THE RADIO-ACOUSTIC POSITION BY THREE TIME INTERVALS.

ITS REDUCTION TO THE PROBLEM OF SNELLIUS POTHENOT.

by

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In our article entitled : *Submarine Phonotelemetry. A new radio-acoustic Line of Positions* published in the *Hydrographic Review* (1st part, vol. IV, N° 2 Nov. 1927 - 2nd part, Vol. V, N° 1, May 1928) we demonstrated how the measurement of the time required for the sound to traverse the distances *PL, PM* and *PN*, separating the point *P* from the three known points *L, M* and *N* (fig. 1), sufficed for the determination of the position of point *P*, *independent* of the knowledge of the velocity of sound. (*)

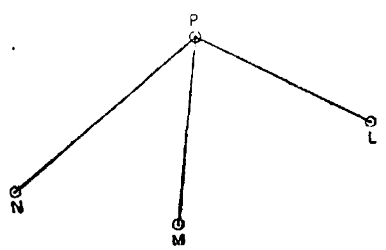

Fig. 1

(*) The determination is based on the hypothesis that the sound is propagated at the same velocity in all directions around the point *P*. If, in the region considered there exists a current, the hypothesis no longer holds, but the problem may then be referred to the ideal case, in which *no* current exists, and very simple corrections applied to the observed time intervals, as stated in the remarks following § 16 in the second part of the above article. The *practical* formula for correcting the observed time interval is given by:

$$\textit{Correction} = - 0^s\,000\,444.\ D.\ \Delta\, v$$

in which *D* represents the distance in *kilometres* (the approximate value of which is always known) and $\Delta\, v$ represents the value of the current component in metres per second in the direction considered. The correction is given in magnitude and sign by the preceding formula if we give $\Delta\, v$ a plus sign when the current increases its value in the direction of propagation of sound and a minus sign in the contrary case.

It is necessary to caution the reader that in the remark made in § 16 quoted above a "lapsus calami" has occurred where it is stated: "the correction should be *positive* when the current increases the velocity of the sound etc., and it should be *negative* in the contrary case". It is necessary to alter these adjectives and to substitute *negative* for *positive* and vice versa.

In the preceding article was also demonstrated the close analogy which exists between this new method of fixing position by radio-acoustic bearings and the above-mentioned problem of SNELLIUS-POTHENOT, based on the measurement of the angles

$$\gamma = LPM \quad \text{and} \quad \delta = MPN$$

beneath which, at the point P, we see the couples of the points LM and MN.

In the present article, we propose to bring out new evidence of the relation existing between the two methods and to demonstrate that the radio-acoustic point which we have proposed may be reduced to the problem of SNELLIUS-POTHENOT, or, in other words, that a knowledge of the time required for the sound to traverse the three distances permits a *very simple* determination of the angles γ and δ and that consequently P may be determined by analytic methods, either by graphic methods employing a station pointer or the usual methods employed in the solution of the SNELLIUS-POTHENOT problem. (*)

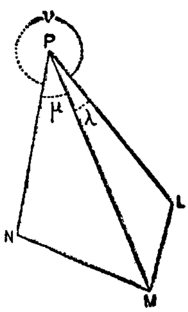

Fig. 2

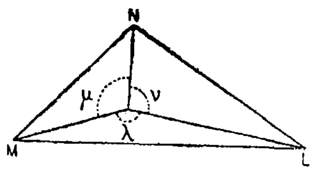

Fig. 2 bis

2. — *Definitions.* Let us consider the three points $L\,M\,N$ (Fig. 2 and fig. 2 *bis*) whose positions are known, and P the point which is to be deter-

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(*) In connection with this subject it would appear advantageous to give here a brief discussion of the name which is attributed to the solution of this problem, which is so very important in its application to topography, hydrography and navigation.

WILLEBROD SNEL, son of RODOLPH SNEL VAN ROYEN, was born at Leiden in the year 1581 (about) and died in 1626. His principal work, for which he is commemorated in history, is the following:

Eratosthenes Batavus — De Terrae ambitus vera quantitate a Willebrodo Snellio suscitatus. Lugduni Batavorum, Apud Iodocum a Colster, 1617.

In this work the problem which we are considering is enunciated on page 203 in the following manner:

Trium locorum intervallis inter se datis, quarti distantiam ab omnibus unica station definire.

The problem of SNELLIUS is better known under the name of the *problem of* POTHENOT, although the latter did not publish it until about 80 years later.

After SNELLIUS, but before Laurent POTHENOT, the same problem was treated by the Englishman John COLLINS (1624-1683) in the *Philosophical Transactions* (March 25, 1671 Vol. 6, page 2093). The solution of the same problem is known by the name of the *auxiliary point solution of* COLLINS.

In France, this problem is known by the term *problème de la carte* and in Germany by the *name Aufgabe des Rückwärtseinschneidens*; in Italy by the name *determinazione del punto a vertice di piramide*, etc.

mined. The lines PL, PM and PN make the three angles λ, μ, ν, on the chart. It is necessary to consider two cases:

1° P lies outside the triangle $L\ M\ N$ or is located on its perimeter (Fig. 2), in this case two of the angles are less than 180° and the third is greater than or equal to 180°. (When it is greater than 180°, P is outside the triangle, when it is equal to 180° it is on the perimeter).

2° P is inside the triangle $L\ M\ N$ (fig. 2 *bis*) and in this case all three angles λ, μ, ν are less than 180°.

In the first case we shall term the *central point* of the three given points $L\ M\ N$ that one which is located on the line which separates the two angles less than 180°. Suppose that an observer, located at point P looks towards the central point. Then of the other two points, one will be located on the right and the other on the left of the observer; we shall therefore call them respectively the *right hand point* and the *left hand point*. By analogy we shall term the *right hand angle* and the *left hand angle* respectively those angles lying to the right and to the left of the line drawn from the point P to the central point. (By definition these angles are less than 180°, or in the special case one is equal to 180° and the other less than 180°).

In the second case the *central point* may be arbitrarily chosen, and therefore the points and angles to the right and left will be determined likewise.

In all that follows we shall designate the points whose positions are known by the letters $A\ B\ C$.

A being the *left hand point*.
B the *central point*.
and C the *right hand point*.

The point to be determined we shall designate as P.

For the rest:

$\gamma = A\ P\ B$ will be the *left hand angle*.
$\delta = B\ P\ C$ the *right hand angle*.

These two angles are the angles we have to consider in the determination of the point P by the Snellius-Pothenot problem.

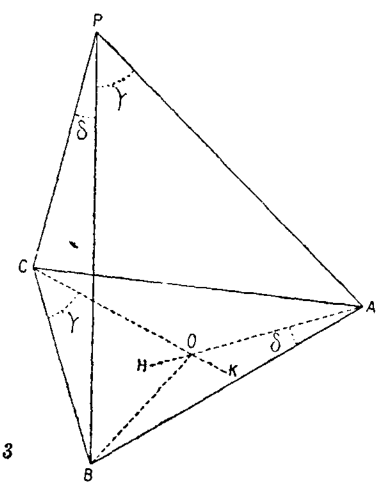

Fig. 3

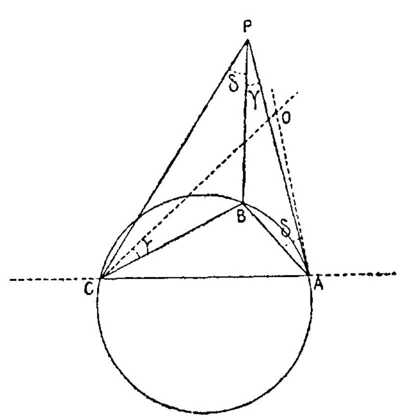

Fig. 4

3. — Adopting the conventions and notations given above, the following general solution will be applicable in every case, regardless of the relative position of the point P with respect to the triangle $A\ B\ C$.

Draw from A (fig. 3) the line $A\ H$, which forms an angle δ with the line AB, to the right of it looking along the direction from A towards B, and from C the line CK which forms an angle γ with the line CB, to the left of it looking along the direction from C towards B. Let O be the point where AH and CK intersect. Join B and O, then in the triangles AOB and COB we have, by construction :

$$B\,A\,O = \delta\ ; \qquad B\,C\,O = \gamma$$

further

$$A\,B\,O = P\,B\,C \qquad \text{and} \qquad C\,B\,O = P\,B\,A$$

consequently the triangles are similar to the triangles $B\ P\ C$ and $A\ P\ B$. (*)

Therefore, we have : (1)
$$\left\{\begin{aligned} A\,O &= \frac{P\,C}{P\,B}\cdot A\,B \\ C\,O &= \frac{P\,A}{P\,B}\cdot C\,B \end{aligned}\right.$$

For the rest if we assume that the straight line AC, which joins the two points, left and right, is prolonged indefinitely and if we consider the two portions of the plan thus divided by the straight line, it is easy to determine in which of these two sections will lie the point O, which we shall call the *auxiliary point*. The distances from this point to A and C respectively are determined by formula (1). Three cases must be considered :

1°. If the point P lies *outside the circle* determined by the three points $A\ B\ C$, then the auxiliary point is located in the half section *containing* B (the central point) (fig. 3 and 4).

2°. If the point P lies *within the circle* $A\ B\ C$, the point O will be located in the section which *does not contain* B (fig. 5).

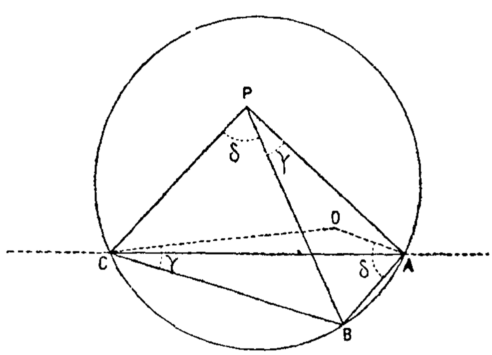

Fig. 5

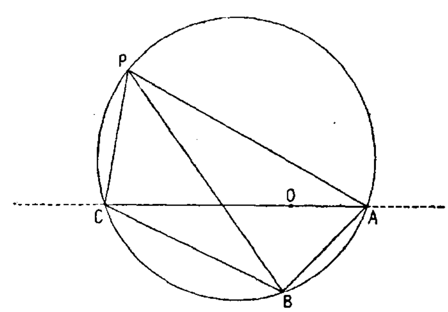

Fig. 6

(*) The proof of this theorem, on which our deductions are based, is very simple and constitutes an easy problem in elementary geometry. In the number of 15th November 1922 of the *Zeitschrift für Vermessungswesens*, the proof is given in a rather complicated and scientific manner by means of the method of vector analysis. (Dr. Franz MULLER. *Neue Lösung des ebenen Rückwärteseinschneidens nach des Friedrichsen Vektormetode*).

3°) In the particular case, where the point P lies *on the circumference of the circle* determined by the points $A\ B\ C$ (fig. 6), the auxiliary point will lie on the line $A\ C$. In this special case we have the well-known *indeterminate case of the problem of* SNELLIUS-POTHENOT, which, as we have already shown in § 18 of our preceding article (*Submarine Phonotelemetry, Hydrographic Review,* Vol. V, N° 2 of May 1928) *coincides with the indeterminate case in the radio-acoustic problem using three time intervals.* In this case the quadrilateral $P\ A\ C\ B$, being actually circumscribable, as well for the problem of SNELLIUS as for the radio-acoustic problem, the two loci of the position given by observation are identical with the circle circumscribed about the triangle $A\ B\ C$. We might note here that in every case we know an *approximate* position of the point P which we may then place on the graph by employing the known values of the three distances PA, PB and PC determined from the approximate value of the velocity of sound. Consequently, the rule just formulated permits us in practice to remove any ambiguity with regard to the position of the auxiliary point.

Finally, let us consider another case which is not without practical interest. The angle formed by the intersection of the straight lines AH and CK (fig. 3), *i. e.* the angle at O of the triangle AOC is equal to the angle of intersection at the point P of the two loci of position obtained from observation, that is, whether by the two circles of Apollonius determined by the measurement of time (radio-acoustic-problem) or the two circles determined by the measurement of the angles γ and δ (problem of SNELLIUS).

4. — If we designate by θ_a, θ_b, θ_c, the times required for the sound to traverse the distances PA, PB and PC respectively, we have (*)

$$\frac{PC}{PB} = \frac{\theta_c}{\theta_b} \quad ; \quad \frac{PA}{PB} = \frac{\theta_a}{\theta_b}$$

Consequently the equation (1) gives

$$(2) \left\{ \begin{aligned} A\,O &= \frac{\theta_c}{\theta_b}\ A\,B \\ C\,O &= \frac{\theta_a}{\theta_b}\ C\,B \end{aligned} \right.$$

The quantities θ_a, θ_b, θ_c, AB, CB are known by hypothesis and therefore, by means of formula (2), we may determine the values of the sides, AO and CO, of the triangle $A\ O\ C$, that is, the distances from the auxiliary point to the left hand right hand points. For the rest, through the application of the rules given in § 3, and knowing the approximate position of P from the graph, we

(*) We assume that the times θ_a, θ_b, θ_c are measured in still water, or where current exists they are corrected for the current. (See foot-note of § 1) may determine whether the auxiliary point lies within that section of the plan containing the central point B, relative to the line AC, or whether it lies in the section which does not contain this point.

Thus the position of the auxiliary point is fully determined and it is therefore easy to construct the angles :

$$BAO = \delta \quad ; \qquad BCO = \gamma$$

These angles being known, the determination of the point P is reduced to the well-known problem of SNELLIUS-POTHENOT, as we wished to prove.

Example :

Given : $BA = 28820{,}0$ metres ; $BC = 19221{,}0$ metres ; $AC = 30752{,}1$ metres.

Consequently : angle $ACB = 65^\circ\ 48'\ 10''$
angle $CAB = 37^\circ\ 28'\ 10''$

Time intervals observed :

Time to traverse PA .... $\theta_a = 26^s\,264$.
" " PB .... $\theta_b = 36^s\,705$.
" " PC .... $\theta_c = 26^s\,996$.

Calculation of the distances of the auxiliary point from the left hand and right hand points.

$$AO = \frac{\theta_c}{\theta_b} \cdot AB$$

θ_c	26.996	*log*	1.4312994
θ_b	36.705		1.5647252
$log \dfrac{\theta_c}{\theta_b}$			$\bar{1}.8665742$
BA	28820	*log*	4.4596940
AO	21196,7		4.3262682

$$CO = \frac{\theta_a}{\theta_b} \cdot BC$$

θ_a	26.264	*log*	1.4193609
θ_b	36.705		1.5647252
$log \dfrac{\theta_a}{\theta_b}$			$\bar{1}.8546357$
BC	19221	*log*	4.2837760
CO	13753,5		4.1384117

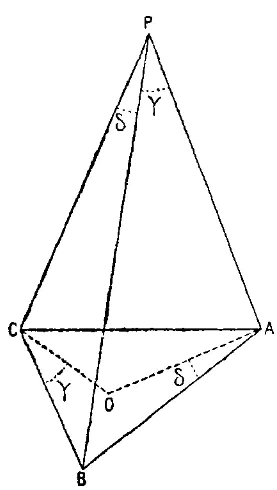

Fig. 7

Resolving the triangle *AOC*, in which we know the three sides (*AC* given; *AO* and *CO* calculated), we obtain:

$$CAO = 22^\circ 20' 25'' \ ; \quad ACO = 35^\circ 51' 38'' \ ; \quad AOC = 121^\circ 47' 57''$$

and finally (see fig. 7, in which the point *P* is located by the approximate distances *PA* and *PC*, see rule in § 3)

$$\gamma = ACB - ACO = \underline{29^\circ 56' 32''}$$

$$\delta = CAB - CAO = \underline{15^\circ 07' 45''}$$

The determination of the point *P* is thus reduced to the problem of SNELLIUS-POTHENOT, which may then be solved either by calculation or by well-known graphic methods. The partisans of the graphic method who stress the importance of *quick solutions* and the reduction of all topographical problems to geometric figures at all costs, may also dispense with the analytic determination of the angles of the triangle *AOC*. In fact, having calculated the distances *AO* and *CO* (the calculation is extremely simple) we may then, (either graphically or on the chart giving the points *A B C*) construct the triangle *AOC* on *AC*, then measure with the vernier scale the angles $\gamma = OCB$ and $\delta = OAB$ and finally with the station pointer, construct the point *P* which is sought by a method analogous to the preceding. It may be assumed that these operations may be carried out without difficulty and in a few minutes even by an operator lacking experience. (*)

5. — It appears that the great simplicity of radio-acoustic position finding, when reduced to the problem of SNELLIUS-POTHENOT, might go far to bring into greater use the submarine phonotelemetric methods in extensive sounding operations.

With an arrangement and a group of instruments, analogous to those adopted by the United States Coast and Geodetic Survey (which are described in the manual entitled *Radio Acoustic Position Finding*, Special Publication N° 146. See also the *Hydrographic Review*, Vol. III N° 1, Nov. 1925, page 51) and employing the well-known acoustic sounding apparatus, the execution of a submarine survey, which with the older methods presents a great many difficulties and takes much time and work without giving assured results of sufficient accuracy, may now be reduced to a simple, rapid and accurate operation. In this connection, it might be well to quote the text of the *Annual Report of the Director, United States Coast and Geodetic Survey,* for for the year 1928, pages 17 and 18 : —

(*) *Remark.* — In this article, as in the preceding articles, we have assumed that the problem is worked out on a plane surface without taking into consideration the curvature of the earth. It would appear, however, that taking the curvature of the earth into consideration in such a case would be equivalent to "splitting straws".

"The recently developed radio-acoustic sound ranging permits the prosecution of hydrographic surveys during foggy and rainy weather and during the hours of darkness. Surveying can be carried on almost regardless of weather conditions and throughout the entire 24 hours of the day. Thus the accomplishment of surveying parties in regions of prevalent adverse weather conditions has been more than doubled."

....

" The master of a ship depending on soundings shown on his chart for the determination of his position requires that the soundings be placed in their correct geographical positions. The radio-acoustic sound ranging method not only speeds up hydrographic surveys by its independence of weather conditions, but also makes possible accurate locations of soundings far offshore where former survey methods made it difficult to chart soundings with any degree of accuracy. Thus the U. S. Coast and Geodetic Survey has been able to decrease the unit cost of surveys and at the same time give the mariner added service. "

January 1929.