sec 2\alpha$ and the radius $a \tan 2\alpha$. These forms are more convenient for obtaining the position of the centre and the radius than the expressions $a \dfrac{K^2+1}{K^2-1}$ and $\dfrac{2aK}{K^2-1}$. The angle is α of course the angle PAB where P is on the position circle which subtends a right angle at AB.

It is easily shown that the tangent from A touches the position circle at the point T where the perpendicular at B to AB meets the circle. Similarly it can be shown that the tangent from O is inclined to AB at the angle 2α and of length a.

3. — The fact that the locus of P is a circle might at first sight appear to promise an easy means of fixing positions by ratios of distance (*i. e.* by ratios of observed times), but unfortunately the practical determination of the

circles of position is not by any means always easy or rapid owing to the inaccessibility of the centre in certain cases. The circles could always be drawn by three points and a set of curves, but the trouble of such a method of plotting would in normal circumstances scarcely be worth while. In isolated cases where an accurate fix is required the method might be valuable.

Theoretically if three stations, either collinear or otherwise were arranged to receive the wireless and sound signals from any point, the point could be fixed by the (*) three circles of position given by the three ratios of times to the three pairs of stations. It is possible to have the system of circles for one pair of stations for the various values of K drawn on the plotting board and those for another pair on tracing paper. In some cases the fix would be bad, *i. e.* the position circles would cut at too small an angle and the use of a transparency would not lead to very accurate plotting. The use of two such systems of circles might, however, be valuable for some purposes.

4. — The system of circles for various values of K for two stations A and B is shewn in fig. 3. The circles, as may be seen from equation (1) of § 2 or otherwise, are a system of non-intersecting coaxal circles, the radical axis being the perpendicular bisector of AB, *i. e.* the straight line $K = 1$.

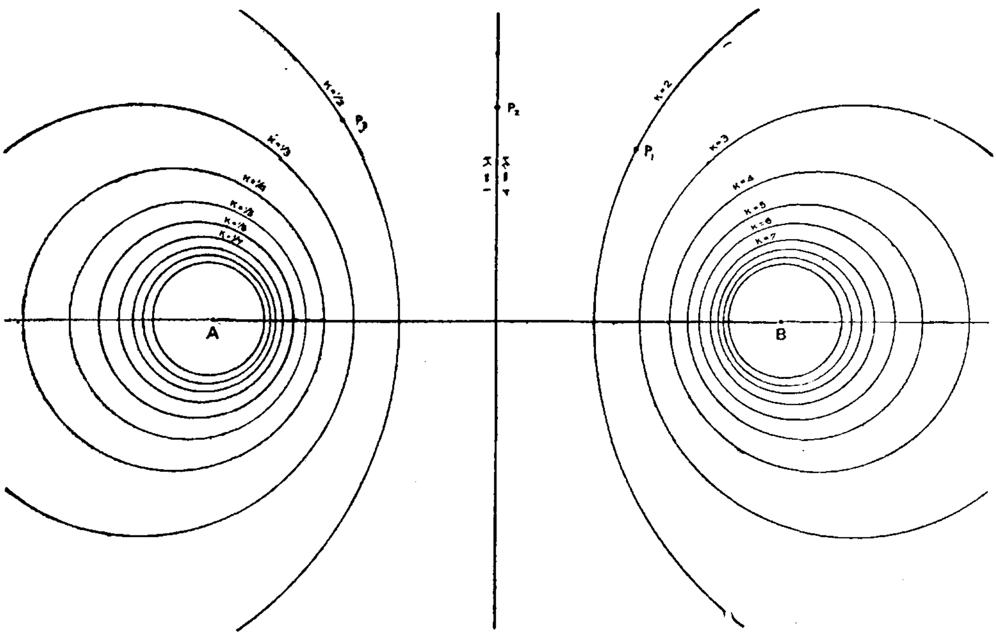

Figure 3

$$AP_1 = 2\,BP_1\ ;\ AP_2 = BP_2\ ;\ AP_3 = \frac{1}{2}\,BP_3.$$

5. — The question of taking account of the effect of current *etc.* can be considered, but to obtain an accurate fix without taking such effects into account a sufficiently laborious process is necessary, involving in general the

(*) Two would of course suffice, but three are available.

calculation of three points on each position circle, or two points on each and their radii, as in many cases the centres may be inaccessible.

Example :

The position circles for fixing a point accurately by this method are shewn in fig. 4. The ratios of the times from P to A and B, B and C, C and A are 0.874, 1.063 and 1.077 respectively, these figures being derived from the distances of P from A, B and C as shewn. The lengths of AB, BC and CA are 12000, 10000 and 19200 ft.

The radii and positions of the centre of the position circles are as given below on the scale of fig. 4, which is 1000 ft. = .25 inch.

> A, B : Radius: 11.094 inches. Centre from O_{AB} : 11.195 inches towards A.
>
> B, C : Radius: 20.437 inches. Centre from O_{BC} : 20.475 inches towards C.
>
> C, A : Radius: 32.423 inches. Centre from O_{CA} : 32.512 inches towards A.

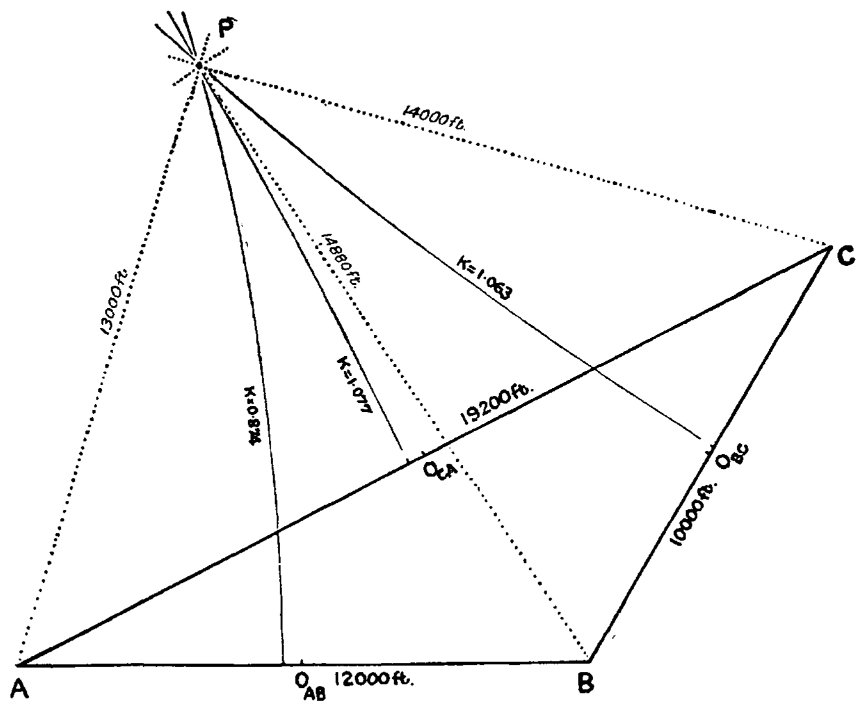

Figure 4

The position circles cannot be drawn conveniently with compasses on the scale of fig. 4, a sheet about 40 inches square being required and on appreciably larger scales the size of the sheet required would be prohibitive. The arcs in fig. 4 have been drawn by means of two points and the radii. One of the points in each case is of course that in which the arc cuts the line joining the two stations, *i. e.* the point M of fig. 2. These points M_{AB}, M_{BC}, and M_{CA} in fig. 4 are .101, .038 and .089 inches from the mid points O_{AB}, O_{BC} and O_{CA} towards A, C and A respectively, these figures being the differences of the distances of centre from the mid point and the radii as above. One other conveniently placed point on each position circle being obtained the arcs for fixing P are drawn by means of curves whose respective radii are equal to those found for the three position circles as above.

It will be observed from the figure that the fix for P by the position circles is rather poor while that by the distances from A, B and C as shewn by the short dotted arcs is good.

6. — It will be noticed that the system of circles which pass through the points A and B, and whose centres therefore lie on the perpendicular bisector of AB, are orthogonal to the system of ratio position circles considered in the preceding paragraphs. That is any circle through the points A and B cuts any one of the ratio position circles for A and B at right angles. This means that a fix obtained by a position circle and one of the other system would always be good, the fixing arcs being always at right angle

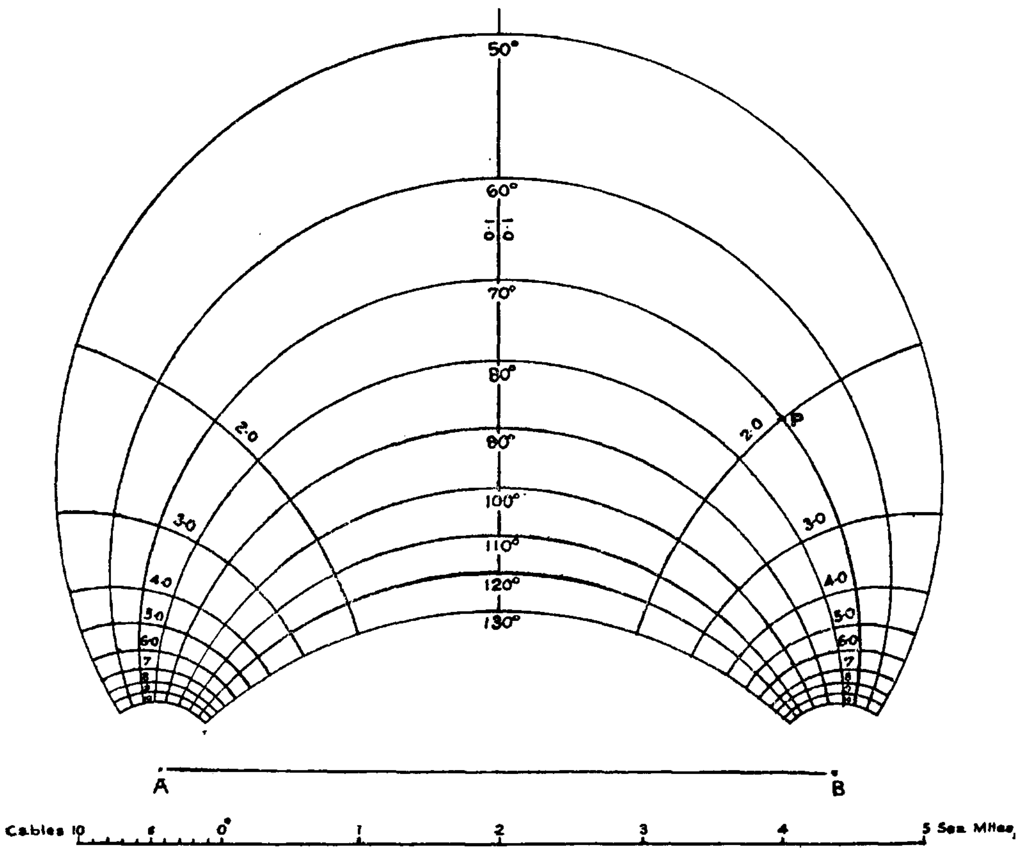

Figure 5

This second system of circles consists of course of the loci of points at which AB subtends a given angle. Thus if the angle PAB were known and also the ratio of times along PA and PB the point P could be effectively fixed by means of a diagram shewing the position circles for the various values of the ratio and the above orthogonal system for the various values of the angle subtended by AB.

The centre of the circle which is the locus of points at which AB subtends a given angle φ, is referred to the axes of fig. 2, the point $(O, a \cot \varphi)$ and its radius a cosec φ, the equation to the circle being:

$$x^2 + y^2 - 2\,ay \cot \varphi - a^2 = 0$$

The portion of the two systems of circles on one side only of a pair of stations would be required in practice, as shewn in outline in fig. 5. The values of the ratio K and of the angle φ are shewn on the respective systems of arcs, the values of K being 1 to 10 and of φ from 50° to 130°. The diagram would shew in practice the values of K for intervals of say 0.1 and those of φ for intervals of 1°.

The values of K in fig. 5 are those of the ratio of the time for the acoustic signal to reach A to that to reach B in the right hand half and vice versa in the left hand half so that K is not less than unity.

As an example suppose that the difference of the bearing of A and B from the ship were 70° and the time ratio of the acoustic signal to A and to B were 2.0. Then the ship would be at P in 5, and her distances from A and B would be 5 miles 1 cable and 2 miles 5 ½ cables.

7. — The angle PAB is the difference between the bearings of A and B from P and should be obtainable by ships carrying wireless direction finding apparatus with considerable accuracy. The use of the two systems of circles therefore offers, apparently, a simple means of accurately fixing ship by the use of two wireless stations, assuming the stations to be in communication and apparatus available for the determination of the ratio of the time taken by the acoustic signal from the ship.

The same diagram of circles would serve for any pair of stations, the copy when used for a particular pair of stations having a scale of distance determined by the actual distance between the stations in question, or a set of scales might be inserted on the diagram for the pairs of stations ordinarily used in various regions. The distances of the ship from the two stations used would then be read from the scale and position of the ship could then be plotted on the chart. The diagram is of course only applicable to distances such that the effect of the curvature of the earth may be taken as inappreciable.

There may be practical difficulties connected with the installation and working of apparatus for furnishing the ratio of times of an acoustic signal to a pair of wireless stations, but the use of this data, in conjunction with the ordinary wireless bearings of stations, obtainable by ships carrying direction finding apparatus, appears worth consideration, as providing a useful means of accurately fixing position when a ship is in the vicinity of wireless stations.