SUBMARINE PHONOTELEMETRY

A NEW RADIO-ACOUSTIC POSITION LINE

Note by Captain L. TONTA, *Director*

1. In these notes the examination of acoustic methods for fixing the position at sea is undertaken, and it is believed that these will be widely used in navigation and for hydrographic surveys.

These methods have been grouped under the single heading " Phonotelemetry ". This word, which is derived from Greek roots and can be adopted in all languages with the necessary orthographic modifications, describes accurately all those methods which are designated in English under the title of " Sound Ranging ", and in French under the title of " Repérage par le Son ".

2. It is believed that the method described in these notes is entirely new in geometrical essentials, in other words, it is based on geometrical principles which, so far, have not been used in phonotelemetry.

The measurements which are necessary in using it are, in fact, identical with those which occur in the application of already known methods among which may be cited, for example, that which is described in the interesting note entitled : —" A radio acoustic method of locating position at sea, *etc.* " by A. B. WOOD and Capt. H. E. BROWNE. (Proc. Physical Society of London. Vol. 35, Part 3, April 1915, 1923, pages 183) and also that in Special Publication N° 107 " Radio acoustic method of position finding in hydrographic surveys " by N. H. HECK, E. A. ECKHARDT and M. KEISER, published by the Coast and Geodetic Survey of the UNITED STATES OF AMERICA (*See Hydrographic Review*, Vol. III, N° 1, November 1925, pages 53-64).

A vessel which desires to ascertain her position explodes a submerged cartridge electrically and at the same instant makes a radio-telegraphic signal. Any stations, the accurate positions of which are known and which are connected to hydrophones placed on the bottom of the sea, register the times of arrival of the radio-telegraphic signal and of the acoustic wave which travels through the water.

The interval θ, which elapses between the two moments of arrival, measures the time which the sound requires to cover the distance between the point of explosion and the hydrophone, seeing that the propagation of the radio-telegraphic wave takes place at such great speed that the interval of time between its emission and the reception of the signal may be neglected.

3. The measurement of the intervals of time θ_a and θ_b which the sound takes to cover the distances D_a and D_b between point P, the origin of sound, and the extremities AB of a known base (fig. 1), determines a line of position for point P, which is *independent of any knowledge of the velocity of sound* and which is based on the sole hypothesis that that velocity remains constant in the two directions PA and PB. In fact, according to this hypothesis, the intervals θ_a and θ_b are proportional to the distances D_a and D_b respectively. On the other hand it is known, from elementary geometry, that the locus of points P, for which the ratio $\frac{D_a}{D_b}$ of distances from two fixed points A and B is constant, is a circle (known as the Circle of Apollonius) the centre of which is on the production of straight line AB, and on that side of the extremity which lies at the shorter distance. The points of intersection M and N and the straight line AB (one intersection being between the points A and B, and the other beyond) are respectively conjugated harmonics of A and B $\left(\frac{\overline{MA}}{\overline{MB}} = -\frac{\overline{NA}}{\overline{NB}}\right)$

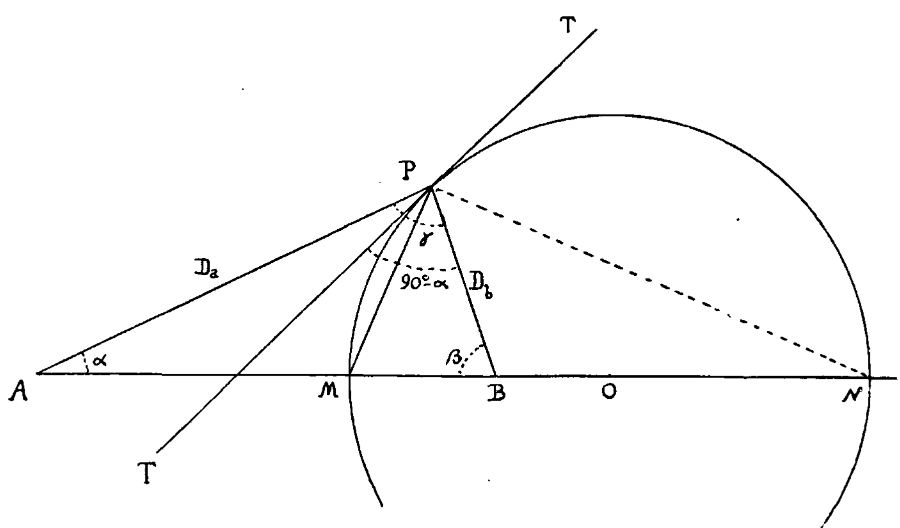

FIG. 1

The straight line PM bisects the interior angle at P of the triangle APB and PN, the perpendicular to PM, therefore bisects the adjacent exterior angle.

Angle $TPB = \tau$ which is formed by the tangent at P to the circle and the *short* side $PB = D_b$ of the triangle APB, $= 90^\circ - \alpha$, α being equal to PAB, which angle is that between the base AB and the other side (the greater) $PA = D_a$.(*)

This angle τ must be reckoned from the side PB towards the interior of angle BPA, so that the intersection of the tangent with the straight line AB will occur between the points A and B, beyond these points on the side of A.

The tangent coincides with the interior bisector PT, (fig. 2), passing through P when the ratio $\frac{D_b}{D_a} = 1$, *i.e.* when $D_b = D_a$.

As the value of the ratio $\frac{D_b}{D_a}$ decreases, the tangent moves away from the bisector and approaches the straight line PT_2 drawn through P perpendicularly to the shorter side PB. In other words, PB being smaller or equal to PA for a given value of angle $\gamma = APB$ (*i.e.* of the difference between the azimuths at which the extremities of base AB lie from P), the tangent lies between the two limit positions PT_1 and PT_2, which correspond respectively to

$$\frac{\theta_b}{\theta_a} = 1, \text{ and } \frac{\theta_b}{\theta_a} = \text{zero},$$

and include the angle $90^\circ - \frac{\gamma}{2}$ between them.

Finally it should be observed that the tangent coincides with the side PA (the greater) when triangle APB has a right angle at B.

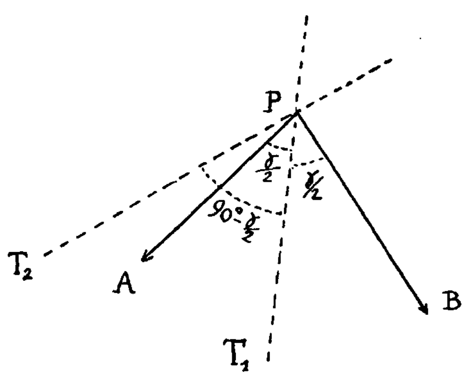

FIG. 2

4. Attention has been directed to this property of the tangent because it has practical importance in the problem dealt with herein.

(*) And therefore, the angle between the tangent and the long side $PA = D_a$ is equal to $90^\circ - \beta$, β being angle PBA.

The velocity of sound through water (which is referred to as v) is not entirely unknown ; on the contrary a very close value v' is always known and with this it is possible to determine the approximate values of the distances D_a and D_b.

$$D'_a = v'\ \theta_a\ , \qquad D'_b = v'\ \theta b$$

As $\frac{D'_a}{D'_b} = \frac{\theta_a}{\theta_b} = \frac{D_a}{D_b}$, point P' (which is obtained by solving the triangle $AP'B$) if $AP' = D'_a$ and $BP' = D_b$, belongs to the position circle MPN (fig. 3) and is close to the point P which is to be determined.

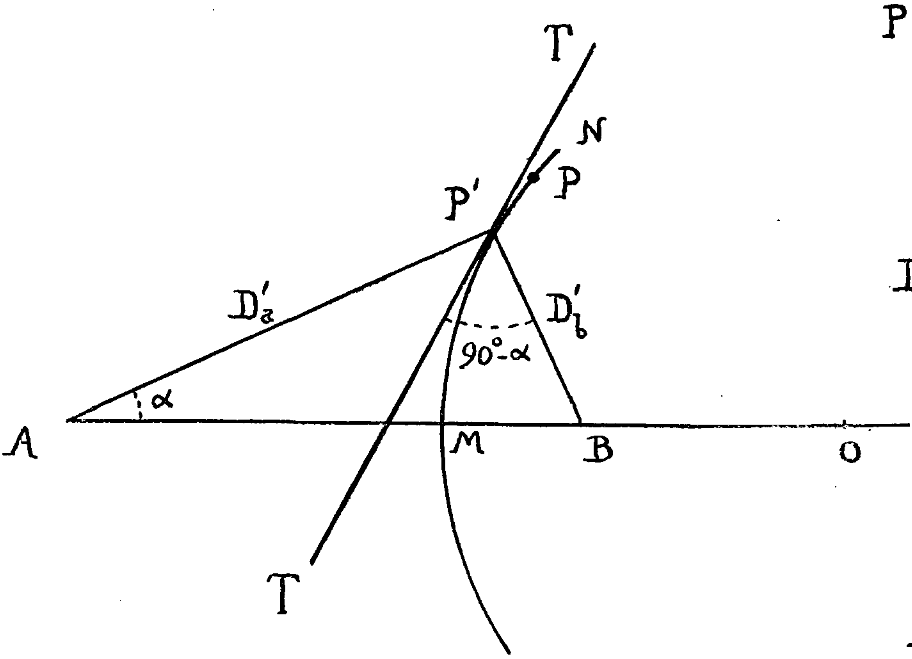

Fig. 3

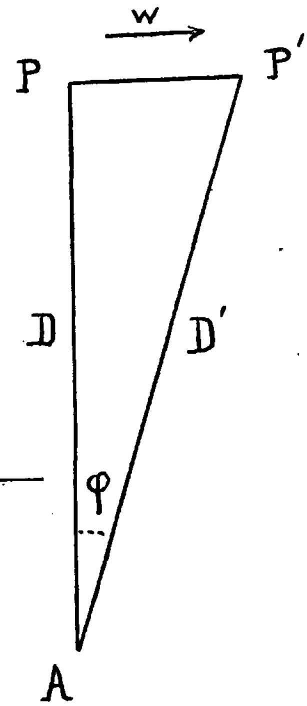

Fig. 4

In the vicinity of P' the differences between the tangent at P' and the circle are negligible ; therefore it follows that this tangent may be substituted for the circle, as the geometrical locus of position P.

The point P' is referred to hereafter by the name of *determinative point* of the position line and triangle $AP'B$ will be called the *position triangle.* Besides tangent TPT' will be called the *position tangent,* or *locus of equal ratio between the distances,* or more briefly " locus of equal $\frac{\theta_a}{\theta_b}$.

INFLUENCE OF CURRENT ON THE VELOCITY OF PROPAGATION OF SOUND IN A GIVEN DIRECTION.

5. The influence of a current on the velocity of propagation of a sound wave along a line joining two fixed points, P and A, will now be examined, the origin of the wave being situated at P (fig. 4).

First assume that a current of velocity w runs in a direction perpendicular to the direction PA. As usual the velocity of sound in water is referred to as v. The sound wave travels *through the moving mass*, radially, at a velocity v, from the point in the *mass of water* where it originated,. This point, which at the moment of emission was at P, is (presumably) carried along by the current in the straight line PC, perpendicular to PA.

If P' is the position attained at the moment at which the sound wave reaches A, the track followed by that wave, at the speed v, *through the moving mass*, will be represented by the line $P'A = D'$, and the interval of time between the emission of the sound at P and the arrival of the sound at A will be :

$$\theta' = \frac{D'}{v}$$

Should it happen that the propagation takes place in still water, the interval would be :

$$\theta = \frac{D}{v}, \; D \text{ being equal to } PA$$

It will now be seen that it may be assumed, *without introducing any sensible error*, that

$$\theta' = \theta$$

i.e. that the interval of time taken by the sound to cover the distance PA is *not* sensibly modified by a current perpendicular to PA. This means that the *velocity of propagation of a sound wave in a given direction is not modified by a current at right angles to that direction.*

6. Now (fig. 4) $\frac{PP'}{P'A} = \frac{w}{v}$; but in the right angled triangle APP'

$$\frac{PP'}{P'A} = \sin \varphi, \text{ if } \varphi = PAP'.$$

Therefore

$$\sin \varphi = \frac{w}{v}$$

As a matter of fact it should be noted that under ordinary circumstances

$$w \leqslant 1{,}5 \text{ metres per second}$$

or in other words, the speed of the current is usually less than 3 knots. In fact v has a value of about 1500 metres per second, and consequently

$$\sin \varphi \leqslant \frac{1}{1000}$$

or $\varphi \leqslant 0{,}001$ radians (approximately 3'.5)

As $D = D' \cos \varphi$

consequently $\theta = \theta' \cos \varphi$

then $\theta' - \theta = \theta' (1 - \cos \varphi)$

But φ being very small, $\cos \varphi = 1 - \frac{\varphi^2}{2}$

and therefore $\theta' - \theta \leqslant \frac{\theta}{2000000}$

Consequently the difference $\theta' - \theta$ is entirely negligible.

7. Let it be assumed that the current, of speed w, is *parallel* to the direction PA. It is evident that the sound wave will cover the distance $D = PA$ at a speed $v + w$ when the current is running from the sonic source towards A, and $v - w$ when the reverse is the case.

8. CONCLUSIONS. — The velocity of propagation of the sound wave in a given direction is *not* modified by the component of the current *perpendicular* to this direction. But it *is* modified by the *parallel* component and is subject to an increase (positive or negative) which is equal to this component.

Therefore the following law may be enounced :—

When there is a current, the velocity of propagation of sound in a given direction is equal to the velocity of propagation in still water, plus the increment (positive or negative) which is equal to the projection of the speed of the current along the direction considered.

DISPLACEMENT OF THE POSITION OF EQUAL $\frac{\theta_a}{\theta_b}$ DUE TO CURRENT.

9. Let triangle APB (fig. 5) be considered. This triangle is formed by the base and the sonic source P'. The influence of the current on the value of the ratio $\frac{\theta_a}{\theta_b}$ will now be determined.

Consider separately the effects produced by the two perpendicular components directed respectively along the bisector of the interior angle at P, of the position triangle and along the perpendicular to this bisector.

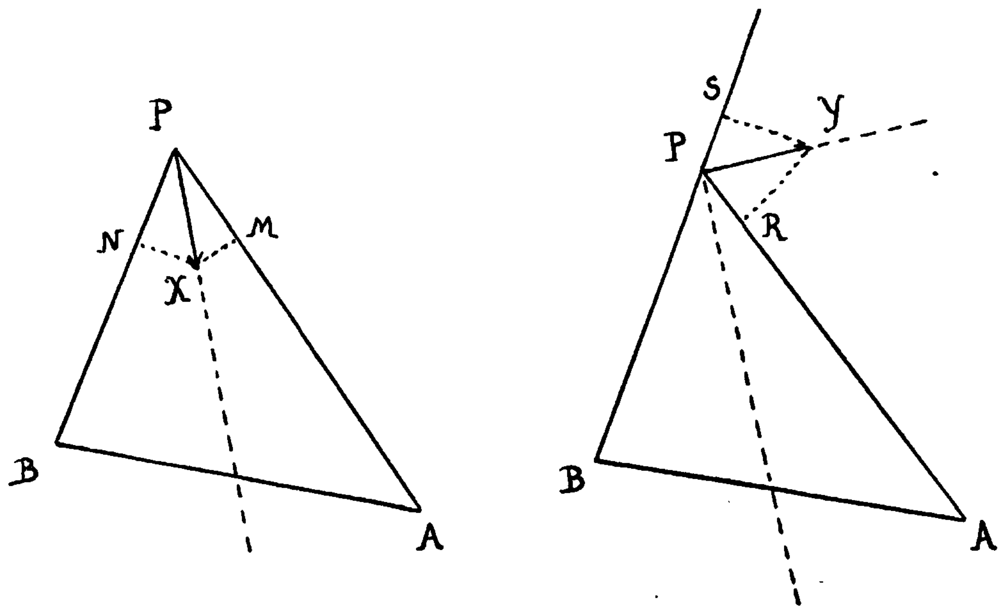

Fig. 5

Component PX (bisector) has *no* influence whatever on the ratio $\frac{\theta_a}{\theta_b}$. In fact, the projections PM and PN of the component PX on the directions PA and PB modify the speed of propagation along PA and PB equally and in the same direction. Therefore the *ratio* of the times taken by the sound to travel over the distances PA and PB remains the same and retains the same value as in still water.

But on the contrary, component PY, which lies along the *perpendicular* to the *bisector*, alters the value of the ratio $\frac{\theta_a}{\theta_b}$ as will now be seen.

10. Let PY (fig. 5) be projected along directions PA and PB; the two projections PR and PS are equal to one another.

Taking $PY = u$, then : —

$$PR = PS = u \sin \frac{\gamma}{2} = u', \ (\gamma = APB)$$

Considering the direction of the two vectors PR and PS, the velocity of propagation along PA is increased by an amount u', and diminished by an amount u' along PB. Therefore the times taken by the sound wave to reach A and B will be :

$$\theta_a = \frac{D_a}{v + u'} \qquad \theta_b = \frac{D_b}{v - u'}$$

$$\frac{\theta_a}{\theta_b} = \frac{D_a}{D_b} \ \frac{v - u'}{v + u'}$$

If there be no current, times θ'_a and θ'_b will be observed to be such that:

$$\frac{\theta'_a}{\theta'_b} = \frac{D_a}{D_b}$$

consequently:

$$\frac{\theta_a}{\theta_b} = \frac{\theta'_a}{\theta'_b} \cdot \frac{v - u'}{v + u'}$$

Therefore the position lines which correspond to the two cases are different.

They are close to one another however, because u' being extremely small with reference to v, factor $\frac{v - u'}{v + u'}$ is very near unity.

In other words, if the position triangle be solved without taking into account the effects of the current on the velocity of propagation of sound in the directions *PA* and *PB*, a determinative point *P'* (fig. 6) will be obtained which does not lie on the real position line *PT*, but is outside though quite close to it.

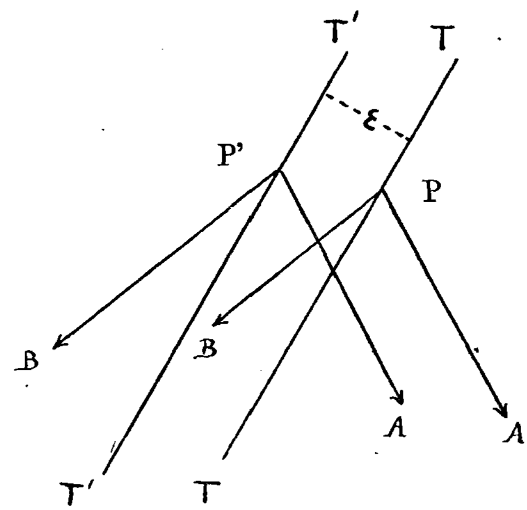

FIG. 6

Seeing how small is the value of the departure *PP'*, the position line *P'T'* passing through *P'* may be considered as parallel to *PT*; therefore the influence of the current appears as a *parallel lateral displacement*, ε of the position line. The amount of this displacement must be determined.

11. It will be now demonstrated that the displacement ε is given by the relation:

(*) $\varepsilon = \frac{u}{v} \cdot \frac{h}{\cos \frac{\gamma}{2}}$, in which:

u = velocity of the component of the current which is perpendicular to the interior bisector at P of the position triangle ;
v = velocity of sound in still water ;
h = height of the vertex of the triangle APB above its base AB ;
γ = angle at P of the position triangle.

By adopting the approximate value of 1500 metres per second for v, the relation (*) shows that *for every knot* of current at right angles to the bisector (u = 0,5 met./sec.) the displacement will be :—

$$\varepsilon = \frac{1}{3} h^{kil.} \sec \frac{\gamma}{2}$$

(in which $h^{kil.}$ is the measure of the height in *kilometres*.)

The formula (*) becomes indeterminate when point P is quite close to the base AB or when it is on the base itself, whereas the departure is clearly defined. Then, observing that in the position triangle :

$$h = \frac{D_a D_b}{q} \sin \gamma \qquad \text{in which } q = AB,$$

the formula (*) may be altered as follows :

$$\varepsilon = 2 \frac{u}{v} \frac{D_a D_b}{q} \sin \frac{\gamma}{2}$$

The departure :

$$\varepsilon = \frac{2}{3} \frac{D_a^{kil.} D_b^{kil.}}{q^{kil.}} \sin \frac{\gamma}{2} \text{ metres.}$$

(in which $D_a^{kil.}$, $D_b^{kil.}$, $q^{kil.}$, are the measures of D_a, D_b and q in kilometres) is that for each knot of current perpendicular to the bisector.

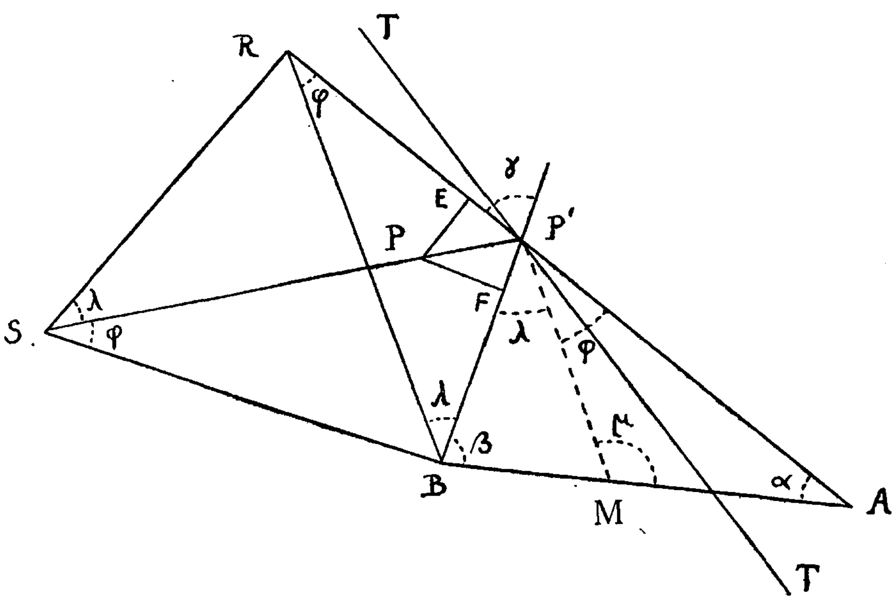

FIG. 7

12. Let (fig. 7) the position triangle $AP'B$ be drawn in which the sides $P'A$ and $P'B$ are determined, *without taking the current into account, i.e.* by making $D'_a = \theta_a v$, $D'_b = \theta_b v$ in which θ_a and θ_b are the observed intervals of time and v the velocity of sound in sea water.

On the same base let triangle APB also be constructed in which the sides PA and PB are fully determined, taking into account the variations of speed of sound owing to the influence of current.

Then :
$$PA = D_a = \theta_a (v + u') = D'_a + \theta_a u'$$
$$PB = D_b = \theta_b (v - u') = D'_b - \theta_b u'$$

Therefore, to obtain P it will be necessary to produce the side $P'A$ of the triangle $AP'B$ to E, so that $P'E = \theta_a u'$, and to reduce side $P'B$ by the quantity $P'F = \theta_b u'$; then with A as centre and radius AE to draw the arc EP and with B as centre and radius BF to draw the arc FP. These two arcs which, by reason of their smallness, may be taken to be the perpendiculars to AE and BF from E and F, determine the required point P by their intersection.

The departure $P'P$ must now be determined and then it must be projected on the perpendicular to the tangent of position $P'T$. This projection gives the required value for the lateral parallel displacement ε, which gives the influence of the current on the position line. $P'P$ is the diagonal of the quadrilateral $PEP'F$. The sides $P'E$ and $P'F$ of this quadrilateral are proportional to the distances D_a and D_b respectively; it is similar to the quadrilateral $PRSB$ which is constructed by applying segment $P'R = AP'$ to the prolongation of the side AP', and then drawing the perpendicular to AP' through R and the perpendicular to BP' through B. The ratio of similarity between these sides of the quadrilateral $PEP'F$ and the homologous sides of $PRSB$ is $\frac{u}{v}$

Therefore :

$$(1) \quad P'P = \frac{u'}{v} P'S$$

The problem is thus reduced to the determination of $P'S$.

It will be noted that the diagonal RB of the quadrilateral $PRSB$ is parallel, on account of its construction, to the median $P'M$ of triangle $AP'B$. Therefore by taking :

$$AP'M = \varphi, \qquad BP'M = \lambda$$

then
$$P'RB = \varphi, \qquad P'BR = \lambda$$

Besides the quadrilateral $PRSB$ being circumscribable,

$$P'SB = P'RB = \varphi, \qquad P'SR = P'BR = \lambda$$

and therefore :
$$BP'S = 90^\circ - \varphi \qquad RP'S = 90^\circ - \lambda$$

In the right angled triangle $P'BS$

$$(2) \quad P'S = \frac{P'B}{\cos BP'S} = \frac{D_b}{\sin \varphi}$$

Let $AB = q, \quad P'M = m, \quad P'AB = \alpha$
Then in triangle $AP'M$:

$$\frac{\sin \varphi}{\frac{1}{2} q} = \frac{\sin \alpha}{m}; \qquad \sin \varphi = \frac{1}{2} \frac{q}{m} \sin \alpha$$

and by substitution in (2)

$$P'S = 2 \frac{D_b}{\sin \alpha} \frac{m}{q}$$

In triangle $AP'B$:

$$\frac{D_b}{\sin \alpha} = \frac{q}{\sin \gamma}, \quad (\gamma = AP'B), \text{ and consequently :}$$

$$P'S = 2 \frac{m}{\sin \gamma}$$

By substituting this value of $P'S$ in formula (1), :

$$P'P = 2 \frac{u'}{v} \frac{m}{\sin \gamma} \quad \text{is obtained.}$$

But $u' = u \sin \frac{\gamma}{2}$, and consequently :

$$(3) \quad PP = \frac{u}{v} \frac{m}{\cos \frac{\gamma}{2}}$$

Noting that the tangent of position $P'T$ and $P'B$ include the angle $90^\circ - \alpha$ (§ 3)

$$\tau = TP'B = 90^\circ - \alpha$$

On the other hand $P'S$ (which coincides with $P'P$) makes with the latter side $P'B$ the angle $90^\circ - \varphi$, and therefore :

$$SP'T = TP'B + BP'S = 180^\circ - (\varphi + \alpha)$$

Let μ be the angle at M in triangle $AP'M$,
then : $\mu = 180^\circ - (\varphi + \alpha)$
therefore $SP'T = \mu$

The projection of $P'P$ on to the perpendicular to the position line $P'T$ is :

$$\varepsilon = P'P \sin (90^\circ - SP'T)$$

therefore $\varepsilon = P'P \sin \mu$

$$\varepsilon = \frac{u}{v} \frac{m \sin \mu}{\cos \frac{\gamma}{2}}$$

But the quantity $m \sin \mu$ is equal to the height h above the base AB of the position triangle $AP'B$, and therefore :

$$\varepsilon = \frac{u}{v} \frac{h}{\cos \frac{\gamma}{2}} \qquad \text{(Q.E.D.)}$$

ERROR OR UNCERTAINTY OF THE POSITION LINE OF EQUAL $\frac{\theta_a}{\theta_b}$ *DUE TO ERRORS IN TIME MEASUREMENT.*

13. An accidental error t_1 in measuring interval θ_a causes a change in distance D_a (fig. 8) equal to vt_1.

Lay off from *P'*, along the side *AP,'* the segment $P'R = vt_1$, which represents the above change; with *A* as centre, and *AR* as radius draw the arc *RP''* ; with *B* as centre, and *BP'* as radius, draw the arc *P'P''*. These two arcs may be assumed to coincide with the perpendiculars dropped from *R* on to the side *AP'* and from *P'* on to the side *BP'*.

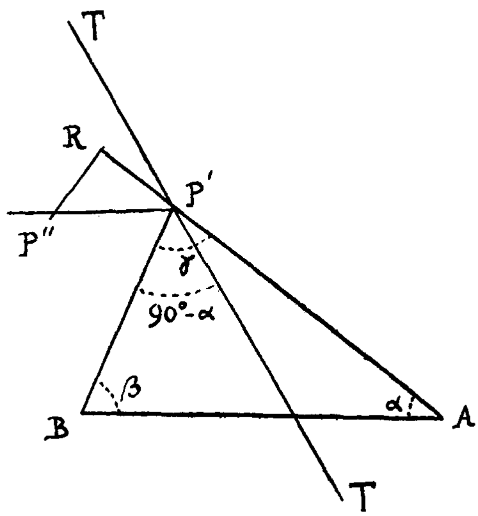

FIG. 8

P'' is the determinative point corresponding to the new value of the distance D_a, distance D_b being unchanged.

In the right angled triangle *P'RP''*, in which $RP''P' = \gamma$,

$$P'P'' = \frac{P'R}{\sin\gamma} = \frac{vt_1}{\sin\gamma}$$

The tangent of position *P'T* lies at an angle of $180° - \alpha$ with *P'P''* and, therefore the projection of *P'P''* on the perpendicular to the tangent is :

$$P'P'' \sin\alpha = vt_1 \frac{\sin\alpha}{\sin\gamma}$$

But $\frac{\sin\alpha}{\sin\gamma} = \frac{D_a}{q}$ and therefore, if e_1 is the error of the position line due to error t_1, then :

$$e_1 = vt_1 \frac{D_b}{q}$$

Let the effect of an accidental error t_2 in measuring the interval θ_b, be considered *i.e.* that the distance D_b be changed by the quantity vt_2, keeping the distance D_a unchanged. By similar reasoning the values e_2 of the position line error due to error t_2 may be obtained.

$$\text{Now :} \qquad e_2 = vt_2 \frac{D_a}{q}$$

If t_1 and t_2 be given the value of the mean errors of measurement, the resulting mean error E_t or *uncertainty* in the position line due to incorrect measurement of time will be obtained.

$$E_t = \sqrt{e_2^2 + e_2^2} \qquad \text{and, assuming } t_1 = t_2 = t,$$

$$E_t = vt \frac{\sqrt{D_a^2 + D_b^2}}{q}$$

If $t = 0{,}01$ sec, then : $vt = 15$ metres.
Therefore : *the mean error, or uncertainty, of the position line for each error of a hundredth of a second in the measurement of the intervals* θ_a *and* θ_b *is equal to*

$$= 15 \frac{\sqrt{D_a^2 + D_b^2}}{q} \text{ metres.}$$

(In the calculation of this formula it is necessary to express D_a, D_b, and q in kilometres ; an approximate value suffices.)

14. In paragraphs 11 and 12, the displacement due to the current, has been calculated, not so much as a correction formula to be used in practice, as to give an idea as to the amount of error which an unknown or ill-determined current may introduce into the determination of a position.

In fact, the use of a formula of correction would only be justified on the hypothesis of a *uniform* current throughout the zone covered by the position triangle, which by reason of the size of the latter, hardly ever is the fact.

It is true that an average current may be considered, the velocity and the direction of which may be assumed, based on measurements taken at various points of the zone. But this estimation may sometimes be arbitrary and in each case may have a large error. It is precisely to determine the error due to an *approximate* measurement of the current, or to the fact that a weak current (or a barely noticeable one) has been neglected, that the formula found in paragraphs 11 and 12 should be used.

In this case, that of a current unknown or accidental by its nature, this error can be calculated for each case and thus the *resulting uncertainty* of the position line of equal $\frac{\theta_a}{\theta_b}$ can be obtained.

Calling this uncertainty E, and taking into account the different causes of error *i.e.* mean displacement ε (§ 11) due to a possible current of velocity u, and direction perpendicular to the bisector, displacements e_1 and e_2 (§ 13) caused by the mean errors t_1 and t_2 in the measurement of intervals θ_a and θ_b.

$$E = \sqrt{\varepsilon^2 + e_1^2, + e_2^2}$$

The following is an example of the application of this formula :

Let $u = \frac{1}{2}$ *knot* ; $t_1 = t_2 = t = 0{,}01$ sec.

After reduction q, D_a and D_b being expressed in kilometres :

$$E = \frac{15}{q}\sqrt{D_a^2 + D_b^2 + \left(\frac{1}{45}\right)^2 D_a^2 D_b^2 \sin^2\frac{\Upsilon}{2}} \quad \text{metres.}$$

NUMERICAL EXAMPLE : (The data are those of the example of the determination of a position line given in the next paragraph.)

$$q = 22{,}9 \text{ kilometres}$$
$$D_a = 45{,}2 \quad »$$
$$D_b = 42{,}0 \quad »$$
$$\frac{\Upsilon}{2} = 15^\circ$$

$$D_a^2 = 2043.04$$
$$D_b^2 = 1764.00$$
$$\left(\frac{1}{45}\right)^2 D_a^2 D_b^2 \sin^2\frac{\Upsilon}{2} = 119.22$$
$$3926.26$$
$$\frac{15}{q} = 0.655$$

$$E = 0.655\sqrt{3926.26}$$
$$= 0.655 \times 62.66$$
$$= \underline{41.04} \text{ metres}$$

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15. *DETERMINATION OF A POSITION LINE OF EQUAL* $\frac{\theta_a}{\theta_b}$

The extremities A and B of the base are known by their rectangular coordinates. The origin O of the coordinates is situated at the point A.

$$A\begin{cases} x_a = o \\ y_a = o \end{cases} \qquad B\begin{cases} x_b = -\ 901.08 \text{ metres} \\ y_b = -\ 22849.24 \quad » \end{cases}$$

With these values of the coordinates it may be calculated that $q = AB = 22876.0$ and the azimuth of B from $A = (AB) = 180°15'30''$.

The *approximate* values of the temperature of water (16° C.) and the salinity (35 o/oo) are known. Thus by adopting (from the Tables by H. N. HECK & Jerry H. SERVICE (*). $v = 1504$ metres,

θ_a = time required by a sound to travel the distance $PA = 30.063$ sec.
θ_b = « « « « « « « « « $PB = 27.949$ sec.

Then :

(*) "*Velocity of Sound in Sea Water*". U.S. *Coast & Geodetic Survey*, Special Publication N° 108, 1924, page 26 (See also *Hydrographic Review*, Vol. III, N° 1, November 1925, pages 69 to 100).

$D_a = 30\,063 \times 1504 = 45214.8$, metres
$D_b = 27\,949 \times 1504 = 42035.3$, metres

D_a	45 214.8		
D_b	42 035.3		
q	22 367.0		
$2s$	110 117.1		
s	55 058.55	*colog*	5.2591752
$s - Da$	9 843.75	*log*	3.9931606
$s - Db$	13 023.25	*log*	4.1147194
$s - q$	32 191.55	*log*	4.5077420
k^2		*log*	7.8747972
k		*log*	3.9373986

$\frac{\beta}{2}$	41°19'54",37	*log tan* $\bar{1}.9442380$	β	32°39'48",74
$\frac{\alpha}{2}$	33°36'54",88	*log tan* $\bar{1}.8226792$	α	67°13'49",76
$\frac{\gamma}{2}$	15°03'10",78	*log tan* $\bar{1}.4296566$	γ	30°06'21",56
				180°00'00",06

The polar coordinates of the determinative point P' referred to the origin A are therefore :

$$(AP') = 182°15'30" - \alpha = 115°01'40".24$$
$$AP' = 45214.8 \text{ metres.}$$

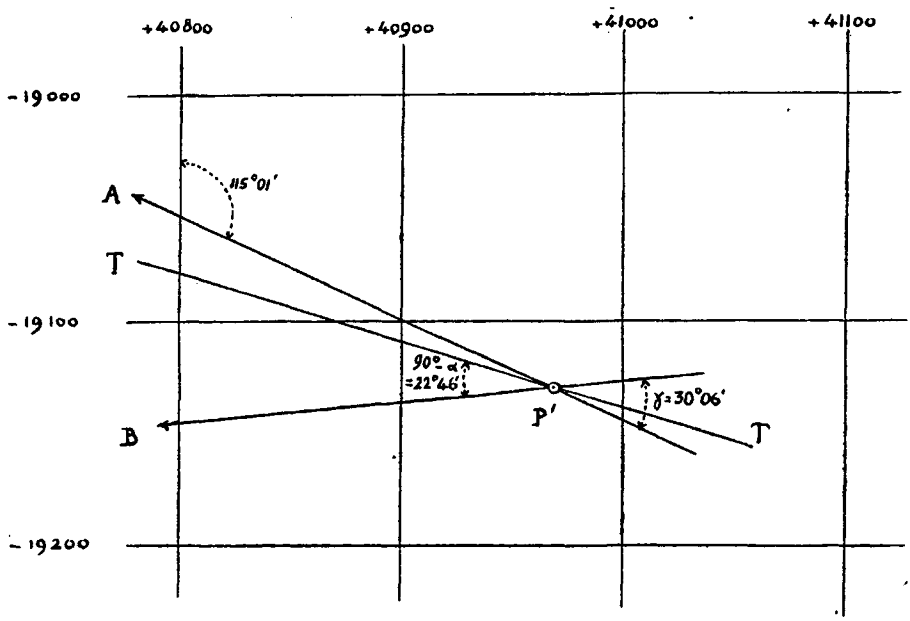

FIG. 9

With these values the coordinates x_p and y_p of P' will be determined

AP' 45214,8	*log*	4.6552806	*log*	4.6552806
(AP') 115°01'40",24	*log sin*	1.9571773	*log cos*	1.6264016
	log x_p	4.6124579	*log* y_p	4.2816822
	x_p +	40969,24	y_p —	19128,55

With these data the determinative point P' may be placed on a large scale diagram (*see* fig. 9). On this diagram, directions $P'A$ and $P'B$ have been drawn from the results of calculation. Finally position tangent $TP'T$ is drawn, which lies with reference to the side $P'B$ (the smaller), at an angle of :—

$$\tau = 90^\circ - \alpha = 90^\circ - 67^\circ 13' 50''$$
$$\theta = 22^\circ 46' 10''$$

If the intervals θ'_a and θ'_b have been calculated at the extremities of another base $A'B'$, wihch is *favourably situated* with reference to AB (*), a second determinative point, P'', may also be calculated by the same process. A second position tangent may be drawn through this point on the same diagram, which, by its intersection with the first will, determine the point P which is required,

September 1927.

To be continued.

(*) It is unnecessary to say that one of the extremities of the new base may coincide (and this will generally be the case) with one of the extremities of the base AB. By this means the position P will be determined by measuring 3 intervals of time.