USE OF AIRCRAFT FOR SURVEYING.

In order to comply with a resolution of the 2nd. International Hydrographic Conference held in Monaco in 1926, the International Hydrographic Bureau addressed the following circular letter to the various States Members :—

CIRCULAR-LETTER N° 34-H of 1926.

USE OF AIRCRAFT FOR SURVEYING.

MONACO
27th. November 1926.

Sir,

I have the honour to inform you that the Second International Hydrographic Conference recently held in Monaco made the following resolution :

" That those Nations which use aerial photography for surveying and have not yet sent reports on the subject to the Bureau, should do so ".

2.— *Should this resolution affect your country, the Bureau will be glad to receive such reports.*

3.— *The subject in question may be logically divided into two sections :*

(A) Apparatus for and method of taking the photographs ;

(B) Apparatus for and method of plotting from the photographs.

It is requested that reports on these two sections be rendered separately.

4.— *Of course the Bureau will be glad to receive further reports from those Nations which have already sent them, if improvements in apparatus or methods have been effected.*

I have the honour to be,
Sir,
Your obedient Servant

Signed : G. SPICER-SIMSON
Secretary General.

So far answers have been received from PORTUGAL, DENMARK, GERMANY, GREAT BRITAIN, JAPAN, the UNITED STATES OF AMERICA, the ARGENTINE REPUBLIC, and from AUSTRALIA.

In addition, it should be noted that reports on the methods employed in FRANCE and in the NETHERLANDS were published in Vol. I. N° 1, pages 73 and 88, of the *Hydrographic Review*.

The reports received from the UNITED STATES OF AMERICA, the ARGENTINE REPUBLIC and from AUSTRALIA are given below.

PORTUGAL and JAPAN have informed the Bureau that their Hydrographic Services have not yet made use of air photographs for surveys.

GERMANY answered that, by the Treaty of Versailles, the Reichsmarine was not allowed to have aircraft, but that commercially numerous appliances had been constructed for taking and for the restitution of photographic views. The "International Company for Photogrammetry" (German Section) deals with all subjects connected with these questions. The results of the studies made in Germany are issued in a periodical pamphlet: "Bildmessung und Luftbildwesen" published by R. REISS, G. m. b. H. at Liebenwerda (Saxony).

In GREAT BRITAIN, the request of the International Hydrographic Bureau was forwarded to the Air Survey Committee, which centralizes all studies on air surveys. The publications of this Committee, which are of general interest, and the special information on those questions which are particularly interesting to it have been forwarded to the International Hydrographic Bureau and will continue to be so forwarded. Particular mention should be made of a very complete bibliography on this subject, but no treatise mentioned therein has special reference to hydrographic surveys. The above documents may be procured from H. M. Stationery Office, Imperial House, Kingsway, London, W. C. 2. and from H. M. Stationery Office, 28, Abingdon Street, London, S. W. 1.

DENMARK informed the Bureau that aerial photography is not used in the Danish Hydrographic Office. The Topographical Service makes use of it but only for revising its charts.

The apparatus used is an aluminium camera fitted with a Zeiss - Protar VII. objective, having a focal length of 26 c/m (11 ins). The diaphragm shutter is situated between the two groups of lenses which compose the objective. A yellow glass is placed in front of the objective. Glass plates only are used; the camera is held by hand as vertically as possible and the flight is made at a height of about 9300 ft. (2800 metres).

The negatives are printed on silver bromide paper by means of a projecting apparatus called the "*perspectograph*" which enlarges the picture to a scale of $\frac{1}{7,500}$. The positives thus obtained are assembled, taking as a basis the original map enlarged to the same scale.

The result obtained from this mosaic is photographed and transferred on to an aluminium plate. A blue print is taken which is handed to the topographer, who is sent to the actual site in order to compare it with the photograph, and then draws up a new draft of the map.

Professor G. CASSINIS, Secretary of the Italian "Photogrammetric Section" formed within the "Italian Aeronautic Association" (A. I. D. A., Rome, Via delle Capelle, 35), has been good enough to send to the Bureau the publications in which he deals with the methods used for photographic air surveys.

The *von Orel* process is in general use at the Italian Military Geographical Institute and at the Stereographical Institute; one method, that of Lieutenant Umberto NISTRI, is used with good results by the Aerophotogrammetric Survey Company (S. A. R. A., Rome, Via Francesco-Negri, 13) and another, that of Lieutenant E. SANTONI, is being studied by the Military Geographical Institute.

REPORT FROM THE HYDROGRAPHIC OFFICE OF THE UNITED STATES OF AMERICA.

SUBJECT: *Use of Aircraft for Hydrographic Surveying.*

From a very unpretentious beginning, the utilization of aircraft in hydrographic surveying has developed until it is the policy of the Hydrographic Office of the United States Navy to profit by all of its possible applications. By thus using aircraft and aerial photographs to the fullest extent the cost of a survey, in both time and money, is considerably reduced, and, in addition, a superior product is obtained.

For a number of years the value of aerial photographs for the purpose of hydrographic surveying was questioned. In about 1916, the value of such photographs was definitely established and an endeavor made to obtain some for experimental purposes. Unfortunately at this time the aeronautical organization of the Navy was extremely small and fully occupied with other and more essential military duties. For these reasons it was not until after the World War with its consequent expansion of the aviation activities of the United States Navy that planes or pilots became available.

In 1919 a small area of about 50 square miles covering Guantanamo Bay was photographed from the air. From this time on the number of instances in which aircraft were used increased in geometrical ratio until for the past three years no survey undertaken by this Office has been without the services of aircraft and aerial photographers.

In addition to work accomplished by naval aircraft for other governmental activities such as topographic surveys for the Interior Department, Department of Agriculture, *etc.*, the following surveys have been and are being aided by aircraft :

North coast of Cuba,
South coast of Cuba,
Harbor of Mariel, Cuba,
Pearl Harbor, T. H., and
Coast of Venezuela.

All projected surveys included the use of photographs taken from planes.

The reason for the adoption of the policy of utilization of aircraft in all Hydrographic Office surveys undertaken by this Office is the two-fold advantage that may be gained by this auxiliary. Not only do the planes and aerial photographs aid the production of the chart technically, but they are also of great value in expediting the logistical side of a survey expedition.

In expediting the logistical supply of a survey the work of the aircraft falls naturally into two headings :

First,— Mechanical aid such as transportation of personnel and supplies to otherwise inaccessible spots or the saving of time that might result by such transfer by air.

Second, — Semi-technical services such as reconnaissance flights and the utilization of photographs taken in advance of the actual surveys by ground methods to aid in determining the location of stations, signals, *etc.*

In considering the first of these aids, the type of plane and its equipment must be considered. Generally speaking, hydrographic surveys that are undertaken by this Office are conducted chiefly in areas previously unsurveyed or where the survey is of such early date as to give doubts as to its accuracy. In addition, these areas generally abound in more or less uninhabited and uncultivated areas. Thus, in addition to the fact that they are along the coast, the natural landing areas which an airplane is capable of landing upon are few and far between.

The working base of the survey has in addition a ship, generally a converted yacht, of the U. S. Navy. Therefore, the utilization of a land plane is generally impracticable and the use of a seaplane more feasible. In most areas, due to the conformation of the coast, it occasionally becomes necessary for the planes to fly inland for a considerable distance where in case of engine or other trouble the water could not be reached. For these and other minor reasons, the use of an amphibian plane in a hydrographic survey is most desirable. The type of plane used up to the present time with the greatest success for such work has been the Lœning amphibian plane modified for aerial photography. It is with this type of plane that practically all of the recent surveys undertaken by the Navy, both for the Hydrographic Office and for other governmental activities, have been conducted.

In the case of the Hydrographic Office survey of the Gulf of Venezuela, an occasion arose for the transfer of a shore party to a spot on the western side of a long narrow lagoon. Due to the presence of an unbroken reef about a mile off shore, it was impossible to transfer the men and their delicate instruments by means of a small boat. The plane, however, was able to embark the party and their equipment, take off from the water, land inside the reef and land the party at the desired point. Had not the plane been available for this service, a march of approximately forty-five miles around the shores of this inlet and through the most dense of tropical jungles would have been necessary. In this one small item a considerable saving of time, money, and labor was made possible. In numerous other instances, the transfer of personnel by air has eliminated the necessity of moving the surveying vessel.

Frequently actual conditions existing on the survey area vary considerably from those on the old charts. Under such conditions and, of course, also when the area has been previously unsurveyed, a reconnaissance flight in which the hydrographic engineers are carried is of great value. Certain areas may be shown or may be discovered to be of little value and their intensive investigation unnecessary.

The other phase of logistical aid comes when aerial photographs are available in advance of the time for the ground work, It is the standard policy of this Office for aerial photographs of an area to be taken during the year previous to the survey of the area by ground methods; whenever the survey area is of such an extent that more than one season is required to complete the work. By a study of these photographs the most favorable positions for triangulation stations, signals, *etc.*, may be determined. Locations which from a study of existing information are apparently most desirable may, when the aerial photographs are studied, turn out to be inaccessible or otherwise undesirable. For instance, in the Gulf of Venezuela survey a certain location along the shore line has been tentatively chosen as the sight of a hundred foot tower. When the aerial photographs, both vertical and oblique, were studied, it became apparent that some other position would have to be selected. The presence of a fairly high bluff with practically no beach extending along the coast line about 1 ½ miles in each direction from the selected site precluded the landing of necessary material in the previously determined vicinity. By means of the information afforded by these photographs, a new site just as advantageously located and much more accessible was selected.

Another instance of this kind occurred along the north coast of Cuba. A certain small island from a study of a chart based on a survey made some fifty years ago appeared to be a logical location for a triangulation station. The aerial photographs, however, indicated that the true position of this island varied considerably from the supposed position and it was therefore unsuitable for a station of this description. A second site more advantageously located was selected and a close inspection of aerial photographs showed that the erection of a tower would be extremely difficult due to the marshy characteristic of the available land. A third position was chosen which, when the actual survey was undertaken, proved readily accessible and very satisfactory. If this advance information had not been available, a considerable expenditure of useless effort would have been made.

In spite of the fact that the photographs, whenever possible, are taken considerably in advance of the ground work, they need not be retaken as the triangulation stations and signals may be plotted readily on them.

In using the aerial photographs for aiding the technical production of the desired charts, slight variations in method are necessary, due to the character of the different surveys. Less refinement is naturally necessary in the case of a running survey than that which obtains when the survey is of secondary or higher order of accuracy. The following method is, however, in general the one used.

In taking the aerial photographs, only the delineation and azimuth of the coast line and such topographic features as will be of assistance to navigation are covered. Generally speaking, this means that the photographic flight is one closely paralleling the shore line. The overlap between individual photographs may therefore be increased without expense over that necessary in topographic mapping where the sides as well as the ends of the individual photographs must overlap. Between 40 and 60 per cent, overlap is the general rule. The plane flies at a certain specified altitude, up to the present time generally 10,000 feet, and the individual pictures cover an area of approximately one square mile. In the vicinity of triangulation stations the shore line is developed by ground methods for the distance of approximately one mile on each side. The result of this work is plotted on a field sheet at the desired scale. The aerial photographs covering this particular area are examined and the relation between the scale of the photographs and the actual area ascertained.

From these photographs by means of a pantograph set to give the necessary reduction, the coast line and other features are drawn on the field sheet. Then the photographs immediately adjacent to that overlapping on the area surveyed by ground methods are studied and three objects visible in both of these two photographs are located by measurement of the distance between these objects and the variation in scale between the two photographs may be discovered. The angles between the common points give the orientation of this second photograph. The photograph is then placed on the pantograph, the necessary change in the pantograph setting made and the shore line of this photograph is drawn and tied up with that obtained by ground method. This process is continued until the second triangulation station and its shore line is reached. With triangulation stations approximately five miles apart, 60 per cent of the shore line work is done by aerial photography. In the vicinity of the triangulation stations the coast is generally more accessible and this portion most adapted to development by ground methods. Thus the most difficult as well as the greater part of the shore line work is done by aerial photography. Frequently the coast line is so difficult of access that it could be done readily by no other known method.

Two features of this method might strike an observer accustomed to dealing with aerial photographs as unusual. First, no attempt is made to form a mosaic from the individual photographs, and second, no apparent attempt is made to eliminate the unavoidable errors incidental to aerial photography.

It is accepted by all who have had to do with aerial photographic surveying, that it is useful only in conjunction with ground controls obtained by the established methods of surveying on the ground, at least for really accurate work. Its greatest field of usefulness for the present appears to be for making preliminary reconnaissance and for filling in detail between control points. For these purposes it is without rival. One reason for this limitation of use is the fact that there are certain distortion errors liable to be present in the photographs which require some consideration.

The errors of distortion are as follows :

a). That due to the lens. This is circularly symmetrical about the centre of the picture.

b). That due to the use of focal plane shutter.
This produces a difference in the scale of the picture in the two directions paralleled and perpendicular to the shutter travel.

c). That due to shrinkage or stretching of film or paper. This is usually circularly symmetrical and uniform for film, but undirectional for paper.

d). That due to difference of elevation of the ground.

e). That due to lack of exact verticality of the optic axis of the camera.

Aside from these distortions or errors, the accuracy with which any point can be located on the ground is limited by the scale, the definition of the photograph, and the means used for measuring.

In all probability the tilt error causes the greatest trouble. Consider that the location on the ground of the point shown at the center of an aerial photograph is known. Then, unless the photograph was made with the optic axis of the camera accurately vertical, the apparent locations on the ground of other points as judged by their positions in the photograph will be in error owing to perspective distortion. The amount of the error for any particular point will depend upon the flying altitude, on the angle of tilt, on the focal length, on the distance, in the photograph, of the image of the point from the center of the plate and on the azimuth of the point relative to the direction of tilt. For given values of the first four variables the error is greatest for that azimuth which coincides with the direction of the tilt. The absolute error from this source is directly proportional to the flying altitude. And, it is also increased with the increase of the angular distance from the optic axis, that is, the greater the angle of view of the camera.

The government specifications for aerial lenses give maximum allowable distortion of .08 millimeter per 100 millimeter focal length at 18 degrees from the optic axis and .02 millimeter per 100 millimeter focal length at 24 degrees from focal axis.

Focal plane shutter distortion can be readily computed for any given set of conditions. Assuming a shutter with .5 inch wide slit giving an exposure of .01 second on an 18 by 24 centimeter plate, the travel will be approximately seven inches so that the time elapsing between the exposure of the first edge and the last edge will be about .14 seconds. If the plane is travelling at 100 feet per second (70 miles per hour) it will have meantime gone 14 feet and there will be 14 feet of error in the relative apparent positions of the objects depicted near the edge of the plate.

The error due to shrinkage of film and paper stock is perplexing in that the error is not a constant; it will, in brief, vary with the particular lot of film or paper used. The shrinkage of film is approximately .5 of one per cent. from all directions to the center; the shrinkage of paper stock varies from .5 of one per cent. to three per cent. and is almost always in one direction only, depending upon the manner in which the paper was flowed over the looms at the paper mill and upon the characteristics of the fibre from which it has been made. Uniform shrinkage in all directions of course produce no distortion, only a uniform decrease in scale. Shrinkage in one direction more than in another, however, produces an effect similar to the error introduced by the focal plane shutter. The absolute error on the ground will be inversely proportional to the scale and directly proportional to the size of the film.

The distortion due to difference of level of the ground or objects upon it is very much complicated by the presence of tilt distortion but for pictures taken vertically it is proportional to the difference of level and to the tangent of the apparent angular distance of the point from the center of the photograph.

These errors are eliminated or minimized by the following methods:

The lens distortion error is greatest on that part of the picture which is farthest from the center.

The large overlap used insures that the parts containing the greatest errors are eliminated.

In the *K1* type of film camera the film is moving during exposure and if the camera is so set that this movement is in the same direction as the movement of the image on the focal plane, the error introduced by the use of a focal plane shutter is reduced to such an extent that it can be safely disregarded.

The shrinkage of the film being symmetrical results in a uniform decrease in the scale of the picture which is eliminated by the use of the pantograph, and the data obtained from the ground control. For the paper shrinkage, shrinkage tests are made on each lot of paper used and the error so determined taken into consideration.

The distortion due to difference of elevation is more difficult to deal with. Generally speaking, only the coastal area is photographed in the making of surveys by this Office and with few exceptions the area so photographed has been of little or no elevation above sea level. In cases where the coast line is bold, special treatment will have to be given to the photographs but this problem has not yet been encountered.

With regard to the errors controlled by the pilot of the plane, considerable progress has been made. The error introduced by difference of altitude at which the plane was flying at the time the photographs were taken is eliminated when the pictures are reproduced at a common scale by means of the pantograph. The tilt error which was and still is the greatest source of annoyance has been considerably reduced by the acquisition of skill on the part of pilots and by increased stability of the type of plane and camera mounting used. Experiments have shown that it is possible for the pilot of a photographic plane to keep the tilt in the direction of flight less than 1/4 of a degree. The tilt in the direction normal to the line of flight is hardest to control but the average error is less than one degree. As the coast line photographs are taken with the water line in the approximate center of the photograph and parallel to the line of flight, the distortion of this line is very small. As yet, no errors which cannot be adjusted when the uncontrolled coast line is tied up to that developed by ground methods, have been encountered and therefore it has not yet been necessary for this Office to adopt any method of rectification or rephotographing of the pictures to eliminate the distortion due to tilt.

The development of instruments and methods of rectification of aerial photographs, stereographic means of determining contours and other new developments to increase the accuracy with which aerial photographs may be used in survey work are being studied with great interest. For the present, however, the method previously outlined is standard for this Office and its opinion is that, for the present at least, more elaborate methods aim at accuracy greater than is justified considering the errors due to lens distortion and use of the focal plane shutter, even when these errors are reduced to the furthermost minimum.

REPORT MADE BY THE HYDROGRAPHIC MISSION OF THE ARGENTINE REPUBLIC.

Hydrographic Mission of the Gulf of St. George, Argentine Republic.

Photographic Work (April 1925) *from the Report*

by

LIEUTENANT RICARDO FITZ SIMON.

I. *MATERIAL.*

A) *Hydroplane.*

The type of aeroplane used was an F. 5. L., with two Liberty motors, each of 400 HP, with a speed of 68 ½ miles (110 kilometers) per hour in still atmosphere.

The plane used for taking the photographs was not suitable for flying at very great altitude. The altitude chosen, approximately 5.000 feet (1.500 metres) entailed loss of time in attaining it and great fatigue to the personnel in charge of the work. On the other hand, the excessive amount of petrol used shortened the length of the flight.

It will be advisable, in the future, to use a smaller aeroplane or seaplane, easily handled and equipped with a single motor, which can rapidly reach the highest altitude and which can carry the pilot and an observer. A larger photographic output would thus be obtained more economically and with a smaller personnel.

B) *Cameras.*

The cameras used were a " K-1" with a " Hawk Eye Aerial" lens having a focal length of 12 inches (305 millimetres) and a fixed aperture f/4.5 ; and a "2-1 " hand camera, fitted with a "Hawk Aerial" lens, focal length 10 inches (254 m/m) and fixed aperture f/4.5, plates and films 4 in × 5 in (10 × 13 c/m). In the K-1 camera, the mechanism for moving the film, and that of the shutter and the air-suction, worked well and the film presented a plane surface when exposed.

The films used in the "K-1" camera were of 70 exposures, each 7ins × 9ins (18 × 24 $^{m}/_{m}$) placed in the chamber of the camera, the edge of the 9 in (24 centimetres) side being perpendicular to the direction of flight.

The films gave good results although they were old. They were orthochromatic and were used without a filter before the lens. In spite of the absence of a filter, the photographs showed no greater effect of the colours of stronger actinic action in the parts corresponding to the outlines of the coast and of the sea, as sometimes happens on account of the sensitiveness of films to green.

II. *PHOTOGRAPHIC RECONNAISSANCE.*

The aerial photographs taken were *vertical* (plate horizontal, optical axis vertical) and *oblique* (optical axis inclined with reference to the horizontal). The first were taken to record the irregularities of the coast, the shoals in the neighbourhood and other details which might be useful to the Surveying Party in charge of the topography. The oblique views, taken with the "2-1" Hand camera, showed the lighting conditions while the vertical pictures were being taken, and recorded isolated details (islands, islets) which it would have been impossible to photograph with the automatic camera.

An examination of the data given by the automatic tide-gauge from the 9th to the 12th of March furnished an exact record of the state of the tide during the photographic surveys. As will be noticed in the Air Service document, the photographs had to be taken in such conditions of weather, light, altitude of the sun, *etc.*, that interruptions occurred, and as a result several views are missing owing to atmospheric disturbances at the altitude of the flight. It was necessary to wait for better weather conditions. During the 13 days that the seaplane remained permanently on the spot, and except for the preliminary flight on the 4th of March, only two other days favourable for photographic surveying occurred, *viz.* the 9th and the 12th of the same month.

Ten aerial soundings were taken. On the 9th, Puerto Nuovo, the San Antonio Peninsula, the reef Basin de Bahia Gil, Valdès Island, Point Guanaco and the entrance to Bahia Gayetano, were photographed. Separate views were taken of Puerto Melo, Bahia Arredondo, *etc.*, in addition.

On the 13th, flights were made during which views were taken from Caleta Saras in Camarones Bay to Cabo dos Bahias, from Cabo dos Bahias to Naufragio Bay; Leones Island with the Lanaud Peninsula and its reefs, Arce Island and Rasa Island and its reefs.

The vertical photographs are, mostly, very sharp.

Taking into account the poor manœuvering power of the seaplane used, its control during these explorations was excellent, both for direction and for altitude attained. The difficulty of control was due to the shape of the hull, which did not allow a convenient observation point to be provided which would have facilitated the aim of the photographic observer in flights parallel to the coast.

It would be advisable to instal the sighting apparatus proposed by Mr.

BONNAFOUX in the planes used for this work; a description of this will be found in the *Annales Hidrograficos* for 1922.

III. *CONSIDERATIONS CONCERNING PHOTOGRAPHS.*

The air photographs of that part of the zone where the Hydrographic Expedition worked make it possible to appreciate the rapidity with which details of the coast may be obtained, the planimetry near the coast, the contours of the rocks at low water, *etc.*, thus reducing the task of the operator on land which requires a great deal of time and attention.

The photographic reconnaissance of the major part of the coast having been made before the work of those who were on land, the surveyors were able, later on, to follow the coast with the photographs obtained, make topographic stations and recognise points on the ground by means of the photographs, thus being able to deduce the correct setting of the photographs and facilitate their restitution. The ordinary method or methods of making topographical surveys being very laborious, hydrographic surveyors used to confine themselves to making the strictly necessary surveys of the coast and surrounding territory only, with the result that hydrographic charts are very diagrammatic in so far as they concern the planimetry of the coast, especially in places where no large-scale surveys, made by the Geographical Service, exist, and to which it is necessary to have recourse. Marine charts usually become almost useless above the high water line.

Air photography permits the topography of the coast to be improved without increasing the work on land, which in itself is another advantage of this method (Volmat).

Comparison of the photographs of the Cabo dos Bahias to Caleta Sara flight, with the configuration shown on the old chart, N° 3, reveals on the latter a lack of topographic details and a few departures from reality.

Thus the photographic method will allow the topography of the charts of our coast to be revised and, in places where the water is clear, it may even be possible to obtain a picture of the bottom near the coast. In places such as the mouth of Puerto Belgrano, the opaqueness of the water makes it impossible to obtain a clear view of the bottom, as has been proved by trials made by our pilots (Photograph of the Manuelita Channel).

Nevertheless, it must be remembered that all resources for obtaining the detail of shoals have not been exhausted, as has been shown by the French Photo-Hydrographic Expedition (Mission Photo-Hydrographique Française) in the Cotentin, where also the waters are somewhat opaque. By operating under certain conditions of tide and wind, many good results were obtained for surveying shoals.

In so far as the study of the photographs obtained by this Expedition are concerned, it can be determined later, after soundings have been made, whether several dark or light spots observed near the coast represent irregularities of the bottom, and this will greatly help the personnel who have to interpret the photographs.

Shoals.

Photography has made it possible to classify these in the following manner :

a) By breakers ; shoals seen at or near low water.

b) By the special sort of sand which is on them : those near the coast in calm water.

c) By eddies : shoals which are not visible at low water.

Typical cases are shown in photographs 73-5 (reef S. E. of Rasa Island) and 55-5 (rocks off Cape dos Bahias). In photograph 14-5 a curve can be seen which is formed by eddies similar to those of Rasa and Dos Bahias. In all three cases soundings will determine whether or not these are due to rocks invisible at low water.

d) By the rips produced over them by Tidal Currents.

The reef to the E. S. E. of the Lanaud Peninsula is shown in this way in the photograph (*). It will be noticed that the rip is less pronounced between the eastern point of the peninsula and the shoal which can be seen on photograph 3-5, whereas to the East of the shoal (photo 2-5) it is very pronounced. The former might indicate shoals at a greater depth than the latter ; at least, the amount of choppiness is less.

Having ascertained the exact height of the tide in that part of the zone, at the moment when the photographs were taken, it will then be possible to determine whether the direction is the same for a rising tide and for a falling tide.

Beaches.

The fluvial erosion, due to mountain streams in the interior, which is shown by the photographs taken when following the coastline, may help to show the deeper places.

With the sediment which they carry in their descent to the coast, the torrents form beaches bare of sea-weed (which covers the rocks battered by breakers) and form channels between the surfaces covered with this weed.

In considering the uniform growth of sea-weed on the rocks which border the coast, soundings will, generally, disclose whether the places stripped of the weed have, at the bottom, sand, shells, mud, *etc.*, which hinders their growth, or whether this is simply due to the fact that the depth is greater.

Sea-weed disappears on bottoms covered with a mixture of stones, shells and sand. This is confirmed to the South of Leone Island anchorage, between this island and the Buque Islands to the S. W. (Photos 11-5 and 12-5).

(*) It is not considered possible to reproduce the photographs.

Currents.

Some photographs (for example *N*-32-5) clearly show the direction of tidal streams (the only ones which exist on our coast). They are disclosed by the rips and races caused by them over shoals, and by the lie of the sea-weed, which has a *combed* appearence. A typical example is given by photos 32-5 and 33-5, which represent the channel between Sola Island and the coast.

Morphology of the coast.

In the zone photographed, various effects of sea erosion may be noticed along the coast. A study of the different aspects of this erosion might be of interest to a geologist. The details of the photographs would provide him with data on different geological formations, *e. g.* on the limits of the zone of attack by the sea and by currents (*see* upper part marked by the darkest colour), on the rocks (lower part bordered by weeds which hide the rocks), and on the composition of the rocks, by the form of erosion.

Air photographs would furnish illustrations for his studies. The results might be incorporated in our Sailing Directions as a supplement which, while not indispensable to the aims of this work, would give Naval Officers a more extensive knowledge of the country which unrolls itself before their eyes, and, as Lieutenant GREEN says in his interesting work "*Earth Features and the Navy Pilot*" (U. S. N. I., Proceedings, Nov., 1923) this would become an interesting professional pastime.

GENERAL REMARKS.

Judging by the trials made, it is considered that air photography will prove a valuable auxiliary in hydrographic surveys, provided that certain conditions, summarised below, can be fulfilled:

1) The necessary air and photographic personnel and equipment should accompany the Hydrographic Expedition from the very beginning of the preliminary work on the ground.

The hydrographic surveyor will require beforehand a photographic reconnaissance of the zone which he is to survey in order to be able to organise his work. During the survey, details of use for topography should be collated by air photographs.

2) The aeronautic installation may be temporary (if no air base exists in the neighbourhood), provided, however, that the equipment is properly cared for. Planes must not be away from their base for too long a time without providing convenient shelter, as this might cause failures in the use of the aeroplane or the seaplane which would be best suited to the requirements of photographic reconnaissance.

3) Another type of aeroplane or seaplane must be procured, better suited to the needs of photographic reconnaissance.

4) The methods of taking photographs should be improved by providing the plane with special appliances which will allow a course to be followed exactly during flights, especially during those which are distant from the coast

5) Laboratory methods should be improved by adopting better means for the taking of negatives and printing of positives. Undoubtedly there exist methods which are more suitable: these should be studied and applied to improve existing systems, especially for use in future photographic surveys in remote districts.

6) The hydrographic surveyor must be informed of the methods of air photography so as to obtain the best possible results from the surveys. Without doubt the photographic work will be placed under the direction of the Hydrographic Office, and thus it is advisable to follow its suggestions.

The task of organising a special personnel charged with the interpretation of the photographs and the office work will be entrusted to the Hydrographic Office, because of its competence in drawing up charts. The Air Service would make the photographic reconnaissance, the results of which, developed in the laboratories of the Aviation Schools, would then be forwarded immediately to the Hydrographic Office. This would not be a burden to the Air Service; on the contrary, it would give it greater experience in flying for, in photographic surveying, rules are followed similar to those used in aerial bombardments, and the pilots would thus have occasion to observe and familiarise themselves with our coast. When it becomes necessary to renew the personnel, and the aviation officer must be transferred to another branch of the Navy, the Hydrographic Branch would be most suitable for him.

7) Wherever aviation is employed the necessary organisation for observing the actual state of the weather and the provision of precise meteorological records of the past must be considered. The observation of wind by sounding balloons at altitudes corresponding to the altitude of flight is absolutely indispensable. By so doing it is possible to avoid the interruptions caused by useless flights, which depress the personnel, and damage to the equipment will be avoided. Meteorological statistics are useful for choosing favourable periods for the satisfactory use of the planes.

In the trials made the value of the statistics based on observations made on Leones Island was appreciated. It is certain that the photographic reconnaissances, which were not made until the end of the favourable period in that zone, would have given better results in the months of January and February.

The preparation of meteorological statistics by observations pertaining thereto (frequency of winds, rain and storms), should precede the possible sending of planes to places near meteorological stations on the coast.

Improvements in the communication of information gathered, and the supply of sounding balloons to the stations, are of great importance for flying, and would give a better general knowledge of the climatology of our coast.

REPORT BY THE NAVAL BOARD OF THE COMMONWEALTH OF AUSTRALIA

APPARATUS FOR AND METHOD OF PLOTTING.

The method of plotting from the photographs and which has been employed wherever possible is to square down on to control points which can be identified and plot or photograph to the required scale.

The work carried out has been largely experimental and great difficulty has been experienced over the isolated reefs on the east coast of Australia where it is usually impossible without a vast expenditure of time and material to establish a sufficient number of control points to co-ordinate the whole work.

Future work of this nature will, it is hoped, be more closely identified with a good trigonometrical survey and it is considered that with a sufficiency of well fixed positions along the coast, the photographic material will be utilised to advantage, ant that it will be possible to construct an accurate mosaic.

A very large amount of the work carried out in 1926 consisted of reconnaissance flights with the object of obtaining information regarding the general position, shape and number of the coral reefs in unexplored portions of the Great Barrier Reef.

The accuracy of this work is probably not great, but experience has shown that by indicating to the surveying ship the rough positions, etc. of the various isolated reefs, much time can be saved to the latter and a degree of confidence instilled, which is of much value when navigating in the midst of reefs of hitherto unknown extent.

APPARATUS FOR AND METHOD OF TAKING THE PHOTOGRAPHS.

Camera « *L* » type 6" *F.L.*

Plates — Wellington Panchromatic 5" x 3 $\frac{7}{8}$"

Metal Magazines.

Paper — Austral Star Contrasty.

The above apparatus is recognised as less suitable than the newer types, It is proposed, however, to replace the plate camera by an automatic film camera at an early date.

RESECTION IN SPACE OF THE POSITION OF AN AEROPLANE AT THE MOMENT WHEN A PHOTOGRAPH IS TAKEN.

by

INGENIEUR HYDROGRAPHE GENERAL P. DE VANSSAY DE BLAVOUS, *Director*.

Use of Aerial Photography for Hydrographic Surveys.

Almost all Services which make surveys now use aerial photographs and find that these offer a great advantage by reducing the time taken in carrying out operations in the field. In certain very wooded or swampy regions, such as in the forests of Canada and in the Mississippi and Irrawaddy deltas, these photographs have enabled very detailed charts to be drawn up, which it would have been almost impossible to do by the ordinary methods.

The Hydrographic Services were by no means the last to employ aerial photography ; a study of this question has already been published in Vol. I of the *Hydrographic Review* and the International Hydrographic Bureau now publishes the information which it has received since then.

The coast line, which is the principal part of the topography of a chart, being entirely on a horizontal plane, is not subject, in aerial photography, to distortion caused by the varying altitude of the other topographical details. This fact is of great advantage to hydrographic surveys. The numerous sinuosities of rocky coast lines are represented on the photographs with a precision which it is impossible to expect from the drawing of a surveyor, however expert he may be, and though their exact representation is not absolutely indispensable to the navigator, it helps him to identify various coastal points on his chart ; besides, it offers the great advantage of giving a trace of the fixed parts of the coast which does not depend on the fancy of the surveyor and which does not vary, as so often happens, with each successive survey. In those portions of the coast which are subject to change, this exact representation will enable the progress of the changes to be followed with precision and the deduction therefrom of very valuable conclusions.

A photograph taken at the instant of particularly low water gives a representation of the uncovered portions, such circumstances being but too seldom favourable during the survey to give sufficient time for the use of ordinary methods. Photographs of the surface of the sea have even been used in searching for rocks (*See* " Annales Hydrographiques " 1919-1920) by taking them either at the moment when, under the influence of a current, the presence of rocks is clearly discernible, or at the moment when conditions of lighting are most favourable and make the rocks appear like blotches on the bottom of the sea. Improved methods may be found ; those now in use are not capable of showing with certainty all dangerous rocks ; but they constitute a great help in this very difficult line of research.

In the various cases mentioned above, the surveyor uses the photograph as a well-made drawing, parts of which may be transferred on to the sheet after a mere change of scale; these parts should be limited by fairly numerous control points and the position of these points should be fixed previously by ordinary surveying methods. It must not be forgotten that the coast line, although in a horizontal plane, is shown distorted on the photograph owing, particularly, to the tilt of the plate at the moment of exposure, the amount of which is generally unknown, and that consequently the scale is not the same in different portions and in different directions. The less certain the horizontal position of the plate, the greater must be the number of control points. In the absence of special apparatus for checking this, it seems that in good photographic conditions the interval between the control points must not exceed two to three centimetres (about 1 inch) on the sheet in order to be certain of not obtaining a survey which has but the appearance of accuracy.

Apparatus for Restitution.

The difficulties are much greater, however, when it is desired to utilise photographic respresentations of hilly country. This problem has been the subject of much study. Naturally the employment and adaptation of methods and appliances which had been invented for using ground photographs and which had already given satisfactory results were tried. Most of these appliances use the stereographic effect, by which and by means of two photographs each taken from a different point in space, an impression of the relief of the ground is obtained and at the same time certainty of being able to identify the same point on the two photographs. Several types of instruments have been constructed and experimented with; others are still on trial. The following may be mentioned: the HEYDE-HUGERSHOFF Autocartograph and Aerocartograph, the ZEISS-BAUERSFELD Stereoplanigraph, the Stereotopometer of Ingénieur PREDHUMEAU, the WILL Autograph, the POIVILLIER machine. Other instruments use the method of double projection such as the ZEISS plastic camera or the NISTRI Photocartograph. Success depends not only on the mechanical perfection of these very delicate instruments but on the photographic cameras which must give perfectly clearly-defined pictures taken by very short exposure so that the movement of the aeroplane during exposure may be neglected. The use of a diaphragm shutter instead of focal plane slit shutter was a great step forward; the study of the shape of the lenses and the manufacture of the plates and films, in order to avoid the distortions arising from them, has given important results also.

The various appliances which have been constructed solve the problem of the complete utilisation of photographs of hilly ground; they are not yet largely used in practice, but this is because of their high cost and their intricacy. The use of mechanical instruments, however, is absolutely necessary if it is desired to obtain from the photograph all the data which it is capable of supplying and to reproduce and fix accurately the infinite amount of detail which it includes. The use of graphs gives quite inadequate precision and would require too much time; the necessary precision could be obtained by calculation but at the cost of much too protracted work.

Position of the aircraft.

Whatever the method employed, a preliminary determination is necessary; that of the conditions under which the photographs were taken, *i. e.* the co-ordinates in space of the point of resection, the directions of the optical axes, and the tilt of the plates. It would not be practicable to ascertain the position of the point of resection by means of theodolites erected at known points on the ground; this determination has to be made by using the known co-ordinates of three points on the ground, clearly visible on the photographs, besides which the focal length of the objective must be known, as well as the position of the principal point of the photograph (base of the perpendicular from the rear nodal point upon the plate). This problem can also be worded thus:— to fit a given triangle on to a given pyramid. Graphic solutions have often been proposed; among others by Captain SACONNEY in his "*Metrophotographie*" by Ingénieur Hydrographe ROLLET DE L'ISLE in the "*Annales Hydrographiques*" of 1917, and by Lt. Colonel L. N. F. I. KING, O. B. E. in "*Graphical Methods of Plotting from Air Photographs*, 1925". They are very valuable for providing approximate values for the required coordinates, but they cannot give the precise coordinates which are necessary for setting the photographs in the restitution apparatus.

Great accuracy in the determination of these coordinates is indispensable if it is desired to make a sort of photographic triangulation, of which the end triangle only of the traverse has been determined by direct measurements. An experiment made in the Netherlands in 1920 (*See* "*Hydrographic Review*", Vol. I. N° 1); proved that such traverses cannot give very accurate results. It is evidently inevitable that the triangles formed by the points on the common parts of several photographs should be badly shaped; they would give points of resection which would not be accurate enough to allow this kind of work to be continued over more than a very small number of triangles. But on the other hand, this method could be used with success, by taking as a basis a system of triangles which need not be very closely spaced, when the sole object is to obtain a topographical drawing from a series of photographs of which some plates only do not contain the three known points which are necessary for the direct determination of the points in space from which they were taken.

Again, the calculation of the point of resection is indispensable whenever precise co-ordinates of a point are required from the photograph. In this way the positions of numerous secondary points could be fixed to a suitable degree of accuracy and thus these secondary points will be connected to the triangulation without requiring further operations on the ground, without even the erection of marks, and without necessarily being visible from the triangulation stations. This might be the case, for example, for points situated at the foot of a cliff, the positions of which may generally be determined only by means of stations at sea or by laying out floating beacons.

Methods of Calculation.

Unfortunately, the calculation of the point of resection is far from being simple. The data given by the photograph enable the pyramid to be defined the apex of which is the point of resection and the base the triangle made by the three points on the photograph, the ground positions of which are known. It can be defined, for example, by the three plane angles at its apex. Each side of the triangle, formed by the three known points on the ground, is the chord of the segment of a circle containing one of these subtended angles. Each of these segments is situated on a tore generated by its rotation around the corresponding side. Consequently the apex of the pyramid is at the point of intersection of three tores; its position is given by a biquadratic equation.

The authors who have dealt with this problem have reduced it to a simple equation by using approximate values of the unknown quantities, employing differential formulae and by making successive approximations. The methods of FINSTERWALDER, of HUGERSHOFF, of FISHER EGGERT and of G. T. Mc CAW (*) may be mentioned. The approximate values are obtained either by graphic construction, or by the " *Bildmesstheodolit* " or " *Photogoniometer* " (**). In spite of the simplifications thus introduced the calculation still remains long and laborious; besides it has the defect of giving no result if the aeroplane, at the moment when the photograph is taken, is situated on the surface of the cylinder erected perpendicularly on the circle circumscribed about the triangle formed by the three points on the ground; it has also the inconvenience of giving an inaccurate, and sometimes even erroneous, resection if the aeroplane is in the vicinity of this cylinder. Consequently the use of these methods of calculation has led to the practice of taking photographs with the optical axis of the camera tilted at about 30° from the vertical in order to make certain of not being on this " danger cylinder ". But oblique photographs, which have certain advantages for the study of relief, offer no advantage if, as is the case in hydrography, accuracy of the positions is the principal aim. Although the representation of a larger surface of ground is obtained by oblique photography, the accuracy is not so great for a large portion of the picture is on much too small a scale.

The method of calculation which is outlined below completely solves the biquadratic problem and gives the four positions in which the aeroplane might have been at the moment of exposure. The selection of that which is correct is generally easy by consideration of the data given by the altimeter and the probable tilt, or by considering a fourth point of which the approximate position at least is known. (***)

The calculation is divided into four parts:

(*) On this subject *see* the bibliography published by the Air Survey Committee, War Office, London.

(**) The Heyde-Hugershoff Photogoniometer has replaced the Porro Photogoniometer exhibited at Milan in 1871.

(***) It would be easy to take curvature of the earth and atmospheric refraction into account; this has not been done in order not to complicate the explanation and because their influence is generally negligible.

1. Calculation of the elements of the triangle formed by the three points on the ground, the co-ordinates of which are known.

2. Calculation of the elements of the pyramid by means of the photographic data ;

3. Calculation of the four positions of the apex of the pyramid in relation to the plane of the three points ;

4. Calculation of these positions in relation to the axes of the co-ordinates.

This calculation is lengthy but does not seem to be longer than those using successive approximations which have been employed up till now. Moreover, parts 1, 2 and 4 are common to all methods, except for a few details. It is not necessary to calculate part 2 if the *Bildmesstheodolite* or *Photogoniometer* be used, and tables of GAUSSIAN logarithms or calculating machines may be used to shorten all these calculations appreciably. Finally, a method is indicated which should very much shorten the calculation of part 3 by the use of an approximate value of the unknown quantity.

Besides giving the four possible positions for the aeroplane, this method enables conditions when the plane is situated on the " danger cylinder " to be understood. In this case, the position is not indeterminate ; on the contrary, the calculation is simpler in this particular case in which the biquadratic equation has a double root. If the aeroplane is in the neighbourhood of the cylinder the equation has two roots differing but slightly and giving two resections fairly near to one another which it is important to distinguish from each other clearly in order that the one which is suitable may be selected. It is necessary to take this into clear consideration even if success, as may be hoped, crowns the efforts which are being made to construct an apparatus which will solve mechanically the problem of finding the point of resection.

Accuracy of resection.

The accuracy with which the resection can be obtained may thus be clearly seen by studying the influence of the various inaccuracies which may occur in the data (*). First, it must be stated that the photograph of hilly country is equivalent to a theodolite station, made at the same place, by means of which the azimuths and zenith distances of all points on the ground shown on the photograph would have been obtained. The altitudes of these points are generally unknown. In this case two photographs taken from two different points of view are necessary in order to ascertain the positions and altitudes. An error in the height of the aeroplane will have no influence on the positions of the points; whereas errors in the three co-ordinates of the aeroplane will have their influence on the calculation of the altitudes of the points. If the positions of the two resections have been calculated by means of different points on the ground, the two photographs provide two distinct values for the altitude of the same point on the ground, the difference be-

(*) *See :* G. CASSINIS, " L'errore di situazione dei punti determinati con procedimenti aerofotogrammetici e ei metodi di triangulazione aerea ". Atti dell'Associazione Italiana di Aerotecnica. Anno IV. N° 3, 1924.

tween which will show the existence and, to a certain degree, the magnitude of the errors of the two positions adopted for the aeroplane.

But if these points appear on the photograph at sea level, their positions can be obtained by a single photograph, but again the accuracy of these positions would then depend on that of the altitude adopted for the aeroplane.

The causes of errors in the determination of the position of the aeroplane are of three kinds :— (1) errors in focal length ; (2) errors in determination of the principal point of the photograph ; and (3) errors in the position of the control points on the ground. These last include errors of measurements taken on the photograph (whatever may be the cause) for it can always be taken for granted that the errors of measurement are nil, but that the co-ordinates of the point on the ground differed from those employed.

These errors produce displacements of the point of resection which can be calculated by means of differentials from equations which will be given ; but the expression of these is not simple and the following considerations will give a better idea of the result :

(1) *Error in Focal Length.* — Let f be the focal length ; df the error in this length. Let $S\,ab$ be the face of the pyramid the apex of which is the point of the resection S and the base of which is the side ab of the triangle of the photograph. If V_c is the angle at the apex, the change in this angle will be :

$$dV_c = \frac{(\overline{Sa}^2 + \overline{Sb}^2)\cos V_c - 2\,\overline{Sa}\,\overline{Sb}}{\overline{Sa}^2\,\overline{Sb}^2 \sin V_c}\; f\; df$$

The point S is situated on a tore generated by the rotation of a segment of the circle containing the angle V_c about the side AB (of the triangle formed by the three points on the ground). If this angle be modified by dV_c the plane which is tangent to the tore at S is moved parallel to itself by the quantity :

$$d_C = \frac{\overline{SA} \times \overline{SB}}{AB}\; d\,V_c$$

Taking h_c and H_C as the heights of the triangles Sab and SAB, starting from the point S, then :

$$d_C = \frac{H_C}{h_c}\left(\frac{\cos V_c}{h_c} - \frac{2\sin V_c}{ab}\right) f\; df$$

is obtained ; but $\dfrac{ab}{2\sin V_c}$ being the radius r_c of the circle circumscribed about the triangle Sab ; this can be expressed :

$$d_C = \frac{H_C}{h_c} \left(\frac{\cos V_c}{h_c} - \frac{1}{r_c} \right) f \quad df$$

The length d_C lies in the same direction as the radius of the circle circumscribed about the triangle SAB which passes through S; it would be nil if the point S were on the diameter of the circle circumscribed about the triangle Sab which is parallel to ab (*See* fig. 1).

The three faces of the pyramid will give the three vectors d_C, d_B, d_A. The planes perpendicular to the extremities of these three vectors generally resect at a point which will be the new position of point S. The displacement of point S, due to error df, is therefore the diagonal of a parallelepiped the parallel faces of which are at distances d_A, d_B, d_C; such displacement must occur so long as df is not nil. If the three vectors d_A, d_B, d_C were in the same plane, the diagonal of the parallelepiped would be a perpendicular to this plane and its length would be infinite in ratio to the lengths of the vectors. In this case this perpendicular would be a common tangent, at the point S, to the three tores generated by the rotation around the three sides AB, BC, CA, of the three segments of the circles containing the angles. An error df will then produce a displacement of S on this tangent which will be very great as compared with df.

The pyramid $SABC$ is then inscribed in a right cylinder the generatrix of which is perpendicular to the plane ABC. This is the " danger " or " critical " cylinder. If we consider the supplementary pyramid of the pyramid $SABC$ and its base on the same plane, it possesses the same properties. The three vectors corresponding to it would be in the same plane as those of the pyramid $SABC$; thus the perpendicular to this plane would be the same. Therefore the projection of this perpendicular on the plane ABC is the radical axis of the circles circumscribed about the triangle ABC and about the base, in the same plane, of the supplementary pyramid; it also passes through the intersection of the Simpson's lines corresponding to these two triangles; it is the same as the projection on the plane ABC of the planes tangent at S to the two spheres circumscribed about the two pyramids. Thus it is easy to determine this line and it is along it that the projection of the point S in the plane ABC will be badly determined if the value of f is not very accurately known. It is better to avoid being situated on this " danger cylinder " when the data of the photographic apparatus are not known exactly.

(2) *Errors in the determination of the principal point of the photograph.* — If ξ and η be the distance from the principal point to the line ab and to the perpendicular at the centre of this line and if these distances are in error to the extents $d\xi$ and $d\eta$, the plane tangent to the tore corresponding to the side SAB will be displaced, parallel to itself, by the quantity:

$$\frac{H_C}{h_c} \left[\left(\frac{\cos V_c}{h_c} - \frac{1}{r_c} \right) \xi \; d\xi - \frac{\eta \; d\eta}{r_c} \right]$$

Three vectors are obtained having the same directions as those due to the *df* error and the same remarks may be applied.

Adding pairs of these vectors if γ be the centre of the side *ab*, their length may be expressed by :

$$\Delta_C = \frac{H_C}{h_c}\left[\cos V_c\, dh_c - \frac{\overline{S\gamma}}{r_c}\, d\overline{S\gamma}\right]$$

Besides, a modification has been produced arising from errors (1) and (2) in the tilt of the plate and in the direction of the tilt.

(3) *Errors in the positions of the control points on the ground.* — The errors do not modify the definition of the pyramid, as did the former; but it will be necessary to displace the pyramid in order to apply it to the new triangle of three points on the ground.

This transfer may be analysed as follows :

(*a*) new obliquity of the pyramid with reference to the plane *ABC* ;

(*b*) translation of this pyramid without changing the obliquity.

The change of obliquity will become very much greater if the apex of the pyramid be situated on the " *danger cylinder* ".

Thus it is obviously necessary to avoid working in the vicinity of this cylinder because then the errors in data involve greater displacement of the point of resection.

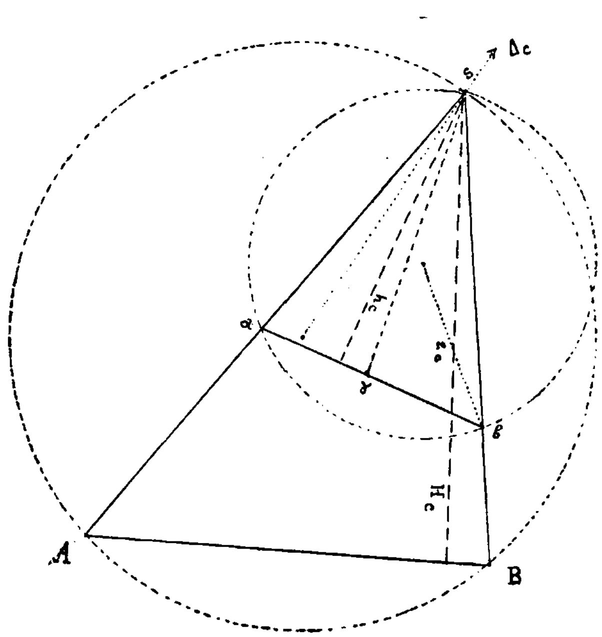

Fig. 1

I. — CALCULATION OF THE BASE TRIANGLE.

Let A, B, C be the three points on the ground the positions of which are known. Let the plane which passes through the point C be assumed to be horizontal. (Fig. 2).

In this plane we know the co-ordinates:

$$A \begin{cases} x_A \\ y_A \\ z_A \end{cases} \qquad B \begin{cases} x_B \\ y_B \\ z_B \end{cases} \qquad C \begin{cases} x_C \\ y_C \\ o \end{cases}$$

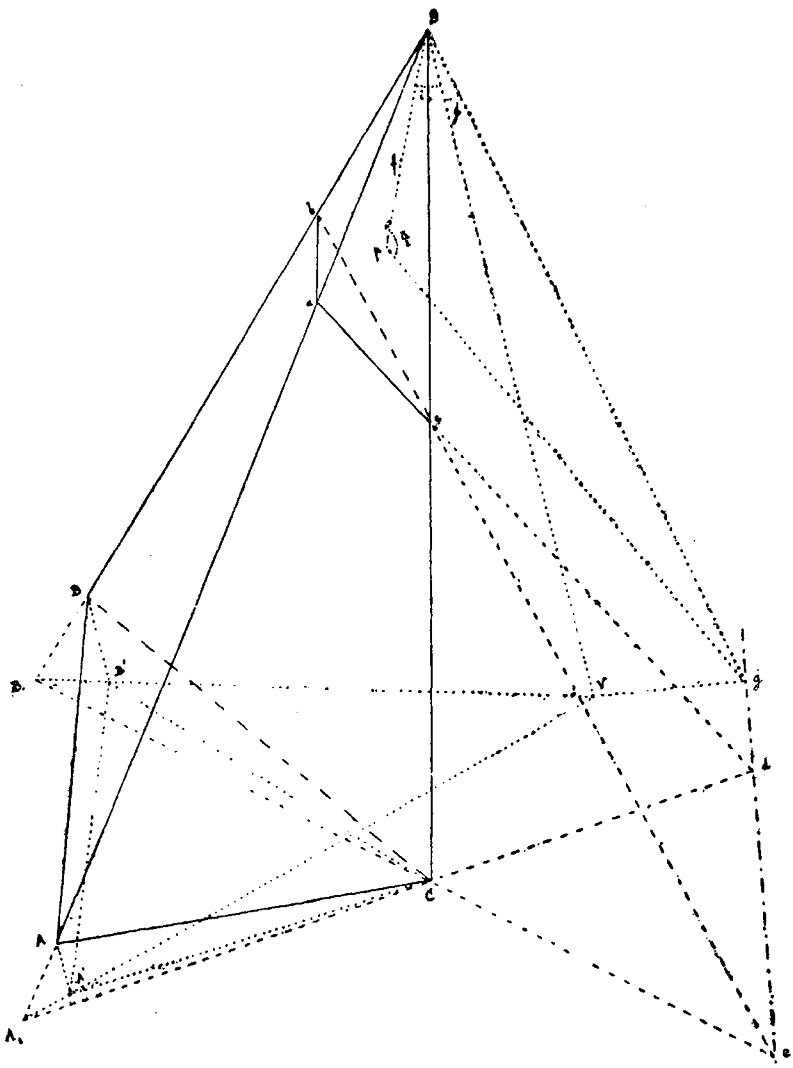

FIG. 2

Let A', B' be the projections of the points A and B on the horizontal plane, The directions and the lengths of the three sides of the triangle A' B' C will be calculated by the formulae : (*)

$$\tan G_{A'C} = \frac{x_C - x_A}{y_C - y_A} \qquad A'C = \frac{x_C - x_A}{\sin G_{A'C}} = \frac{y_C - y_C}{\cos G_{A'C}}$$

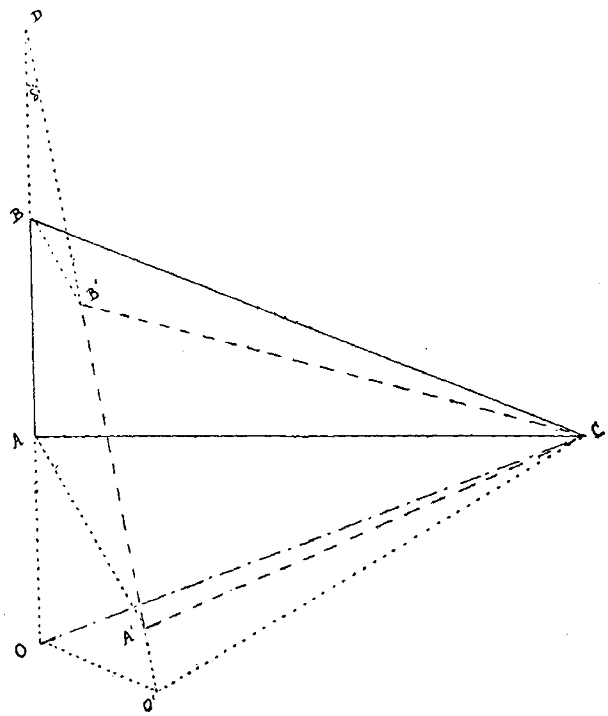

FIG. 3

Let CO' be the perpendicular dropped from C onto A' B' (fig. 3) then :

$$O'C = \quad A'C \sin (G_{A'C} - G_{A'B'})$$
$$O'A' = - A'C \cos (G_{A'C} - G_{A'B'})$$
$$O'B' = - B'C \cdot \cos (G_{B'C} - G_{A'B'})$$

Let $O'O$ be the perpendicular dropped from the point O' onto AB.
The line CO will be perpendicular to AB.
The angle δ between the lines AB and $A'B'$ is given by :

$$\tan \delta = \frac{z_A - z_B}{A'B'}$$

(*) The symbols : *tan* $G_{A'C}$, *sin* $G_{A'C}$, etc. will be used hereafar to designate the tangent, sine, *etc.* of the direction $A'C$ *etc.*

Then let :—

$$OA = O'A' \cos \delta - z_A \sin \delta.$$
$$OB = O'B' \cos \delta - z_B \sin \delta.$$
$$O'O = O'A' \sin \delta + z_A \cos \delta.$$

be calculated.

The angle between the lines CO' and CO and the length OC are obtained by :

$$\tan \widehat{O'CO} = \frac{O'O}{O'C} \qquad OC = \frac{O'O}{\sin \widehat{O'CO}} = \frac{O'C}{\cos \widehat{O'CO}}$$

The angle ε between the planes ABC and $A'B'C$ is given by :

$$\cos \varepsilon = \cos \delta \cos \widehat{O'CO}$$

Finally obtain the angle $\widehat{O'CD}$ by the formula :

$$\tan \widehat{O'CD} = \frac{O'B' + z_B \cot g\, \delta}{O'C}$$

from which :

$$\cos G_{DC} = -\sin (G_{A'B'} + \widehat{O'CD})$$
$$\sin G_{DC} = \cos (G_{A'B'} + \widehat{O'CD})$$

is deduced :

The triangle ABC will be defined by the lengths OA, OB, OC referred to as Y_A, Y_B, Y_C respectively.

Now take new axes of co-ordinates : OC being the axis for the X's ; OAB that for the Y's and a perpendicular to the plane ABC being the axis of the Z's. The change from this system to the former is made by means of the following formulae :

$$x = x_A + \left(x_C - x_A + \frac{x_B - x_A}{AB} Y_A \right) \frac{X}{X_C} + \frac{x_B - x_A}{AB} (Y - Y_A) - Z \sin \varepsilon \cos G_{DC}$$

$$y = y_A + \left(y_C - y_A + \frac{y_B - y_A}{AB} Y_A \right) \frac{X}{X_C} + \frac{y_B - y_A}{AB} (Y - Y_A) + Z \sin \varepsilon \sin G_{DC}$$

$$z = z_A + \left(- z_A + \frac{z_B - z_A}{AB} Y_A \right) \frac{X}{X_C} + \frac{z_B - z_A}{AB} (Y - Y_A) + Z \cos \varepsilon.$$

The denomination of the points ABC may always be chosen so that $\overline{OC}$ lies in a positive direction. If it were otherwise, however, it would be sufficient to make X_C a negative.

II. — CALCULATION OF THE ELEMENTS OF THE PYRAMID.

Figure (4) shows the pyramid the base of which is the triangle *abc* formed by the three points on the photograph of which the positions on the ground are known.

Sp is the perpendicular to the plate from the optical centre of the objective; its length is *f*.

c_1 is the projection of the point *c* on to the plane *Sab*, which is adopted as the plane of the figure.

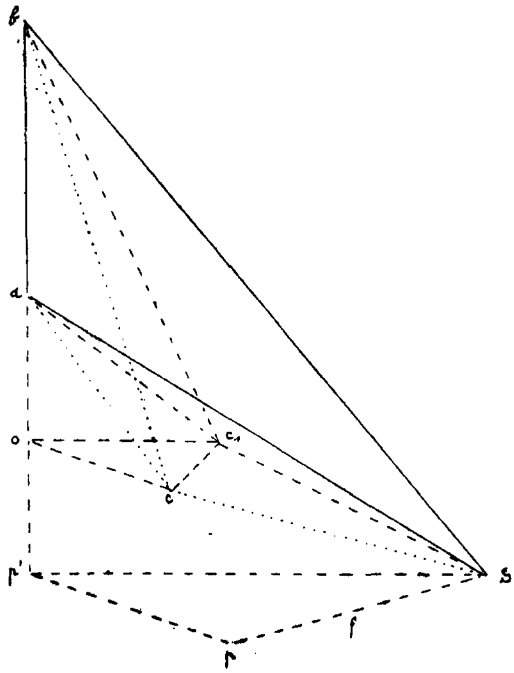

FIG. 4

From the points *p* and *c* drop perpendiculars *pp'* and *co* on to the side *ab*. The lengths *p'p*, *oc*, *p'a*, *p'b*, *p'o* may be measured on the photograph. Then calculate:

$$\tan \widehat{p\,p'\,S} = \frac{f}{p\,p'}$$

from which

$$p'\,S = \frac{f}{\sin \widehat{p\,p'\,S}} = \frac{p\,p'}{\cos \widehat{p\,p'\,S}}$$

is obtained:

Next, calculate:

$$\tan \widehat{p'\,S\,a} = \frac{p'\,a}{p'\,S} \qquad \tan \widehat{p'\,S\,b} = \frac{p'\,b}{p'\,S}$$

$$\widehat{a S b} = \widehat{p' S b} - \widehat{p' S a}$$

$$o c_1 = o c \cos \widehat{p p' S}$$

from which is obtained :

$$\tan \widehat{p' S c_1} = \frac{p' o}{p'S - o c_1 \cos \widehat{p p' S}}$$

$$S c_1 = \frac{p' o}{\sin \widehat{p' S c_1}} = \frac{p' S - o c_1 \cos \widehat{p p' S}}{\cos \widehat{p' S c_1}}$$

The pyramid will now be defined by means of the angles

$$\alpha = \widehat{c_1 S a} \qquad \beta = \widehat{c_1 S b}$$

which the edges Sa and Sb of the pyramid make with the projection of the third edge Sc on the face Sab. These angles are reckoned positively clockwise. They are found by the formulae :

$$\alpha = \widehat{p' S a} - \widehat{p' S c_1} \qquad \beta = \widehat{p' S b} - \widehat{p' S c_1}$$

These two angles define the pyramid with the addition of the angle θ, which the edge Sc makes with the face Sab; this last angle is calculated by the formula :

$$\tan \theta = \frac{o c \sin \widehat{p p' S}}{S c_1}$$

If the lengths of Sa and Sb are required, they may be calculated by the formulae ·

$$S a = \frac{p' a}{\sin \widehat{p' S a}} \qquad S b = \frac{p' b}{\sin \widehat{p' S b}} \qquad S c = \frac{S c_1}{\cos \theta}$$

III. — CALCULATION OF THE DIRECTION OF THE HORIZONTALS AND OF THE TILT OF THE OPTICAL AXIS.

The co-ordinates X_S, Y_S, Z_S of the apex S (position of the aeroplane) will be determined in Chapter IV : and the co-ordinates x_S, y_S, z_S will then be deduced therefrom by the formulae of Chapter I.

x_S and y_S are the co-ordinates of the point V, the projection of the point S, on the horizontal plane, z_S is the height of the point S above this plane.

The co-ordinates of the points A_1 and B_1, which are the intersections of the edges SA and SB on the horizontal plane, will be :

$$x_{A_1} = x_S + (x_A - x_S)\frac{z_S}{z_S - z_A} \qquad y_{A_1} = y_S + (y_A - y_S)\frac{z_S}{z_S - z_A}$$

$$x_{B_1} = x_S + (x_B - x_S)\frac{z_S}{z_S - z_B} \qquad y_{B_1} = y_S + (y_B - y_S)\frac{z_S}{z_S - z_B}$$

but it is not necessary to calculate them.

The directions of the lines VA' and VB' and their lengths will be given by :

$$\tan G_{VA'} = \frac{x_A - x_S}{y_A - y_S} \qquad VA' = \frac{x_A - x_S}{\sin G_{VA'}} = \frac{y_B - y_S}{\cos G_{VA'}}$$

The angles VSA, VSB, VSC may then be calculated by the formulae :

$$\tan \widehat{VSA} = \frac{VA'}{z_S - z_A} \qquad \tan \widehat{VSB} = \frac{VB'}{z_S - z_B} \qquad \tan \widehat{VSC} = \frac{VC}{z_S}$$

The co-ordinates of the points a, b, and c (fig. 2) are then obtained.

$$x_a = x_s + \overline{Sa}\sin\widehat{VSA}\sin G_{VA'} \quad x_b = x_s + \overline{Sb}\sin\widehat{VSB}\sin G_{VB'} \quad x_c = x_s + \overline{Sc}\sin\widehat{VSC}\sin G_{VC}$$

$$y_a = y_s + \overline{Sa}\sin\widehat{VSA}\cos G_{VA'} \quad y_b = y_s + \overline{Sb}\sin\widehat{VSB}\cos G_{VB'} \quad y_c = y_s + \overline{Sc}\sin\widehat{VSC}\cos G_{VC}$$

$$z_a = z_s - \overline{Sa}\cos\widehat{VSA} \qquad z_b = z_s - \overline{Sb}\cos\widehat{VSB} \qquad z_c = z_s - \overline{Sc}\cos\widehat{VSC}$$

For the direction in which the vertical plane SpV lies there is deduced therefrom :

$$\tan G_{SpV} = -\frac{y_a(z_b - z_c) + y_b(z_c - z_a) + y_c(z_a - z_b)}{x_a(z_b - z_c) + x_b(z_c - z_a) + x_c(z_a - z_b)}$$

This formula is more easily calculated by putting it in the following form :

$$\tan G_{SpV} = -\frac{\dfrac{y_b - y_c}{z_b - z_c} - \dfrac{y_a - y_c}{z_a - z_c}}{\dfrac{x_b - x_c}{z_b - z_c} - \dfrac{x_a - x_c}{z_a - z_c}}$$

It will not be necessary to calculate x_a, y_a, z_a, but only $x_a - x_S$, $y_a - y_S$, $z_a - z_S$.

Now in the triangle *abc* of the photograph, let *d* and *e* be the points of intersection of the lines *ac* and *bc* with the horizontal plane. Then:

$$\overline{cd} = \overline{ac}\,\frac{z_c}{z_a - z_c} \qquad \overline{ce} = \overline{bc}\,\frac{z_c}{z_b - z_c}$$

In order to obtain the direction of the horizontals in the photograph, it will be sufficient to lay off, from the point *c* along the sides *ac* and *bc*, lengths in the ratio:

$$\frac{\overline{ac}}{\overline{bc}} \qquad \frac{z_b - z_c}{z_a - z_c}$$

It remains only to calculate the tilt *i* of the optical axis.
The co-ordinates of the points *d* and *e* are:

$$x_d = x_c - z_c\,\frac{x_a - x_c}{z_a - z_c} \qquad x_e = x_c - z_e\,\frac{x_b - x_c}{z_a - z_c}$$

$$y_d = y_c - z_c\,\frac{y_a - y_c}{z_a - z_c} \qquad y_e = y_c - z_c\,\frac{y_b - y_c}{z_a - z_c}$$

Let *g* be the base of the perpendicular dropped from the point *V* on to the line *e d* (fig. 2).

The direction of *Vg* is G_{Spv} which has already been calculated.

By projecting on to *Vg* the lengths *Vd* and *Ve* the expressions:

$$Vg = (x_d - x_s)\,sin\,G_{SpV} + (y_d - y_s)\,cos\,G_{SpV} = (x_e - x_s)\,sin\,G_{SpV} + (y_e - y_s)\,cos\,G_{SpV}$$

are obtained.

Let *j* be the angle *VSg*, which gives:

$$tan\,j = \frac{Vg}{z_s} = \frac{x_d - x_s}{z_s}\,\sin\,G_{SpV} + \frac{y_d - y_s}{z_s}\,\cos\,G_{SpV}$$

The angle $i+j$ will be given by the formula:

$$\cos\,(i+j) = \frac{f}{z_S}\,\cos\,j.$$

Thus *all* the elements defining the position of the photographic plate will have been calculated. It is possible to deduce therefrom the position on the ground of any point on the plate, at least assuming the altitude of this point to be nil; *i. e.* we can calculate the intersection with the horizontal plane of the radius *Sm* which joins the point *S* to a point *m* on the plate.

For this purpose the point *m* of the plate will be defined by its co-ordinates with reference to the axes *o c* and *o b.* (*see* fig. 4), namely ξ and η. The co-ordinates of this point with reference to the former axes will be:

$$x_m = x_a + \left[(x_b - x_a) \frac{oa}{ab} + (x_c - x_a) \right] \frac{\xi}{oc} + (x_b - x_a) \frac{\eta - oa}{ab}$$

$$y_m = y_a + \left[(y_b - y_a) \frac{oa}{ab} + (y_c - y_a) \right] \frac{\xi}{oc} + (y_b - y_a) \frac{\eta - oa}{ab}$$

$$z_m = z_a + \left[(z_b - z_a) \frac{oa}{ab} + (z_c - z_a) \right] \frac{\xi}{oc} + (z_b - z_a) \frac{\eta - oa}{ab}$$

The co-ordinates of the points *a, b* and *c* have been calculated.

The co-ordinates of the required point on the ground will be :

$$x = x_S - z_S \frac{x_m - x_S}{z_m - z_S} \qquad y = y_S - z_S \frac{y_m - y_S}{z_m - z_S}$$

This formula can also be used to select, from among the four positions of point S which the calculation has provided, that which gives for a fourth point on the photograph the position on the ground nearest to that which is known approximately. This choice will usually be decided however by the knowledge of the approximate altitude which is obtained by reading the altimeter.

IV. — TO FIT THE TRIANGLE *ABC* ON THE PYRAMID.

In Chapter I the elements have been calculated of the triangle *ABC* which is to be fitted on the pyramid, defined as just stated.

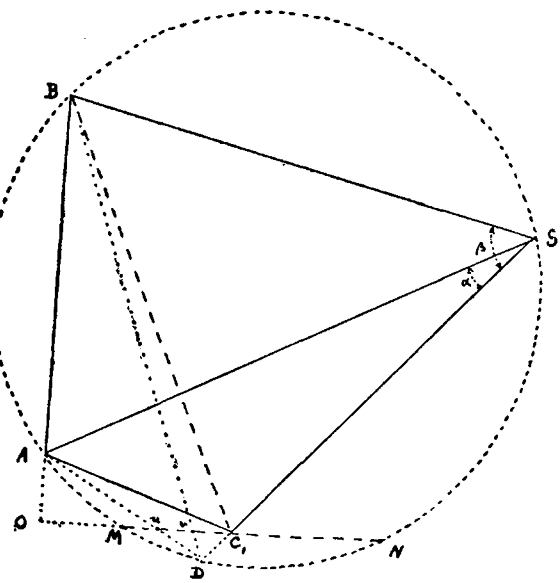

FIG. 5

Taking the plane of the face $\widehat{SAB}$ as the plane of the figure, the apex *S* will be on a point of the segment of the circle containing the angle $\widehat{ASB}$ described on *AB*. This angle, which is equal to $\widehat{aSb}$ has just been calculated.

Let C_1 be the projection of the point *C* on the plane of the face *SAB*. This point is on the perpendicular drawn from the point *O* on to the straight line *AB*. The position of the point *O* is known and *OA* and *OB* have been calculated in Chapter I. The projection OC_1 of *OC*, which is equal to *OC* cos ω will be taken as the unknown quantity; *OC* is the true height of the triangle and ω the angle between the plane *ABC* and the face *SAB*.

The line OC_1 cuts the segment at two points *M* and *N* and the line SC_1 cuts it at the point *D*. The angle α was calculated in Chapter II. The arc *AD* is represented by $2\,\alpha$. Thus the position of the point *D*, as well as those of the points *M* and *N* may be calculated. The unknown quantity x may then be obtained by the equation :

$$DC_1 \times C_1 S = MC_1 \times C_1 N$$

$$C_1 S = \sqrt{\overline{OC}^2 - x^2}\ cot\ \theta \qquad MC_1 = x - OM \qquad C_1N = ON - x$$

Let x_D and y_D be the co-ordinates of the point *D* with reference to the

2 axes ON and OB. For determining x the following biquadratic equation is obtained :

$$\left(\overline{OC}^2 - x^2\right)\left[(x - x_D)^2 + y_D^2\right] = (x - OM)^2 (x - ON)^2 tg^2\,\theta$$

The solving of this equation gives the position of the point C_1 and, consequently, that of point S which is at the intersection of the straight line DC_1 and of the circle.

The quantities OM, ON, x_D, y_D will first be calculated.

$$\text{As}: \quad OM \times ON = oA \times oB \quad \text{and} \quad oM + oN = A\,B \cot(\beta - \alpha)$$

OM and ON are the roots of a quadratic equation. It is not always necessary to calculate them, for the biquadratic equation which gives x contains only their sum and their product. Moreover OM and ON may be imaginary quantities, but their sum and their product are real.

The co-ordinates of the point D will be given by the formulae :

$$\begin{cases} x_D = \overline{A\,B}\,\dfrac{\tan\alpha\,\tan\beta}{\tan\beta - \tan\alpha} = \overline{AB}\,\dfrac{\sin\beta\,\sin\alpha}{\sin(\beta-\alpha)} \\[2ex] y_D = \dfrac{\overline{OA}\tan\beta - \overline{OB}\tan\alpha}{\tan\beta - \tan\alpha} = OA - x_D \cot\beta. \end{cases}$$

It will be recalled that in Chapter I, OA and OB were called Y_A and Y_B ; OC was called X_C.

In order to solve the biquadratic equation the angle ω between the plane ABC and the face SAB will be taken as the unknown quantity. Then (always adopting the absolute value of X_C) :

$$x = X_c \cos\omega$$

The equation can thus be put in the following form :

$$\cos^4\omega - 2\,\xi\cos^3\omega + (\xi^2 + A)\cos^2\omega + 2\,B\cos\omega + C = O$$

putting :

$$\xi = \frac{(OM + ON - x_D)\sin^2\theta + x_D}{X_C}$$

$$A = \left(\frac{OM + ON - x_D}{X_C}\right)^2 \sin^2\theta\cos^2\theta + 2\frac{OM.ON}{X_C^2}\sin^2\theta + \left(\frac{y_D^2}{X_C^2} - 1\right)\cos^2\theta$$

$$B = \frac{x_D}{X_C}\cos^2\theta - \frac{OM + ON}{X_C}\,\frac{OM.\,ON}{X_C^2}\,\sin^2\theta$$

$$C = \left(\frac{OM.\,ON}{X_C^2}\right)^2 \sin^2\theta - \frac{x_D^2 + y_D^2}{X_C^2}\cos^2\theta$$

We can also consider the biquadratic equation of x as the equation to the abscissae of the intersections of two conics the equations of which are: If $A > O$ a parabola whose axis is parallel to the axis of the y's and a circle:

$$y\sqrt{A} = x\left(\frac{x}{X_C} - \xi\right)$$

$$x^2 + y^2 + 2\frac{B}{A}X_C x + \frac{C}{A}X_C^2 = 0.$$

If $A < O$; a parabola and an equilateral hyperbola

$$y\sqrt{-A} = x\left(\frac{x}{X_C} - \xi\right)$$

$$x^2 - y^2 + 2\frac{B}{A}X_C x + \frac{C}{A}X_C^2 = 0.$$

Substituting for these two conics a system of two straight lines the equation of which are:

$$x^2(1+\lambda) \pm y^2 + 2\left(\frac{B}{A} - \lambda\frac{\xi}{2}\right)X_C x - \lambda\sqrt{\pm A}\,X_C y + \frac{C}{A}X_C^2 = 0$$

If λ is a root of the equation:

$$A\lambda^3 + (\xi^2 + A)\lambda^2 - 4\frac{C + B\xi}{A}\lambda + 4\frac{B^2 - AC}{A^2} = 0$$

the biquatratic equation of x can then be put in the form:

$$\left[2A(1+\lambda)x - \lambda A\xi + 2B\right]^2 + 4A(1+\lambda)\left[x(x-\xi) - \lambda\frac{A}{2}\right]^2 = 0$$

To solve the cubic equation of λ, the unknown auxiliary μ is taken:

$$\mu = \lambda + \frac{\xi^2 + A}{3A}$$

The equation will become:

$$\mu^3 - \mu\frac{(\xi^2+A)^2 + 12(C+B\xi)}{3A^2} + 2\frac{(\xi^2+A)^3 + 18(\xi^2+A)(C+B\xi) + 54(B^2-AC)}{27A^3} = 0$$

Putting:

$$D = 1 + 12\frac{C + B\xi}{(\xi^2 + A)^2}$$

$$E = D + 6\frac{C + B\xi}{(\xi^2 + A)^2} + 54\frac{B^2 - AC}{(\xi^2 + A)^3}$$

If μ is one of the roots of the equation, the roots of the biquadratic equation of x will be, taking :

$$\eta^2 = \frac{\xi^2 - A\,(2 + 3\,\mu)}{3} = \frac{\xi^2 + A}{3} - A - A\,\mu.$$

$$\frac{x_1}{X_C} = \frac{\xi}{2} + \frac{\eta}{2} + \sqrt{\frac{\eta^2 + \frac{3}{2} A\,\mu}{2} - \frac{A\,\xi + 2B}{2\,\eta}}$$

$$\frac{x_2}{X_C} = \frac{\xi}{2} + \frac{\eta}{2} - \sqrt{\frac{\eta^2 + \frac{3}{2} A\,\mu}{2} - \frac{A\,\xi + 2B}{2\,\eta}}$$

$$\frac{x_3}{X_C} = \frac{\xi}{2} - \frac{\eta}{2} + \sqrt{\frac{\eta^2 + \frac{3}{2} A\,\mu}{2} + \frac{A\,\xi + 2B}{2\,\eta}}$$

$$\frac{x_4}{X_C} = \frac{\xi}{2} - \frac{\eta}{2} - \sqrt{\frac{\eta^2 + \frac{3}{2} A\,\mu}{2} + \frac{A\,\xi + 2B}{2\,\eta}}$$

The equation of μ will have three true roots if $D^3 > E^2$. An auxiliary angle φ is then calculated by the formula :

$$tg\,\varphi = -\frac{\sqrt{D^3 - E^2}}{E} \qquad \cos\varphi = -\frac{E}{D\sqrt{D}}$$

and for μ the value

$$\mu = 2\,\frac{\xi^2 + A}{3\,A}\sqrt{D}\,\cos\frac{\varphi}{3}$$

will be assumed.

The biquadratic equation of x must have at least two true roots since the photograph belongs to a true position of the aeroplane. It will have four true roots if $D^3 > E^2$.

These roots are always comprised between $-\overline{OC}$ and $+\overline{OC}$, as may be seen from the first form of the biquadratic equation.

If $D^3 < E^2$, the equation of μ has but one true root.

If $D < o$, taking $\tan^2\psi = -\frac{D^3}{E^2}$

then :

$$\mu = \sqrt[3]{\frac{2E}{\cos\psi}} \left[\sqrt[3]{\sin^2\frac{\psi}{2}} - \sqrt[3]{\cos^2\frac{\psi}{2}} \right] \frac{\xi^2 + A}{3A}$$

If $D > o$, taking $\sin^2\psi = \frac{D^3}{E^2}$

then :

$$\mu = \sqrt[3]{2E} \left[\sqrt[3]{\sin^2\frac{\psi}{2}} + \sqrt[3]{\cos^2\frac{\psi}{2}} \right] \frac{\xi^2 + A}{3A}$$

Two of the four roots will certainly be imaginary, for if not, there would be three groups of real straight lines and consequently the equation of μ would have three real roots. On the other hand, there must be two real roots, the problem clearly giving one and the biquadratic equation must give an even number of roots. If $A\,\xi + 2B > o$, the roots, x_1 and x_2 will be imaginary. If $A\,\xi + 2B < o$, the roots x_3 and x_4 will be imaginary.

Where $D_3 = E_2$ the equation of μ will have a double root and consequently this will also be the case with the biquadratic equation. Then $\varphi = \pi$; the double root is :

$$\mu = \frac{\xi^2 + A}{3A}\sqrt{D} = \frac{\xi^2 + A}{3A}\frac{E}{D}$$

and the single root is :

$$\mu = -2\frac{\xi^2 + A}{3A}\sqrt{D} = -2\frac{\xi^2 + A}{3A}\frac{E}{D}$$

The double root corresponds to two lines which intersect each other at the tangent point of the two conics. This point corresponds to a double root of the biquadratic equation of x, which has in this case a double root and two true single roots.

The double root may be expressed :

$$\frac{x}{X_C} = \frac{\xi}{2} - \frac{3}{2}\,\frac{A\,\xi + 2B}{2A + (\xi^2 + A)\frac{E}{D} - \xi^2}$$

Certain of the true roots found for the biquadratic equation might not be suitable for the problem if the position found for the point C_1 (fig. 5) leads to a point S situated on the portion of the circle which corresponds to the angle $\pi - (\beta - \alpha)$ and not to the angle $\beta - \alpha$. This is what will happen if the point C_1 be situated on the segment of the lines $u\ v$ which the two straight lines DA and DB determine on the line OC_1 ; *i. e.* if x falls between the two values $\dfrac{Y_A\ x_D}{Y_A - y_D}$ and $\dfrac{Y_B\ x_D}{Y_B - y_D}$. This may happen, but it will very often occur that none of the roots fall within this space and that there may actually be four possible positions for the aeroplane from which a choice must be made. It may also happen that one of the roots must be rejected as utilising the prolongation of the pyramid on the other side of the apex S.

The co-ordinates of the point S will be given by the intersection of the line DC_1 with the circle. They are :

$$u = x - \frac{(x-x_D)\,(x-OM)\,(x-ON)}{(x-x_D)^2 + y_D^2} = x - (x - x_D)\,\frac{\sin^2\omega\ \cot g^2\,\theta}{\dfrac{x-OM}{X_C}\ \dfrac{x-ON}{X_C}}$$

$$v = y_D\,\frac{(x-OM)\quad(x-ON)}{(x-x_D)^2 + y_D^2} = \frac{x-u}{x-x_D}\,y_D$$

With reference to the plane ABC, the co-ordinates of the point S will be :

$$X_S = u\cos\omega \qquad Y_S = v \qquad Z_S = u\sin\omega.$$

and the formulae in Chapter I permit the calculation of the co-ordinates x_S, y_S, z_S, of the point S with reference to the horizontal plane. If X_C were negative, it would be necessary to take :

$$X_S = -u\cos\omega \qquad Y_S = v \qquad Z_S = u\sin\omega$$

2nd method of Calculation.

The line DC_1 and consequently the position of the point S will be ill-defined if the points D and C_1 are very near to one another. More precise results will be obtained in this case by calculating, not the unknown quantity x or $\cos\omega$, but the angle γ made by the line DC_1S with the line of direction OAB. These angles being read positively clockwise, the angle of the line of direction OAB with the line DB will be $-\alpha$, and with the line DA it will be $-\beta$.

Thus the relations

$$\frac{OM + ON}{y_D} = \overline{AB}\,\frac{1 + tan\,\alpha\,tan\,\beta}{\overline{OA}\,tan\,\beta - \overline{OB}\,tan\alpha}$$

$$\frac{OM.\,ON}{y_D^2} = \overline{OA}\;\overline{OB}\left(\frac{tan\,\beta - tan\,\alpha}{\overline{OA}\,tan\,\beta - \overline{OB}\,tan\,\alpha}\right)^2$$

$$x_D = \overline{AB}\,\frac{tan\,\alpha\,tan\,\beta}{tan\beta - tan\,\alpha} \qquad y_D = \frac{\overline{OA}\,tan\,\beta - \overline{OB}tan\,\alpha}{tan\,\beta\; - \;tan\,\alpha} \qquad \frac{x}{y_D} = \overline{AB}\,\frac{tan\,\alpha\,tan\,\beta}{\overline{OA}\,tan\,\beta - \overline{OB}\,tan\,\alpha} - tan\gamma$$

are obtained.

The biquadratic equation then becomes :

$$tan^4\,\gamma - 2\,\xi\,tan^3\,\lambda + (\xi 2 + A)\,tan^2\,\lambda + 2B\,tan\,\lambda + C = O.$$

Calling ξ, A, B, C, the co-efficients as in the equation of cos ω, although they have not the same value

Taking :

$$q = \frac{\overline{AB}}{\overline{OA}\,tan\,\beta - \overline{OB}\,tan\,\alpha}$$

then :

$$\xi = q\,(tan\,\alpha\,tan\,\beta - \sin^2\,\theta)$$

$$A = 1 + \sin^2\theta + 2q\sin^2\theta(tan\,\alpha + tan\,\beta) + (q\sin^2\theta)^2 cotg^2\theta - \left[q\,(tan\,\beta - tan\,\alpha)\,\frac{X_c}{AB}\right]^2\cos^2\,\theta$$

$$B = -\,\xi + q^2\,(tg\,\alpha + tan\,\beta)\,(1 - tg\,\alpha\,tan\,\beta)\,\sin^2\,\theta$$

$$C = q^2 tan^2\alpha\,tan^2\beta\cos^2\theta - \left[q\,(tan\,\beta - tan\alpha)\,\frac{X_c}{AB}\right]^2\cos^2\theta + \left[1 + q(tan\,\alpha + tan\,\beta)\right]^2\sin^2\,\theta$$

The remainder of the calculation is made as for the equation of cos ω tan γ once known, x and cos ω may be deduced therefrom

The co-ordinates of the point S will be :

$$u = \overline{AB}\,\frac{\cos^2\gamma}{tan\,\beta - tan\,\alpha}\,(tan\,\gamma + tan\,\alpha)\,(tan\gamma + tan\beta)$$

$$v = (u - x)\,cot\,\gamma =$$

$$\frac{\cos^2\gamma}{tan\,\beta\; - \;tan\,\alpha}$$

$$[\,(OA\,tan\,\beta - OB\,tan\,\alpha)\,tan^2\,\gamma + \overline{AB}\,(1 - tan\,\alpha\,tan\,\beta)\,tan\,\gamma + OB\,tan\,\beta - OA\,tan\,\alpha\,]$$

3rd method of Calculation.

The second procedure, in which the calculation is the most rapid, is inconvenient as it necessitates the use of very large numbers when y_D is small. In fact the quantity $OA \tan\beta - OB \tan\alpha$ will be very small and two of the roots $\tan\gamma$ will be very large.

It is better in this case to take $\cot\gamma$ as the unknown quantity. A biquadratic equation will again be obtained and it may be expressed in the form :

$$\cot^4\gamma - 2\xi \cot^3\gamma + (\xi^2 + A)\cot^2\gamma + 2B\cot\gamma + C = 0$$

Taking :

$$p = \frac{OA \tan\beta - OB \tan\alpha}{AB}$$

$$g = \tan^2\alpha \tan^2\beta \cos^2\theta - (\tan\beta - \tan\alpha)^2 \frac{X_C^2}{\overline{AB}^2}\cos^2\theta + (\tan\alpha + \tan\beta + p)^2 \sin^2\theta$$

the values of the co-efficients will be :

$$\xi = \frac{p(\tan\alpha \tan\beta - \sin^2\theta) - (\tan\alpha + \tan\beta)(1 - \tan\alpha \tan\beta)\sin^2\theta}{g}$$

$$A = \frac{p^2(1+\sin^2\theta) + 2p(\tan\alpha + \tan\beta)\sin^2\theta - \dfrac{X_C^2}{\overline{AB}^2}(\tan\beta - \tan\alpha)^2\cos^2\theta + \tan^2\alpha\tan^2\beta}{g}$$

$$+ \frac{\sin^2\theta\,(1 - 2\tan\alpha\tan\beta) - \xi^2}{g}$$

$$B = -\frac{p}{g}(\tan\alpha \tan\beta - \sin^2\theta) \qquad C = \frac{p^2}{g}$$

Danger Cylinder

Another expression will now be given for the position of the base of the perpendicular dropped from the point S on to the plane ABC.

The co-ordinates of the centre of the circle circumscribed about the triangle ABC are :

$$\frac{X_C^2 + Y_A Y_B}{2X_C} \quad \text{et} \quad \frac{Y_A + Y_B}{2}$$

Its radius is :

$$R^2 = \frac{(X_C^2 + Y_A^2)\ (X_C^2 + Y_B^2)}{4X_C^2}$$

The distance to the centre of this circle from the base of the perpendicular dropped from S on to the plane ABC will be :

$$\rho^2 = \left(X_S - \frac{X_C^2 + Y_A Y_B}{2X_C}\right)^2 + \left(Y_S - \frac{Y_A + Y_B}{2}\right)^2$$

This may be written :

$$\rho^2 - R^2 = X_S\left(X_S - X_C - \frac{Y_A\ Y_B}{X_C}\right) + \left(Y_S - Y_A\right)\ \left(Y_S - Y_B\right)$$

If the fact be taken into account that the point S is on the circle which passes through the points A, B ; M and N, and $Y_3 = v$, then :

$$(Y_S - Y_A)\ (Y_S - Y_B) = u\ (OM + ON - u)$$

and thus :

$$\rho^2 - R^2 = \frac{u}{X_C}\left[\frac{ux^2}{X_C} - x\,X_C - \frac{x\ OM.\ ON}{X_C} + X_C\,(OM + ON) - u\,X_C\right]$$

or, sustituting its value instead of u :

$$\rho^2 - R^2 = \left[\frac{x}{X_C} - \frac{(x-x_D)(x-OM)(x-ON)}{X_C[(x-x_D)^2 + y_D^2]}\right](x-OM)(x-ON)\left[\frac{(X_C^2 - x^2)\ (x-x_D)}{X_C[(x-x_D)^2 + y_D^2]} + \frac{x(x^2 - OM.ON) - X_C^2(2x - OM - ON)}{X_C\,(x-OM)\ (x-ON)}\right]$$

Note, however, that the biquadratic equation, which has given us x may be written :

$$X_C^2 = x^2 + \frac{(x - OM)^2\ (x - ON)^2}{(x - x_D)^2 + y_D^2}\ tan^2\,\theta$$

This equation may be considered as defining an expression X_{CI}, a function of x. By obtaining its derivative and eliminating $\tan^2\,\theta$, then :

$$X_C\,\frac{\delta X_C}{\delta x} = \frac{X_C^2\ (2x - OM - ON) - x\ (x^2 - OM.ON)}{(x - OM)\ (x - ON)} - \frac{(X_C^2 - x^2)\ (x - x_D)}{(x - x_D)^2 + y_D^2}$$

It will be seen that the last factor of $\rho^2 - R^2$ is none other than $-\dfrac{dX_C}{dx}$

Therefore :

$$\rho^2 = R^2 - \frac{u}{X_C} (x - OM) \ (x - ON) \ \frac{\delta X_C}{\delta x}$$

The expressions u, x-oM, x-oN cannot cancel out in the cases which occur in practice ; but ρ^2 becomes equal to R^2 if $\dfrac{dX_C}{dx}$ is nil, *i. e.* when x is a double root of the biquadratic equation. In this case the point S will be found on the cylinder circumscribed about the triangle ABC, the generatrices of which are perpendicular to the plane of this triangle. It is known then that :

$$D^3 = E^2$$

is the relation between the elements of the pyramid and those of the base triangle.

But this relation may be verified without the point S being on the cylinder circumscribed about the triangle. It shows that there are three possible and clearly defined positions for the point S, one on the cylinder and the two others elsewhere ; the two conics which have been considered, the intersection of which gives the values of x, will then be tangent ; but their point of contact exactly defines the double root, but only on condition that these conics are not replaced by the tangent at the common point in a method of successive approximations.

V. — METHOD BY SUCCESSIVE APPROXIMATIONS.

The solution of the biquadratic equation entails rather lengthy calculation. Nevertheless this is not appreciably longer than those called for by the method of successive approximations employed up till now and, besides, it presents the very important advantage of providing all the solutions of the problem, whether there be a double root or not.

But the form given above to the biquadratic equation allows a method of successive approximations to be used which is rapid and applicable to all cases.

Supposing that approximate values of X_s, Z_s, are available. The approximate value ω of the angle made by the plane ABC with the face SAB can be deduced therefrom by the formula :

$$\frac{Zs}{Xs} = \tan \omega$$

Then the exact values of OM, ON, X_D, Y_D are calculated; and again x by the relation:

$$x = X_C \cos \omega$$

An approximate value of x is thus obtained. The biquadratic equation may be written:

$$X_C \cot \theta = \frac{(x - OM) \quad (x - ON)}{\sin \omega \sqrt{(x - x_D)^2 + y_D^2}}$$

Give to x two round figure values, which are in the neighbourhood of the approximate value just found, and deduce therefrom the two corresponding values of ω. The second term of the preceding equation can be rapidly calculated for these values of x and of ω; this will give two values of the quantities X_C cot θ which differ from the true quantity already known. Simple proportion will give a value of x nearer to the true one. Transferring this value, and that of sin ω deduced from it, into the second term, a value of X_C cot θ will be found which differs very little from the precise value and a second proportior will give a definite value for x.

Taking as ordinates the three values of X_C cot θ which have just been calculated and which correspond to three values of x which are taken as abscissae, the value of the approximation may be judged and at the same time it may be seen whether two roots are not close to one another which, of course, may be easily determined by the same process.

Having thus determined x the calculation of X_S, Y_S, Z_S is made as has been shown. The calculations indicated in Chapters I and III would also have to be carried out

An approximate value of x may equally be obtained by means of a graph. In Chapter IV it is shown that the four roots x may be given by the intersection of a parabola with either a circle or an equilateral hyperbola. A rough graph of these curves will give us approximate values for the four roots.

Another graph which gives the same result is as follows: The points of intersection are sought of the circle:

$$x^2 + y^2 = X_C^2$$

and of the curve:

$$y^2 = \frac{(x - OM)^2 \, (x - ON)^2 \tan^2 \theta}{(x - x_D)^2 + y_D^2}$$

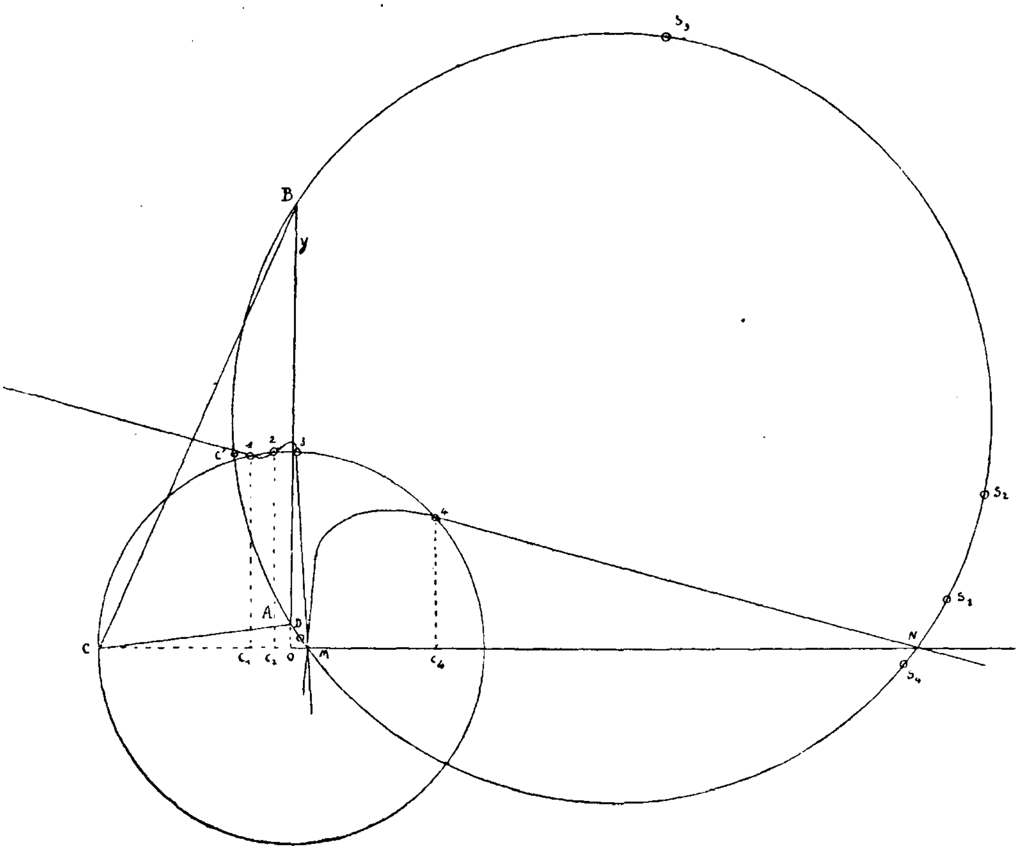

FIG. 6

This curve is very easily constructed. For this purpose take (fig. 5 and 6) the edge SC of the pyramid and lay it on to the plane SAB by turning it round its projection on this plane. For a given position of the point S on the segment of the circle containing the angle, a straight line SC' is thus obtained which lies at an angle θ with SC and which consequently passes through a fixed point C' situated on the circumference at a point which makes the arc DC' equal to $2\ \theta$.

If a perpendicular to SC_1 be drawn at the point C_1, and also a perpendicular to ON, the length of the segment perpendicular to SC_1 which lies between SC_1, and SC' is $C_1S \tan \theta$ it suffices to lay off this length, from C_1, on the perpendicular to ON in order to obtain a point of the curve.

This curve of the 4th degree, which is symmetrical with reference to ON, admits points M and N as double points.

Its construction was demonstrated by Captain SACONNEY in 1913 in his " Métrophotographie ".

The curve corresponding to data taken from Report N° 1-1923 of the Air Survey Committee, page 107, is given here as also is the complete calculation of the possible points of resection.

The data given include four control points. The curve and the first calculation employ three of them. The calculation is made by means of the equation which gives $\tan \gamma$. A second calculation has been made using the first two and the fourth control points. The method by successive approximations has been used, adopting as approximate position the second position obtained by the calculation, this position being the only one of the four which is satisfactory to the system of four control points.

Thus two positions are obtained for point S which are about 57^{m} (187 ft) apart and differ from the positions given by the Air Survey by about 39^{m} (128 ft.) and 23^{m} (74 ½ ft.) respectively. The Air Survey, using the positions of the four points for its calculations, has more than enough data at its disposal. Besides, the calculation has been made assuming that the four points were in the same plane, which is not quite correct.

The difference of 57^{m} (187 ft.) between the two positions of the point S_2 which have been found seems to be explicable only by an inaccuracy in the focal length used or in the position of the principal point of the photograph, unless some error has occurred in our calculation.

Special Cases.

I. A special case is that where:

$$Y_D = o$$

i e when the point D coincides with one of the points M or N The previous reasoning no longer applies. But yet there may still be four solutions.

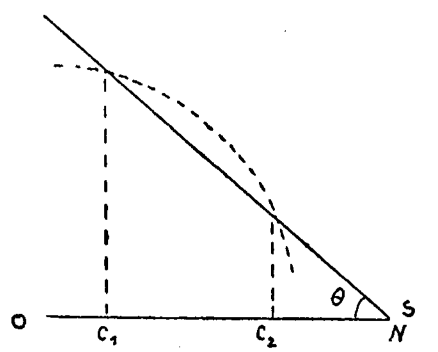

Fig. 7

Let it be supposed that D and M coincide. The point S may be placed at N. Turn the edge SC about ON. With the point O as centre and with radius X_C, draw the arc of a circle. If this meets the edge which was turned there will be two solutions C_1 and C_2, the abscissa x_{12} of which will be given by the formula:

$$x_{12} = \overline{ON} \sin^2 \theta \pm \cos \theta \sqrt{X_C^2 - \overline{ON}^2 \sin^2 \theta}$$

They correspond to two values ω_1 and ω_2 of the angle of the plane ABC to the face SAB, such that :

$$\cos \omega_{12} = \frac{x_{12}}{X_C}$$

The co-ordinate of the point S will be :

$$X_{S_{12}} = \frac{\overline{ON}^2}{X_C} \sin^2 \theta \pm \overline{ON} \cos \theta \sqrt{1 - \frac{\overline{ON}^2}{X_C^2} \sin^2 \theta} = \overline{ON} \cos \omega_{12}$$

$$Y_{S_{12}} = 0$$

$$Z_{S_{12}} = \overline{ON} \sin\theta \sqrt{1 + \frac{\overline{ON}^2}{X_C^2}(\cos^2 \theta - \sin^2 \theta) \mp 2\frac{ON}{X_C} \cos \theta \sqrt{1 - \frac{\overline{ON}^2}{X_C^2} \sin^2\theta}} = \overline{ON} \sin \omega_{12}$$

The upper and lower signs correspond to one another. (*See* Figs. 7 and 8). But there are still two other solutions. Take ω_3 as the angle of inclination of the plane ABC such that the point C is projected at M which is also the point D. Then :

$$\cos \omega_3 = \frac{OM}{X_C}$$

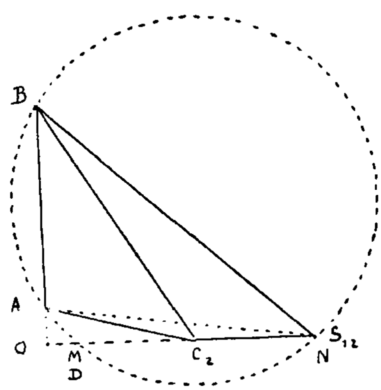

Fig. 8

The length MS should be equal to :

$$X_C \sin \omega_3 \; cot\; \theta = \sqrt{X_C^2 - \overline{OM}^2} \; cot\; \theta$$

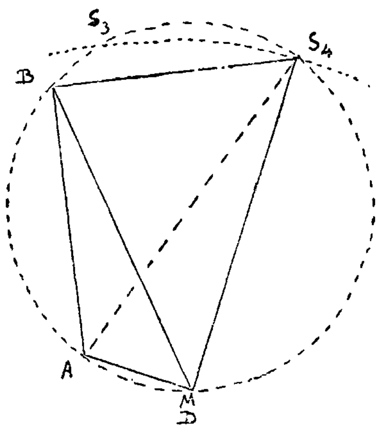

Fig. 9

Besides, the point S should be on the circle MAB. Therefore it will be found at the intersection of this circle and of another circle, the centre of which is M and the radius MS. These circles meet at two points which give two solutions S_3 and S_4 and which generally are suitable to the problem (*See* Fig. 9). Therefore, in the case where the point D coincides with the point M, the problem of placing the triangle ABC on a given pyramid still involves four solutions, but two of these solutions correspond to the same angle of the plane ABC to the face SAB. Further, if the base of the perpendicular dropped from the point S onto the plane ABC be considered, it will be seen that the bases of the perpendiculars corresponding to the apices S_1 and S_2 are situated on the perpendicular to the base of the triangle taking C as apex. Conversely, if one of the bases of the perpendiculars is on this line there are always two. It may also be taken from this remark that, if one of the bases is the point of intersection of the perpendiculars to the three sides of the triangle ABC from its angles, there will be another base on each of these perpendiculars.

The same angle ω_3 of the plane ABC to the face SAB corresponds to two separate solutions; there are, in all, four distinct positions for the pyramid. Two of these positions would be superimposed if points C_1 and C_2 coincided, *i. e.* if:

$$X_C = ON \sin \theta.$$

The straight line SC would then be perpendicular to the plane SAB and the point S would be situated actually on the right cylinder circumscribed about the triangle ABC.

There will still be a double solution if S_3 and S_4 coincide. This will happen if:

$$\sqrt{X_C^2 - \overline{OM}^2} \cot \theta = \frac{AB}{\sin (\beta - \alpha)}$$

then :

$$OM = OB \tan \alpha \; ; \qquad OA \tan \beta = OB \operatorname{tg} \alpha$$

$$\tan \gamma = \frac{1 - \tan \alpha \tan \beta}{\tan \alpha + \tan \beta} = \cot (\alpha + \beta) \qquad \gamma = \frac{\pi}{2} - (\alpha + \beta)$$

$$X_C^2 = \frac{\overline{AB}^2}{(\tan \beta - \tan \alpha)^2 \cos^2 \theta} \left[\sin^2 \theta \, (1 + \tan^2 \alpha + \tan^2 \beta) + \tan^2 \alpha \tan^2 \beta \right]$$

$$X_{S_{34}} = \left[OM + \overline{AB} \frac{\cos (\beta + \alpha)}{\sin (\beta - \alpha)} \right] \cos \omega_3 = \frac{\overline{OA} . \overline{OB}}{X_C}$$

$$Y_{S_{34}} = \overline{A\,B} \frac{\sin (\beta + \alpha)}{\sin (\beta - \alpha)} = \overline{OB} + \overline{OA}$$

$$Z_{S_{34}} = \left[OM + \overline{AB} \frac{\cos (\beta + \alpha)}{\sin (\beta - \alpha)} \right] \sin \omega_3 = \overline{OA} \cot \alpha \sqrt{1 - \frac{\overline{OB}^2}{X_C^2} \tan^2 \alpha}$$

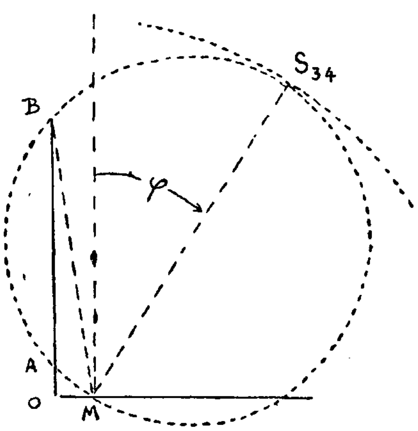

Fig. 10

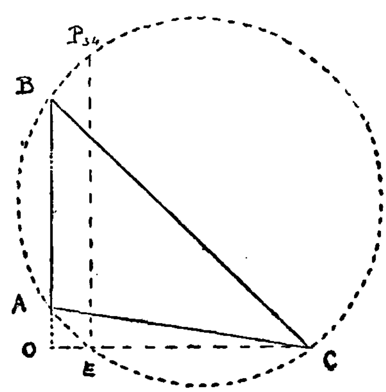

Fig. 11

The point with Xs_{34} and Ys_{34} as co-ordinates is then on the circle circumscribed about the triangle ABC; as might have been expected, it is the symmetrical of the point E with reference to the diameter which is perpendicular to $\overline{AB}$.

The case where the point D coincides with the point M has been examined. If, however, it should coincide with the point N, S cannot be placed at M, for, from this, solutions would result which are inadmissible in the photographic problem dealt with; two solutions S_3 and S_4 can, however, still be found as in the preceding case, the centre of second circle this time being N and not M.

In the case where Y_D has a value, though very small, X_D will have a value very close to that of CM and the biquadratic equation of x will have two roots approximateing closely to the value OM which is given on the line OMN by two points C_3 and C_4 they being very near the point D, so that the lines DC_3 and DC_4 meet the circle $BAMN$ at two points S_3 and S_4, very far apart from each other, as happened in the extreme case where Y_D was nil. The biquadratic equation of tan γ will then provide the positions of the point S in this case without any ambiguity.

II. Lieutenant E. SANTONI proposed (*) to use the sun as a third control point. A special fitting registers the image of the sun on the plate as well as the time indicated by a chronometer at the moment of exposure. The time thus registered enables the azimuth and the altitude of the sun at this moment, to be calculated.

This particular case may be dealt with in the same way as the general case by considering the point which has been referred to as B as being infinitely distant in a known direction.

The pyramid will be defined, as before, by the angles α, β and θ.

The segment of the circle containing the angle β-α on the face ABS, will be a straight line AMS (*See* Fig. 12) lying at an angle β - α to the line OAB.

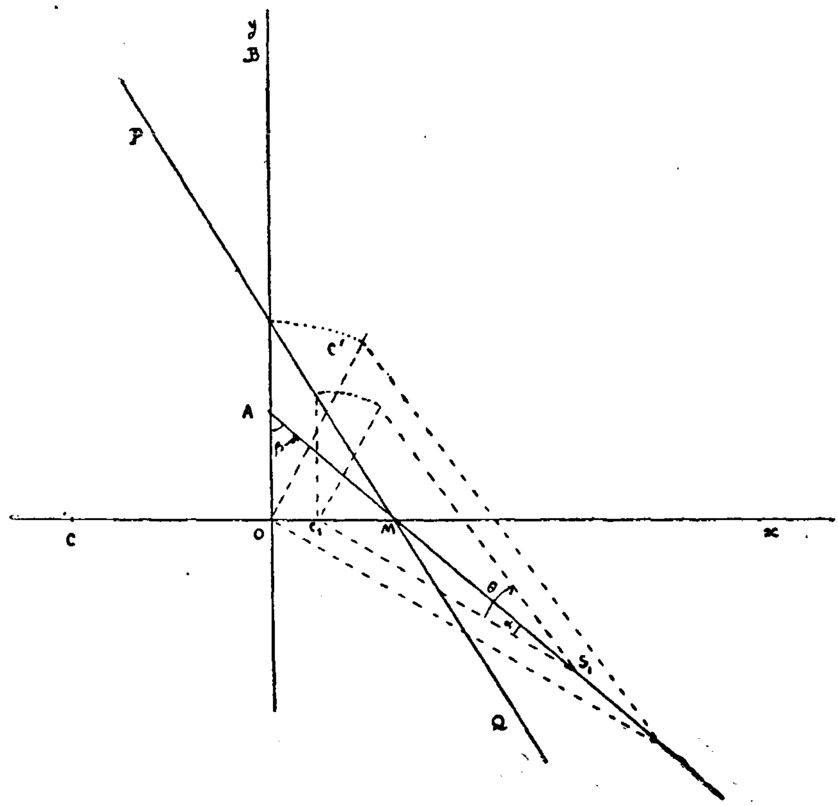

FIG. 12

(*) Fotogrammetria aerea col metodo SANTONI, *extract from the* "Atti della I Settimana Aerotecnica", Rome 23rd. to 29th, Nov. 1925.

By using the construction employed for Fig. 6, instead of obtaining the curve of the 4th degree, a line PMQ (and its symmetrical with reference to the axis of the x's) will be obtained, the intersection of which with the circle of radius OC will give two abscissa points similar to C_1. A line drawn through C_1 and lying at an angle α to the line AM meets this line at a point S_1 which when plotted, gives one of the two positions of the point of resection.

It will be easily found that the equation of the line PMQ is:

$$x \cos(\beta-\alpha) + y \sin \alpha \cot \theta = \overline{OA} \sin(\beta-\alpha).$$

That of the circle is $x^2 + y^2 = X_C^2$.

Thus the abscissae of the points similar to C_1 are given by the equation :

$$x = \frac{\overline{OA} \sin(\beta-\alpha)\cos(\beta-\alpha) \pm \sin\alpha \cot\theta \sqrt{X_C^2\,[\cos^2(\beta-\alpha) + \sin^2\alpha \cot^2\theta] - \overline{OA}^2 \sin^2(\beta-\alpha)}}{\cos^2(\beta-\alpha) + \sin^2\alpha \cot^2\theta}$$

The co-ordinates u and v of the point S will be those of the intersection of the two lines:

$$AMS \qquad u = (OA - v) \tan(\beta - \alpha)$$

and

$$\text{et } C_1 S \qquad x - u = v \tan\beta$$

Then :

$$u = \frac{OA \tan\beta - x}{\tan\beta - \tan(\beta-\alpha)} \tan(\beta-\alpha) = \frac{OA \sin\beta - x\cos\beta}{\sin\alpha} \sin(\beta-\alpha)$$

$$v = \frac{x - OA \tan(\beta-\alpha)}{\tan\beta - \tan(\beta-\alpha)} = \frac{x \cos(\beta-\alpha) - OA \sin(\beta-\alpha)}{\sin\alpha} \cos\beta$$

As before, the co-ordinates, after re-erection of the pyramid, will be :

$$X_S = u \cos\omega = \frac{u\,x}{X_C}$$

$$Y_S = v$$

$$Z_S = u \sin\omega$$

The circle circumscribed about the triangle ABC becomes, in this case, a straight line AC, and the " danger cylinder " becomes the plane passing through AC and through a perpendicular to the plane formed by AC and the direction of the sun.

If the point S is situated in this plane, the distance of the line PQ from the point O is then equal to OC ; the two roots of the quadratic equation are reduced to a single root which is clearly defined. The only case where it would be indeterminate is when the line AC happens to be perpendicular to the direction of the sun. In this case, point S may be on any point in the segment of the circle drawn on AC containing the angle β-α.

III. If the point S is in the same plane as the three control points A, B, C, the angle referred to θ will be nil and the three points a, b, c, will be in a straight line on the photograph. It is only when the points A, B, C are in the same vertical plane that they will be in a straight line on the chart. The resection is in the plane of the three points and the position thereof is given by the intersection of two segments of circles containing the angles. If the points A, B, C were on the same straight line in space, the position of the point of resection would be indeterminate but on a circle the plane of which would be perpendicular to this straight line.

Calcul de la Position de l'Avion d'après les éléments relevés sur la photographie.
Les points de la photographie sont rapportés aux axes de la plaque.

longueur focale = 254 mm

données extraites du "Report of the Air Survey Committee. N°1 - 1923".

$x_a = +43.^{mm}44 \quad x_b = -73.79 \quad x_c = +95.37$
$y_a = +64.39 \quad y_b = -50.27 \quad y_c = +13.57$

Calculation of the Resection from data taken from the photograph.
points on the photo are referred to the axes of the plate.

focal length 254 m.m

$x_A = +1184.^m61 \quad x_B = +1396.90 \quad x_C = +978.10$
$y_A = -491.64 \quad y_B = -33.53 \quad y_C = -408.25$
$z_A = +5.48 \quad z_B = +5.48 \quad z_C = 0.$

numerical data are taken from "Report of the Air Survey Committee. N°1 - 1923".

I. - Calcul du Triangle de Base. - Calculation of the base triangle.

$x_c = +978.10$
$x_A = +1184.61$
$x_c - x_A = -206.51$
$\log(x_c - x_A) = 2.3149411$
$\log \sin G_{A'C} = \bar{1}.9671991$
$\log A'C = 2.3477420$
$\log \sin \text{diff.} = \bar{1}.9994536$
$\log O'C = 2.3471956$
$\log OO' = 0.7387806$
$\log tg\, \widehat{O'CO} = \bar{2}.3915850$
$\widehat{O'CO} = 1°24'40''$

$y_c = -408.25$
$y_A = -491.64$
$y_c - y_A = +83.39$
$\log(y_c - y_A) = 1.9211140$
$\log \cos G_{A'C} = \bar{1}.5733720$
$= 2.3477420$
$\log \cos \text{diff.} = \bar{2}.7001866$
$\log A'O' = 1.0479286$
$OA = +11.^m1668$

2.3149411
1.9211140
$tg \cdot 0.3938271$
$G_{A'C} = 291°59'21''$
$G_{B'A'} = 204\ 51\ 47$
$\text{diff} = 87\ 07\ 34.$

$x_c = +978.10$
$x_B = +1396.90$
$x_c - x_B = -418.80$
$\log(x_c - x_B) = 2.6220067$
$\log \sin G_{B'C} = \bar{1}.8722947$
$\log B'C = 2.7497120$
$\log \sin \text{diff} = \bar{1}.5974836$
$\log O'C = 2.3471956$
$\log \cos \widehat{O'CO} = \bar{1}.9998683$
$\log OC = 2.3473273$
$OC = +222.^m50$

$y_c = -408.25$
$y_B = -33.53$
$y_c - y_B = -374.72$
$\log(y_c - y_B) = 2.5737069$
$\log \cos G_{B'C} = \bar{1}.8239949$
$= 2.7497120$
$\log \cos \text{diff.} = \bar{1}.9630005$
$\log B'O' = 2.7127125$
$OB = +516.^m075$

2.6220067
2.5737069
$tg - 0.0482998$
$G_{B'C} = 228°10'46''$
$G_{B'A'} = 204\ 51\ 47$
$\text{diff} = 23\ 18\ 59$

$AB = 504.^m908$

$x_A = +1184.61$
$x_B = +1396.90$
$x_A - x_B = -212.29$
$\log(x_A - x_B) = 2.3269295$
$\log \sin G_{B'A'} = \bar{1}.6237175$
$\log B'A' = 2.7032120$

$y_A = -491.64$
$y_B = -33.53$
$y_A - y_B = -458.11$
$\log(y_A - y_B) = 2.6609698$
$\log \cos G_{B'A'} = \bar{1}.9577578$
$= 2.7032120$

2.3269295
2.6609698
$tg.\ \bar{1}.6659597$
$G_{B'A'} = 204°51'47''$

III. - Calcul des éléments de la Pyramide. - Calculation of the elements of the Pyramid.

$x_b = -73.79$
$x_a = +43.44$
$x_b - x_a = -117.23$
$\log(x_b - x_a) = 2.0690388$
$\log \sin G_{ab} = \bar{1}.8542451$
$\log ab = 2.2147937$

$y_b = -50.27$
$y_a = +64.39$
$y_b - y_a = -114.66$
$\log(y_b - y_a) = 2.0594119$
$\log \cos G_{ab} = \bar{1}.8446182$
$= 2.2147937$

2.0690388
2.0594119
$tg.\ 0.0096269$
$G_{ab} = 45°38'06''$

$\log x_b = 1.8679975$
$\log \sin G_{pb} = \bar{1}.9172129$
$\log pb = 1.9507846$
$\log \sin \widehat{bpp'} = \bar{1}.9932170$
$\log p'b = 1.9440016$

$\log y_b = 1.7013089$
$\log \cos G_{pb} = \bar{1}.7505243$
$= 1.9507846$
$\log \cos = \bar{1}.2439507$
$\log pp' = 1.1947353$

$\log f = 2.4048337$
$\log \sin \widehat{pp's} = \bar{1}.9991764$
$\log p's = 2.4056573$
$\log p'b = 1.9440016$
$\log tg\, \widehat{p'sb} = \bar{1}.5383443$
$\widehat{p'sb} = +19°03'21''$
$\widehat{p'sc} = -17\ 16\ 23$
$\beta = +36\ 19\ 44$

$\log \cos = \bar{2}.7890779$
$= 2.4056574$
$\log p'a = 1.8812618$
$\log tg\, \widehat{p'sa} = \bar{1}.4756045$
$\widehat{p'sa} = -16°38'39''$
$= -17\ 16\ 23$
$\alpha = +0\ 37\ 44$
$\beta - \alpha = 35\ 42\ 00$

$\log x_b = 1.8679975$
$\log y_b = 1.7013089$
$\log tg\, G_{pb} = 0.1666886$
$G_{pb} = 235°44'06''$
$G_{pp'} = 315\ 38\ 06$
$\widehat{bpp'} = 79\ 54\ 00$

$\log f = 2.4048337$
$\log pp' = 1.1947353$
$\log tg\, \widehat{pp's} = 1.2100985$
$\widehat{pp's} = 86°28'21''$

$p's = 254.48$
$cc' \cos pp's = 4.47$
$p's - cc' \cos pp's = 250.01$

$\theta = 15°28'42''$

$\log x_a = 1.6378898$
$\log \sin G_{pa} = \bar{1}.7476193$
$\log pa = 1.8902705$
$\log \sin \widehat{app'} = \bar{1}.9909913$
$\log p'a = 1.8812618$

$\log x_c = 1.9794118$
$\log \sin G_{pc} = \bar{1}.9956476$
$\log pc = 1.9837642$
$\log \sin \widehat{pcc'} = \bar{1}.9064809$
$\log p'c' = 1.8906451$
$p'c' = -77.74$

$\log cc' = 0.8611758$
$\log \sin \widehat{pp's} = \bar{1}.9991764$
$\log cc' \sin \widehat{pp's} = 1.8603522$
$\log S\bar{c} = 2.4179991$
$\log tg\, \theta = \bar{1}.4423531$

$\log y_a = 1.8088184$
$\log \cos G_{pa} = \bar{1}.9185479$
$= 1.8902705$
$\log \cos \widehat{app'} = \bar{1}.3044648$
$\log pp' = 1.1947353$

$\log y_c = 1.1325798$
$\log \cos G_{pc} = \bar{1}.1488155$
$= 1.9837643$
$\log \cos \widehat{pcc'} = \bar{1}.7719909$
1.7557552

$pp' = 56.98$
15.66
$cc' = 72.64$

$\log x_a = 1.6378898$
$\log y_a = 1.8088184$
$\log tg\, G_{pa} = \bar{1}.8290714$
$G_{pa} = 34°00'18''$
$G_{pp'} = 315\ 38\ 06$
$\widehat{app'} = 78\ 22\ 12$

$\log x_c = 1.9794118$
$\log y_c = 1.1325798$
$\log tg\, G_{pc} = 0.8468320$
$G_{pc} = 81°54'07''$
$G_{pp'} = 315\ 38\ 06$
$\widehat{pcc'} = 53\ 43\ 59$

$\log cc' = 0.8511758$
$\log \cos \widehat{pp's} = \bar{2}.7890779$
$\log cc' \cos \widehat{pp's} = 0.6502537$

IV. - Calcul des paramètres. - Calculation of the parameters.

$$q = \frac{AB}{OA\, tg\beta - OB\, tg\alpha}$$

$\log OA = 1.0479286$
$\log tg\, \beta = \bar{1}.8664938$
$\log OA\, tg\beta = 0.9144224$
$OA\, tg\beta = +8.21148$
$\log OB = 2.7127125$
$\log tg\, \alpha = \bar{2}.0404677$
$\log OB\, tg\alpha = 0.7532802$
$OB\, tg\alpha = +5.66605$
$\text{diff} = +2.54543$

$\log tg\alpha = \bar{2}.0404677$
$\log tg\beta = \bar{1}.8664938$
$\log tg\alpha\, tg\beta = \bar{3}.9069615$
$tg\alpha\, tg\beta = 0.0080716$
$1 - tg\alpha\, tg\beta = 0.9919284$

$\log AB = 2.7032123$
$\log \text{diff} = 0.4057611$
$\log q = 2.2974512$

$tg\alpha = +0.0109766$
$tg\beta = +0.7353495$
$tg\alpha + tg\beta = +0.7463261$
$tg\beta - tg\alpha = +0.7243729$

$\log AB = 2.7032123$
$\log X_c = 2.3473273$
$\log \frac{X_c}{AB} = \bar{1}.6441150$
$\log(tg\beta - tg\alpha) = \bar{1}.8599622$
$\log q = 2.2974512$
$\log[\ ] = 1.8015284$

$\log \sin\theta = \bar{1}.4263089$
$\log \sin^2\theta = \bar{2}.8526178$
$\sin^2\theta = 0.0712226$
$tg\alpha\, tg\beta = 0.0080716$
$tg\alpha\, tg\beta - \sin^2\theta = -0.0631510$
$\log = \bar{2}.8003802\ n$
$\log q = 2.2974512$
$\log \xi = 1.0978314\ n$
$\xi = -12.52655$

$\log \cos\theta = \bar{1}.9839558$
$\log \cos^2\theta = \bar{1}.9679116$
$\log \sin^2\theta = \bar{2}.8526178$
$\log \cot g^2\theta = 1.1152938$

$\log q = 2.2974512$
$\log \sin^2\theta = \bar{2}.8526178$
$\log q \sin^2\theta = 1.1500690$

$2\log q = 4.5949024$
$2\log \cos\theta = \bar{1}.9679116$
$\log q^2 \cos^2\theta = 4.5628140$

$2\log[\ ] = 3.6030558$
$\log \cos^2\theta = \bar{1}.9679116$
$\log 4^e\text{term} = 3.5709684 \quad = -3723.6462$
$2\log q \sin^2\theta = 2.3001380$
$\log \cot g^2\theta = 1.1152938$
$\log 3^e\text{term} = 3.4154318 \quad = +2602.7461$
$\log 2 = 0.3010300$
$\log(tg\alpha + tg\beta) = \bar{1}.8729286$
$\log q \sin^2\theta = 1.1500690$
$\log 2^e\text{term} = 1.3240276 \quad = +21.0876$
$1 + \sin^2\theta = +1.0712$
$A = -1098.7413$

$\log q^2 = 4.5949024$
$\log \sin^2\theta = \bar{2}.8526178$
$\log(tg\alpha + tg\beta) = \bar{1}.8729286$
$\log(1 - tg\alpha\, tg\beta) = \bar{1}.9964083$
$\log 2^e\text{term} = 3.3168571$
$2^e\text{term} = 2074.2309$
$-\xi = +12.5266$
$B = +2086.7575$

$\log q = 2.2974512$
$\log(tg\alpha + tg\beta) = \bar{1}.8729286$
$\log = 2.1703798$
$q(tg\alpha + tg\beta) = 148.0402$
$-d^o + 1 = 149.0402$
$\log. = 2.1733034$
$2\log. = 4.3466068$
$\log \sin^2\theta = \bar{2}.8526178$
$\log 3^e\text{term} = 3.1992246$
$3^e\text{term} = +1582.0661$
$4^e\text{term of } A = -3723.6462$
$1^{er}\text{term} = +2.3809$
$C = -2139.1992$

$\log q^2 \cos^2\theta = 4.5628140$
$2\log tg\alpha\, tg\beta = \bar{5}.8139230$
$\log 1^{er}\text{term} = 0.3767370$

$2\log \xi = 2.1956628$
$\xi^2 = 156.9144$
$A = -1098.7413$
$\xi^2 + A = -941.8269$
$\log(\xi^2 + A) = 2.9739711\ n$

$\log \xi = 1.0978314\ n$
$\log B = 3.3194720$
$\log B\xi = 4.4173034\ n$
$B\xi = -26139.8674$
$C = -2139.1992$
$C + B\xi = -28279.0666$
$\log(C + B\xi) = 4.4514650\ n$
$\log 12 = 1.0791812$
$5.5306462\ n$
$2\log(\xi^2 + A) = 5.9479422$
$\bar{1}.5827040\ n$
$= -0.382564$
$D = +0.617436$

$2\log B = 6.6389440$
$\log A = 3.0408955\ n$
$\log C = 3.3302512\ n$
$\log AC = 6.3711467$
$AC = 2350426.5$
$B^2 = 4354557.0$
$B^2 - AC = 2004130.5$
$\log = 6.3019260$
$\log 54 = 1.7323938$
8.0343198
$3\log(\xi^2 + A) = 8.9219133\ n$
$\bar{1}.1124065\ n$
$= -0.129541$
$\tfrac{1}{2} = -0.191282$
$D = +0.617436$
$E = +0.296613$

Calcul des Racines. - Calculation of the Roots.

$\log D = \bar{1}.7905920$
$3\log D = \bar{1}.3717760$
$D^3 = 0.235384$
$E^2 = 0.087979$
$D^3 - E^2 = 0.147405$
$\log(D^3 - E^2) = \bar{1}.1685122$
$\tfrac{1}{2} = \bar{1}.5842561$
$\log E = \bar{1}.4721902$
$\log tg\, \varphi = 0.1120659\ n$
$\varphi = 127°41'18''.1$
$\tfrac{1}{3}\varphi = 42\ 33\ 46$

$\log E = \bar{1}.4721902$
$2\log E = \bar{2}.9443804$

$\log(\xi^2 + A) = 2.9739711\ n$
$\tfrac{1}{2}\log D = \bar{1}.8952960$
$\log \cos \frac{\varphi}{3} = \bar{1}.8671944$
$\log \frac{3}{2} A\mu = 2.7364615\ n$
$\frac{3}{2} A\mu = -545.0826$
$3A\mu = -1090.1652$
$A\mu = -363.3884$

$\tfrac{1}{3}(\xi^2 + A) = -313.9423$
$-A = +1098.7413$
$-A\mu = +363.3884$
$y^2 = +1148.1874$
$2\log y = 3.0600127$

$\xi = -12.52665$
$y = +33.88484$
$\xi + y = +21.35829$
$\xi - y = -46.41139$
$\tfrac{1}{2}(\xi + y) = +10.67915$
$\tfrac{1}{2}(\xi - y) = -23.20569$

$\log \text{diff} = 1.5667650$
$\tfrac{1}{2}\log \text{diff} = 0.7833825$
$\sqrt{\text{diff}} = 6.07271$

$y^2 = +1148.1874$
$\frac{3}{2} A\mu = -545.0826$
$y^2 + \frac{3}{2} A\mu = +603.1048$
$\tfrac{1}{2} = +301.5524$
$\frac{A\xi + 2B}{2y} = +264.6746$
$\text{diff.} = +36.8778$
$\text{summ.} = +566.2270$
$\log \text{sum.} = 2.7529906$
$\tfrac{1}{2}\log \text{sum.} = 1.3764953$
$\sqrt{\text{sum.}} = 23.79552$

$\log \xi = 1.0978314\ n$
$\log A = 3.0408955\ n$
$\log A\xi = 4.1387269$
$A\xi = +13763.4367$
$2B = +4173.5150$
$A\xi + 2B = +17936.9517$
$\tfrac{1}{2} = 8968.4759$
$\log = 3.9527187$
$\log y = 1.5300064$
$\log \frac{A\xi + 2B}{2y} = 2.4227123$
$\frac{A\xi + 2B}{2y} = +264.6746$

$\tfrac{1}{2}(\xi + y) = +10.67915$
$+\sqrt{\text{diff}} = +6.07271$
$tg\gamma_1 = +16.75186$

$\tfrac{1}{2}(\xi + y) = +10.67915$
$-\sqrt{\text{diff.}} = -6.07271$
$tg\gamma_2 = +4.60644$

$\tfrac{1}{2}(\xi - y) = -23.20569$
$+\sqrt{\text{sum.}} = +23.79552$
$tg\gamma_3 = +0.58983$

$\tfrac{1}{2}(\xi - y) = -23.20569$
$-\sqrt{\text{sum.}} = -23.59552$
$tg\gamma_4 = -47.00121$

V. - Calcul des Coordonnées - Calculation of the Coordinates.

$tg\gamma_1 =$	$+16.75186$
$tg\alpha =$	$+0.0109766$
$tg\gamma + tg\alpha =$	$+16.7628366$
$\log(tg\gamma + tg\alpha) :$	1.2243475
$\log(tg\gamma + tg\beta) :$	1.2427205
$\log \cos^2\gamma_1$	$= \bar{3}\ 5503290$
$\log \frac{AB}{tg\beta - tg\alpha}$	$= 2.8432501$
$\log u_1$	$= 2.8606471$
$q\, tg\alpha\, tg\beta$	$= +1.60108$
$-tg\gamma_1$	$= -16.75186$
$\frac{x_1}{y_D}$	$= -15.15078$
$\log \frac{x_1}{y_D}$	$= 1.1804350\ n$
$\log y_D$	$= 0.5457989$
$\log x_1$	$= 1.7262339\ n$
$\log X_c$	$= 2.3473273$
$\log \frac{x_1}{X_c}$	$= \bar{1}.3789066\ n$
$\log u_1$	$= 2.8606471$
$\log X_1$	$= 2.2395537\ n$
X_1	$= +173.6016$

	$= +16.75186$
$tg\beta$	$= +0.7353495$
$tg\gamma + tg\beta =$	$+17.4872095$
$\log tg\gamma_1 :$	1.2240631
$\log \cos\gamma_1 =$	$\bar{2}.7751645$
$u_1 =$	$+725.5162$
$x_1 =$	-53.2395
$u_1 - x_1 =$	$+778.7557$
$\log(u_1 - x_1) :$	2.8914012
$\log tg\gamma_1 =$	1.2240631
$\log \frac{u_1 - x_1}{tg\gamma_1} :$	1.6673381
Y_1	$= +46.8770$
$\log \sin\omega_1 =$	$\bar{1}.9871971$
$\log u_1$	$: 2.8606471$
$\log Z_1$	$= 2.8478442$
Z_1	$= +704.4403$

$tg\gamma_2 =$		
	$+4.60644$	$+4.60644$
	$+0.0109766$	$+0.7353495$
	$+4.6174166$	$+5.5317895$
	0.6643991	0.6633654
	0.7276867	
	$\bar{2}.6532700$	$\bar{1}.3266350$
	2.8432501	
	2.8886059	$+773.7593$
		-10.5608
	$+1.60108$	$+784.3201$
	-4.60644	2.8944934
	-3.00536	0.6633654
		2.2311280
	$0.4778965\ n$	
	0.5457989	$Y_2 = +170.266$
	$1.0236954\ n$	
	2.3473273	
	$\bar{2}.6763681\ n$	$\bar{1}.9995102$
	2.8886059	2.8886059
	$1.5649740\ n$	2.8881161
	$X_2 = +36.7260$	$Z_2 = +772.8871$

$tg\gamma_3 =$		
	$+0.58983$	$+0.58983$
	$+0.0109766$	$+0.7353495$
	$+0.6008066$	$+1.3251795$
	$\bar{1}.7787345$	$\bar{1}.7707269$
	0.1222747	
	$\bar{1}.8703426$	$\bar{1}.9351713$
	2.8432501	
	2.6146021	$+411.7201$
		$+3.5535$
	$+1.60108$	$+408.1666$
	-0.58983	2.6108375
	$+1.01125$	$\bar{1}.7707269$
		2.8401106
	0.0048586	
	0.5457989	$Y_3 = +692.007$
	0.5506575	
	2.3473273	
	$\bar{2}.2033302$	$\bar{1}.9999446$
	2.6146021	2.6146021
	0.8179323	2.6145467
	$X_3 = -6.5756$	$Z_3 = +411.6676$

$tg\gamma_4 =$		
	-47.00121	-47.00121
	$+0.0109766$	$+0.7353495$
	-46.9902334	-46.2658605
	$1.6720076\ n$	$1.6721090\ n$
	$1.6652607\ n$	
	$\bar{4}.6555852$	$\bar{2}.3277926$
	2.8432501	
	2.8361036	$+685.6517$
		$+170.7873$
		$+514.8644$
	$+1.60108$	
	$+47.00121$	2.7116929
	$+48.60229$	$1.6721090\ n$
		$1.0395839\ n$
	1.6866567	
	0.5457989	$Y_4 = -10.9543$
	2.2324556	
	2.3473273	
	$\bar{1}.8851283$	$\bar{1}.8068197$
	2.8361036	2.8361036
	2.7212319	2.6429233
	$X_4 = -526.2982$	$Z_4 = +439.4640$

$\log AB =$	2.7032123
$\log(tg\beta - tg\alpha) =$	$\bar{1}.8599622$
$\log \frac{AB}{tg\beta - tg\alpha} :$	2.8432501
$\log q$	$: 2.2974512$
$\log tg\alpha\, tg\beta$	$: \bar{3}.9069615$
$\log q\, tg\alpha\, tg\beta$	$: 0.2044127$
$q\, tg\alpha\, tg\beta$	$: 1.60108$
$\log diff.$	$: 0.4057611$
$\log(tg\beta - tg\alpha)$	$= \bar{1}.8599622$
$\log y_D$	$= 0.5457989$

VI - Changement de Coordonnées. - Transformation of Coordinates.

$\log \cos G_{DC} =$	$\bar{1}.9577578$
$\log \sin \varepsilon =$	$\bar{2}.3914533$
$\log coeff. Z :$	$\bar{2}.3492111\ n$

$\log \sin G_{DC} =$	$\bar{1}.6237175$
	$\bar{2}.3914533$
	$\bar{2}.0151708$

$\log \cos \varepsilon :$	$\bar{1}.9998683$
	$\bar{1}.9998683$

$\log x_B - x_A$	$= 2.3269295$
$\log AB$	$: 2.7032123$
$\log coeff. Y$	$: \bar{1}.6237172$
$\log Y_A$	$: 1.0479286$
	0.6715458
$\frac{x_B - x_A}{AB} Y_A$	$: +4.695$
x_A	$: +1184.61$
$x_A - [\ ]$	$: +1179.915$
x_c	$: +978.10$
$x_c - x_A + \frac{x_B - x_A}{AB} Y_A$	$: -201.815$
$\log$	$: 2.3049535\ n$
$\log X_c$	$: 2.3473273\ n$
$\log coeff. X$	$: \bar{1}.9576262$

$\log y_B - y_A$	$= 2.6609698$
	$: 2.7032123$
	$: \bar{1}.9577575$
	$: 1.0479286$
	1.0056861
$\frac{y_B - y_A}{AB} Y_A$	$: +10.132$
y_A	$: -491.64$
$y_A - [\ ]$	$: -501.772$
y_c	$: -408.25$
	$+93.522$
$\log$	$: 1.9709138$
	$: 2.3473273\ n$
	$: 1.6235865\ n$

$\log z_B - z_A$	$=$
	$=$
$\frac{z_B - z_A}{AB} Y_A$	$= 0$
z_A	$= +5.48$
$z_A - [\ ]$	$= +5.48$
z_c	$= 0$
	-5.48
$\log$	$: 0.7387806\ n$
	$: 2.3473273\ n$
	$: 2.3914533$

$\log X_1 = 2.2395537$	$\log X_2 = 1.5649740$	$\log X_3 = 0.8179323\ n$	$\log X_4 = 2.7212319\ n$
$\log coeff\ X = \bar{1}.9576262$	$\bar{1}.9576262$	$\bar{1}.9576262$	$\bar{1}.9576262$
2.1971799	1.5226002	$0.7755585\ n$	$2.6788581\ n$
$\log Y_1 : 1.6673381$	$\log Y_2 = 2.2311280$	$\log Y_3 = 2.8401106$	$\log Y_4 : 1.0395839\ n$
$\log coeff\ Y : \bar{1}.6237172$	$\bar{1}.6237172$	$\bar{1}.6237172$	$\bar{1}.6237172$
1.2910553	1.8548452	2.4638278	$0.6633011\ n$
$\log Z_1 = 2.8478442$	$\log Z_2 = 2.8881161$	$\log Z_3 = 2.6145467$	$\log Z_4 = 2.6429283$
$\log coeff\ Z : \bar{2}.3492111\ n$	$\bar{2}.3492111\ n$	$\bar{2}.3492111\ n$	$\bar{2}.3492111\ n$
$1.1970553\ n$	$1.2373272\ n$	$0.9637578\ n$	$0.9921344\ n$

$x_A - \frac{x_B - x_A}{AB} Y_A$:	$+1179.95$	$+1179.95$	$+1179.95$	$+1179.95$
term. X :	$+157.463$	$+33.312$	-5.964	-477.373
term. Y :	$+49.546$	$+71.589$	$+290.956$	-4.606
term. Z :	-15.742	-17.271	-9.799	-9.821
	$x_{S_1} = +1371.182$	$x_{S_2} = +1267.545$	$x_{S_3} = +1455.708$	$x_{S_4} = +688.115$

$\log X_1 = 2.2395537$	$\log X_2 = 1.5649740$	$\log X_3 = 0.8179323\ n$	$\log X_4 = 2.7212319\ n$
$\log coeff\ X = \bar{1}.6235865\ n$	$\bar{1}.6235865\ n$	$\bar{1}.6235865\ n$	$\bar{1}.6235865\ n$
$1.8631402\ n$	$1.1885605\ n$	0.4415188	2.3448184
$\log Y_1 : 1.6673381$	$\log Y_2 : 2.2311280$	$\log Y_3 : 2.8401106$	$\log Y_4 : 1.0395839\ n$
$\log coeff\ Y : \bar{1}.9577575$	$\bar{1}.9577575$	$\bar{1}.9577575$	$\bar{1}.9577575$
1.6250956	2.1888855	2.7978681	$0.9973414\ n$
$\log Z_1 = 2.8478442$	$\log Z_2 = 2.8881161$	$\log Z_3 : 2.6145467$	$\log Z_4 = 2.6429333$
$\log coeff.\ Z = \bar{2}.0151708$	$\bar{2}.0151708$	$\bar{2}.0151708$	$\bar{2}.0151708$
0.8630150	0.9032869	0.6297175	0.6580941

$y_A - \frac{y_B - y_A}{AB} Y_A$:	-501.772	-501.772	-501.772	-501.772
term. X :	-72.970	-15.437	$+2.764$	$+221.217$
term. Y :	$+42.179$	$+154.485$	$+627.868$	-9.939
term. Z :	$+7.295$	$+8.004$	$+4.263$	$+4.551$
	$y_{S_1} = -525.268$	$y_{S_2} = -354.720$	$y_{S_3} = +133.123$	$y_{S_4} = -285.943$

$\log X_1 : 2.2395537$	$\log X_2 : 1.5649740$	$\log X_3 : 0.8179323\ n$	$\log X_4 : 2.7212319\ n$
$\log coeff\ X : \bar{2}.3914533$	$\bar{2}.3914533$	$\bar{2}.3914533$	$\bar{2}.3914533$
0.6310070	$\bar{1}.9564273$	$\bar{1}.2093856\ n$	$1.1126852\ n$
$\log Z_1 : 2.8478442$	$\log Z_2 : 2.8881161$	$\log Z_3 : 2.6145467$	$\log Z_4 : 2.6429233$
$\cos \varepsilon : \bar{1}.9998683$	$\bar{1}.9998683$	$\bar{1}.9998683$	$\bar{1}.9998683$
2.8477125	2.8879844	2.6144150	2.6427916

z_A =	$+5.48$	$+5.48$	$+5.48$	$+5.48$
term. X =	$+4.276$	$+0.905$	-0.162	-12.962
term. Z =	$+704.227$	$+772.653$	$+411.543$	$+439.331$
	$z_{S_1} = +713.983$	$z_{S_2} = +779.038$	$z_{S_3} = +416.861$	$z_{S_4} = +431.849$

Calcul par approximations successives des valeurs de x pour la pyramide passant par les points A, B et C'.
Calculation by successive approximation of values of x for the pyramid based on the points A, B and C'.

(On a pris comme position approchée de S la position S_2 du calcul précédent.)
(Position S_2 found by the preceeding calculation is taken as approximate position for S)

$x_{c'} = +1161.075$ $\quad x_{c'} = +1^{mm}82$
$y_{c'} = -73.609$ $\quad y_{c'} = -63^{mm}70$
$z_{c'} = 0$

Coordonnées approchées de S. – Approximate coordinates of S.

+170 266
−381 398

$X'_S = X_2 = +36.726$ $\quad Y'_S = Y_2 - \overline{cc'} = -211.132$ $\quad Z'_S = Z_2 = +772.8871$

$x_c = +978.10$
$-x_{c'} = -1161.075$
$x_c - x_{c'} = -182.975$

$y_c = -408.25$
$-y_{c'} = +73.609$
$y_c - y_{c'} = -334.641$

$\log x_c - x_{c'} = 2.2623918\ n$ $\quad 2.2623918\ n$
$\log y_c - y_{c'} = 2.5245792\ n$ $\quad 2.5245792\ n$
$\log tg\ G_{\overline{cc'}} = \bar{1}.7378126$ $\quad lg\ sin\ G_{\overline{cc'}} = \bar{1}.6810133\ n$ $\quad \log cos: \bar{1}.9432007\ n$
$c'c = +381.398$ $\quad \log c'c = 2.5813785$ $\quad 2.5813785$

I – Calcul du triangle de base ABC'. – Calculation of the base triangle ABC'.

$x_{c'} = +1161.08$
$x_A = +1184.61$
$x_{c'} - x_A = -23.53$
$\log x_{c'} - x_A = 1.3716219\ n$
$\log sin\ G_{AC'} = \bar{1}.7497275\ n$
$\log A'C' = 2.6218944$
$\log sin = \bar{1}.6728149$
$\log 0'C' = 2.2947093$
$= 2.2947105$
$\log 00' = 0.7387806$
$\log tg\ \widehat{O'CO} = \bar{2}.4440701$

$y_{c'} = -73.61$
$y_A = -491.64$
$y_{c'} - y_A = +418.03$
$\log y_{c'} - y_A = 2.6212076$
$\log cos\ G_{AC'} = \bar{1}.9993131$
$= 2.6218944$
$\log cos = \bar{1}.9455928\ n$
$\log A0' = 2.5674872\ n$
$0A = -369^{m}.392$
y_A
$\widehat{O'CO} = \varepsilon = 1°35'33''$

$\log x_{c'} - x_A = 1.3716219\ n$
$\log y_{c'} - y_A = 2.6212075$
$\log tg\ G_{AC'} = \bar{2}.7504144\ n$
$G_{AC'} = 356°46'42''$
$G_{BA'} = 204\ 51\ 47$
diff $= 28\ 05\ 05$

$x_{c'} = +1161.08$
$x_B = +1396.90$
$x_{c'} - x_B = -235.82$
$\log x_{c'} - x_B = 2.3725806\ n$
$\log sin\ G_{BC'} = \bar{1}.9938163$
$\log B'C' = 2.3787643$
$\log sin = \bar{1}.9159474$
$\log 0'C' = 2.2947117$
$\log cos\ \widehat{OCO} = \bar{1}.9998322$
$\log OC' = 2.2948795$
$X_c \quad OC' = -197^{m}188$

$y_{c'} = -73.61$
$y_B = -33.53$
$y_{c'} - y_B = -40.08$
$\log y_{c'} - y_B = 1.6029277\ n$
$\log cos\ G_{BC'} = \bar{1}.2241634$
2.3787643
$\log cos: \bar{1}.7532566$
$\log B0' = 2.1320209$
$OB = +135^{m}525$
y_B

$\log x_{c'} - x_B = 2.3725806\ n$
$\log y_{c'} - y_B = 1.6029277\ n$
$\log tg\ G_{BC'} = 0.7696529$
$G_{BC'} = 260°21'15''$
$G_{BA'} = 204\ 51\ 47$
diff $= 55\ 29\ 28$

$AB = 504^{m}91$

III – Calcul des éléments de la pyramide Sabc'. – Calculation of the elements of the pyramid Sabc'.

$\log$ diff $= 2.3990504$
$\log cos = \bar{1}.9936316$
$\log Sc_1 = 2.4054188$

2.3990504
$\log p'c'_1 = 1.6358811$
$\log tg\ \widehat{p'Sc_1} = \bar{1}.2368307$

$p'S = 254.48$
$c'_1c' \cos \widehat{pp'S} = 3.84$
diff $= 250.64$

$\widehat{p'Sb} = +19°03'21''$
$\widehat{p'Sc_1} = +9\ 47\ 17$
$\beta = +9\ 16\ 04$

$\widehat{p'Sa} = -16°38'39''$
$+9\ 47\ 17$
$\alpha = -26\ 25\ 56$

$\beta - \alpha = 35°42'00''$

$\log x_{c'} = 0.2600714$
$\log sin\ G_{pc'} = \bar{2}.4557549$
$\log pc' = 1.8043165$
$\log sin = \bar{1}.8315646$
$\log p'c'_1 = 1.6358811$
$p'c'_1 = 43.24$

$\log y_{c'} = 1.8041394\ n$
$\log cos\ G_{pc'} = \bar{1}.9998228\ n$
1.8043166
$\log cos = \bar{1}.8660386$
1.6703552
46.81
$pp' = 15.66$
$c'_1c' = 62.47$

$\log x'_{c} = 0.2600714$
$\log y_{c'} = 1.8041394\ n$
$\log tg\ G_{pc'} = \bar{2}.4559320\ n$
$G_{pc'} = 178°21'48''$
$G_{pp'} = 315\ 38\ 06$
$\widehat{pcc'} = 42\ 43\ 42$
$\log c'_1c' = 1.7956715$
$\log cos\ \widehat{pp'S} = \bar{2}.7890779$
$\log c'_1c' \cos \widehat{pp'S} = 0.5847494$

$\log c'_1c' = 1.7956715$
$\log sin\ \widehat{pp'S} = \bar{1}.9997764$
$\log c'_1c' sin\ \widehat{pp'S} = 1.7948479$
$\log Sc_1 = 2.4054188$
$\log tg\ \theta = \bar{1}.3894291$
$\theta = 13°46'28''$

Calcul de : (Calculation of) OM . ON . x_D . y_D . $X_c \cot g\ \theta$.

$\log OA = 2.5674872\ n$
$\log OB = 2.1320209$
$\log OM \cdot ON = 4.6995081\ n$
$OM \cdot ON = -50061.851$

$\log AB = 2.7032217$
$\log tg(\beta - \alpha) = \bar{1}.8564708$
$\log OM + ON = 2.8467509$
$OM + ON = 702.669$

$\log AB = 2.7032217$
$\log sin\ \alpha = \bar{1}.6484955\ n$
$\log sin\ \beta = \bar{1}.2069575$
$1.5586747\ n$
$\log sin(\beta - \alpha) = \bar{1}.7660715$
$\log x_D = 1.7926032\ n$
$x_D = -62.0302$
$x_D^2 = +3847.745$

$\log x_D = 1.7926032\ n$
$\log tg\ \beta = \bar{1}.2126639$
$2.5799393\ n$
$x_D \cot g\ \beta = -380.136$
$OA = -369.392$
$y_D = +10.844$
$y_D^2 = +117.5923$

$\log Z_S = 2.8881161$
$\log X_S = 1.5649736$
$\log tg\ \omega = 1.3231425$
$\log X_c = 2.2948795\ n$
$\log tg\ \theta = \bar{1}.3894291$
$\log = 2.9054504\ n$
$X_c \cot g\ \theta = -804.360$

$\log X_c = 2.2948795\ n$
$\log cos\ \omega = \bar{2}.6763717$
$\log x = 0.9712512\ n$
$x = -9.3595$

Calcul de x par approximations successives.
Calculation of x by successive approximations.

$$X_c \cot g\ \theta = \frac{x^2 - (OM + ON)x + OM \cdot ON}{\sin \omega \sqrt{(x - x_D)^2 + y_D^2}} \qquad x = X_c \cos \omega$$

$x = -6.9$

$\log x = 0.8388491\ n$
$\log X_c = 2.2948795\ n$
$\log cos\ \omega = \bar{2}.5439696$
$\log sin\ \omega = \bar{1}.9997340$
$\log \sqrt{\ } = 1.7496312$
$\log$ den. $= 1.7493652$

$x = -6.9$
$x_D = -62.0302$
$x - x_D = +55.1302$
$(x - x_D)^2 = 3039.3196$
$y_D^2 = 117.5923$
$+3156.9119$
$\log = 3.4992624$
$\log \sqrt{\ } = 1.7496312$

$x^2 = +47.61$
$-(OM+ON)x = 4848.42$
$OM \cdot ON = -50061.85$
num. $= -45165.82$
$\log$ num. $= 4.6548099\ n$
$\log$ den. $= 1.7493652$
$2.9054447\ n$
$X_c \cot g\ \theta = -804.3492$

$x = -9$

$\log x = 0.9542425\ n$
$\log X_c = 2.2948795$
$\log cos\ \omega = \bar{2}.6593630$
$\log sin\ \omega = \bar{1}.9995472$
$\log \sqrt{\ } = 1.7334185$
$\log$ den. $= 1.7329657$

$x = -9$
$x_D = -62.0302$
$x - x_D = +53.0302$
$(x - x_D)^2 = 2812.2013$
$y_D^2 = 117.5923$
$+2929.7936$
$\log = 3.4668370$
$\log \sqrt{\ } = 1.7334185$

$x^2 = +81$
$-(OM+ON)x = +6324.021$
$OM \cdot ON = -50061.85$
num. $= -43656.83$
$\log$ num. $= 4.6400522\ n$
$\log$ den. $= 1.7329657$
$2.9070865\ n$
$X_c \cot g\ \theta = -807.3959$

P.P.
$X_c \cot g\ \theta = -804.360$
$x = -6.907$

Calcul de : (Calculation of) $X_S \cdot Y_S \cdot Z_S$.

$$u = x - (x - x_D)\frac{x^2 - (OM + ON)x + OM \cdot ON}{(x - x_D)^2 + y_D^2} \qquad v = \frac{x - u}{x - x_D} y_D \qquad X_S = -u \cos \omega \qquad Y_S = v \qquad Z_S = u \sin \omega$$

$x = -6.907$
$\log x = 0.8392895\ n$
$\log X_c = 2.2948795\ n$
$\log cos\ \omega = \bar{2}.5444100$
$\log sin\ \omega = \bar{1}.9997333$

$x = -6.907$
$x_D = -63.0302$
$x - x_D = 55.1232$
$(x - x_D)^2 = 3038.5671$
$y_D^2 = 117.5923$
den. $= 3156.1594$

$x^2 = +47.707$
$-(OM+ON)x = 4853.337$
$OM \cdot ON = -50061.851$
-45160.807
$\log = 4.6547617\ n$
$\log (x - x_D) = 1.7413344$

$\log$ num. $= 6.3960961\ n$
$\log$ den. $= 3.4991589$
$2.8969372\ n$
-788.7460
$x = -6.907$
$u = +781.839$

$x - u = -788.746$
$\log (x - u) = 2.8969372\ n$
$\log (x - x_D) = 1.7413344$
$1.1556028\ n$
$\log y_D = 1.0351895$
$\log v = 2.1907923\ n$

$\log u = 2.8931173$
$\log cos\ \omega = \bar{2}.5444100$
$\log X_S = 1.4375273$
$X_S = -27.3859$ $\quad Y_S = -155.164$

$\log u = 2.8931173$
$\log sin\ \omega = \bar{1}.9997333$
$\log Z_S = 2.8928506$
$Z_S = +781.359$

Transformation de Coordonnées – Transformation of Coordinates.

$\log X_S = 1.4375273$
$\log X_c = 2.2948795\ n$
$\log \frac{X}{X_c} = \bar{1}.1426478$

$Y_S = -155.164$
$Y_A = -369.392$
$Y - Y_A = +214.228$
$\log = 2.3308762$

$Z = +781.359$
$\log Z = 2.8928506$

$\log cos\ G_{DC} = \bar{1}.9577578$
$\log sin\ \varepsilon = \bar{2}.4439009$
$\log Z = 2.8928506$
1.2945093
19.702

$\log sin\ G_{DC} = \bar{1}.6237175$
$= \bar{2}.4439009$
$= 2.8928506$
0.9604690
9.130

$\log cos\ \varepsilon = \bar{1}.9998323$
$= 2.8928506$
2.8926829
781.057

$\log \frac{x_B - x_A}{AB} = \bar{1}.6237172$
$\log Y_A = 2.5674872\ n$
$2.1912044\ n$
-155.312
$x'_c - x_A = -23.535$
-178.847
$\log = 2.2524816\ n$
$\log \frac{X}{X_c} = \bar{1}.1426478$
$1.3951294\ n$

$= \bar{1}.6237172$
$\log (Y - Y_A) = 2.3308762$
1.9545934
$+90.073$

$\log \frac{y_B - y_A}{AB} = \bar{1}.9577575$
$\log Y_A = 2.5674872\ n$
$2.5252447\ n$
-335.154
$y'_c - y_A = +418.031$
$+82.877$
$\log = 1.9184340$
$\log \frac{X}{X_c} = 1.1426478$
1.0610818

$= \bar{1}.9577575$
$\log (Y - Y_A) = 2.3308762$
2.2886337
$+194.372$

$\log Z_A = 0.7387806$
$\log \frac{X}{X_c} = 1.1426478$
$\bar{1}.8814284$

	x	y	z
	$x_A = +1184.61$	$y_A = -491.64$	$z_A = +5.48$
2e terme =	$+24.839$	-11.510	$+0.761$
3e terme =	$+90.073$	$+194.372$	0
4e terme =	-19.702	$+9.130$	$+781.057$
	$x'_{S_2} = +1279.32$	$y'_{S_2} = -299.65$	$z'_{S_2} = +787.30$
	$x_{S_2} = +1267.55$	$y_{S_2} = -354.72$	$z_{S_2} = +779.04$

résultats obtenus par l'"Air Survey Committee";
results as obtained by "Air Survey Committee";
$x_{AS} = +1281.68$ $\quad y_{AS} = -337$ $\quad z_{AS} = +776.77$