THE POSITION AT SEA BY RADIOGONIOMETRIC BEARINGS

by

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In *The Hydrographic Review*, Vol. X, No. 2, November 1933, we showed some ways of utilizing wireless bearings taken from a ship ; we propose to give now a compendium of the formulae which enable the calculations to be made the most quickly, whether the ship takes a bearing of a transmitting station or is given the bearing from a shore station. We shall finally give a method, which we believe to be partly new, that enables us, by means of very short tables or of the diagrams reprinted here, to avoid computation altogether.

I. — COMPUTATION OF THE POSITION.

(a) *Bearing taken by the ship.* — Let M be the ship and Q the transmitting station (Fig. 1). The formulae which we gave on page 94 of the above-quoted Review bring out the various properties of the curve of equal azimuth (or azimuth curve) ; they also show the analogy between the straight line of equal azimuth (or azimuth line) and the position line in altitude, and enable us to determine the estimated position, which is the foot of the normal dropped from the D.R. position on to the azimuth curve, and the tangent MT to the curve at this point. In practice this method only requires the working of the following formulae, which are applicable in every case and always give the best solution.

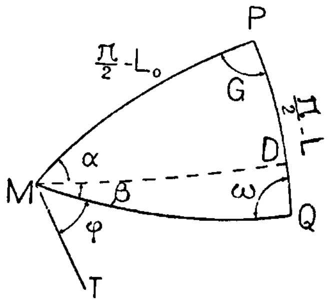

Fig. 1.

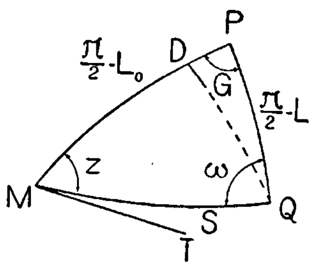

Fig. 2.

$$\cot \alpha = \tan \varphi = \sin L_0 \tan G.$$

$$\tan PD = \cot L_0 \cos G.$$

$$\cot \beta = \cos L_0 \sin G \tan (L+PD). \qquad dZ = Z - (\alpha+\beta).$$

$$\delta = \frac{\cot L_0 \cos \alpha}{\sin Z} dZ, \text{ in the direction } \beta.$$

Bearing sent from a shore station. — When the station Q takes a bearing ω (*) of the ship M, the ship is on the great circle MQ whose azimuth at Q is ω (Fig. 2). The distance δ of the D.R. position from this great circle is equal to $\sin S\, d\omega$, where S is the arc of the great circle MQ.

The quickest method of calculation is with the following formulae.

$$\tan PD = \cot L \cos G$$

$$\tan Z = \frac{\sin PD \tan G}{\cos (L_0 + PD)}$$

$$\sin \omega_e = \frac{\sin Z \cos L_0}{\cos L} \qquad d\omega = \omega - \omega_e$$

$$\delta = \frac{\sin G \cos L}{\sin Z} d\,\omega, \text{ in the direction } Z + 90^\circ.$$

(b) *Bearing taken on board. — When the angle* $Z + \varphi$ *is less than* 45° *or more than* 135°, we can try to find for what value of G the curve of equal azimuth Z meets the D.R. parallel L_0.

The calculation is done by decomposing the spherical triangle into two rectangular triangles by an arc PR perpendicular to MQ. Let p be the angle MPR (Fig. 3).

We first calculate φ with the estimated values L_0 and G so as to see whether the case is favourable for the use of this method. This value of φ will be sufficient for drawing the tangent, only a short length of which need be used. The following are the formulae :

$$\tan \varphi = \sin L_0 \tan G$$

$$\cot p = \sin L_0 \tan Z$$

$$\cos (G - p) = \tan L \cot L_0 \cos p.$$

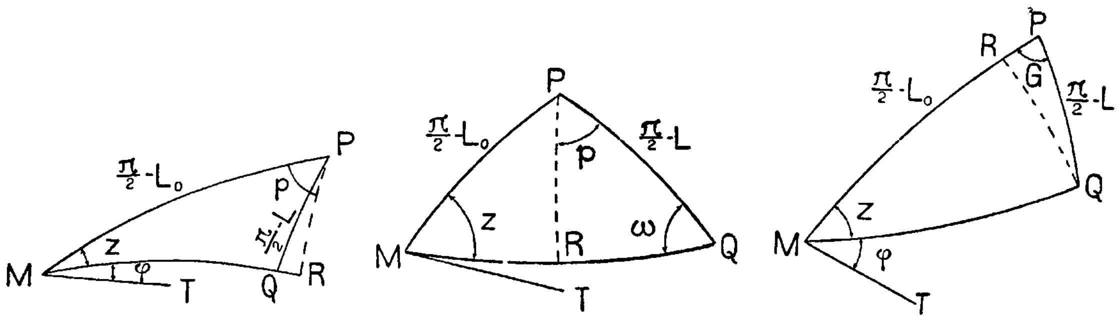

Fig. 3. Fig. 4. Fig. 5.

Bearing sent from a shore station. — When the angle Z *is less than* 45° *or more than* 135°. The calculation is almost identical with the above (Fig. 4). We determine G for the estimated value of L_0 :

(*) It is convenient here to reckon the azimuth from 0° to 180° from north towards east or west, and always to consider G as positive.

$$\cot p = \sin L \tan \omega$$
$$\cos (G-p) = \tan L_0 \cot L \cos p$$
$$\tan Z = \frac{\cot (G-p)}{\sin L_0}$$

(c) *Bearing taken on board. — When the angle* Z + φ *is between* 45° *and* 135°, we shall try to see for what value of L_0 the curve of azimuth Z meets the D.R. meridian G (Fig. 5).

The spherical triangle will be decomposed into two rectangular triangles by means of an arc QR perpendicular to MP. We calculate φ as in (b) to find out whether this method must be used. The following are the formulae.

$$\tan \varphi = \sin L_0 \tan G$$
$$\tan PR = \cot L \cos G$$
$$\cos (L_0 + PR) = \tan G \cot Z \sin PR.$$

Bearing sent by a shore station. — When the angle Z *is between* 45° *and* 135°, a process will be employed almost identical with process (b), but determining L_0 for the estimated value of G (Fig. 4).

$$\cot p = \sin L \tan \omega$$
$$\tan L_0 = \tan L \frac{\cos (G-p)}{\cos p}$$
$$\cot Z = \sin L_0 \tan (G-p).$$

As the tangent MT need only be used for a small part of its length, Z need not be determined very exactly. Hence Z can be calculated with the estimated values of L_0, and G, and consequently immediately after the calculation of p. According to the value found for Z, we shall know if cos $(G-p)$ must be calculated as in (b) or tan L_0 as here in (c).

Note. — If the measurement of the azimuths Z or ω were within an accuracy of at most 1/4°, it might be necessary, before applying methods (a), (b) or (c), to add to the observed value the quantity ε which we have shown how to calculate on pp. 88 to 93 of the above-quoted Review. This quantity allows for the flattening of the earth; its expression at the distances appropriate to direction finding is:

$$\varepsilon = -\frac{e^2}{2} \cos^2 L_0 \sin 2Z + s \frac{e^2}{4} \sin 2L_0 \sin Z,$$

when the ship, in latitude L_0, takes a bearing Z of a station at a distance s; or

$$\varepsilon = -\frac{e^2}{2} \cos^2 L \sin 2\omega + s \frac{e^2}{4} \sin 2L \sin \omega,$$

when a station, in latitude L, takes a bearing ω of a ship at a distance s.

The following table gives in minutes of arc the value of the first term of ε which is independent of the distance. The second term is proportional to the distance; it is always less than 2' if the distance does not attain 1,200 miles; it has the same sign as the latitude.

Z or ω / L_0 or L						Sign of the 1st term
	0	10	20	30	40	—
	90	80	70	60	50	—
	90	100	110	120	130	+
	180	170	160	150	140	+
+ or —						
0	0	3',9	7',4	10',0	11',4	
10	0	3,8	7,2	9,7	11,0	
20	0	3,5	6,5	8,8	10,0	
30	0	3,0	5,6	7,5	8,5	
40	0	2,3	4,3	5,8	6,6	
50	0	1,6	3,1	4,1	4,7	
60	0	1,0	1,9	2,5	2,9	
70	0	0,5	0,9	1,2	1,3	
80	0	0,1	0,2	0,3	0,3	
90	0	0	0	0	0	

II. — USE OF MERCATOR'S PROJECTION.

In present-day practice, wireless bearings cannot give the ship's position with an accuracy comparable with that of astronomical observations at much more than 1,000 nautical miles. To obtain satisfactory precision at greater distances it would be necessary for the bearings to be more accurate than it is certain that the physical conditions of the wave transmission will ever permit of. Thus one may generally be content with a graphical construction, made on a MERCATOR's chart capable of holding at the same time the positions of the transmitting and the receiving stations. This construction may be based on the use of developments such as those on p. 97 of the above-quoted Review, which give the difference between the loxodromic and the observed orthodromic azimuths. In this case we shall confine ourselves to the first two terms of the development; for, if it became necessary to take it further, the calculation to be done would become longer than the one we have just shown in (I), besides giving a less accurate result. When, then, we give this third term, it will be only by way of reminder.

(a) We shall always find the conditions favourable, provided the development be permissible, if we express $A-Z$ by a development in powers of the estimated distance between the ship M and the station Q. The orthodromic distance could only be found by a rather long and unnecessary calculation; the loxodromic distance would not be as convenient in use as that which we are going to adopt. We shall measure on MERCATOR's chart the rectilinear length MQ in angular values of the equator, as we do for G, by taking it on the longitude scale of the chart. We shall call the measurement thus obtained d, which is merely a parameter for purposes of calculation but which can be obtained very easily and exactly and which enables us to say in all rigour that for a given value of d the locus of the point M on a MERCATOR's chart is a circle of centre Q and radius d. We have then :

$$G = d \sin A = d \sin (Z + A - Z) \;;$$

which enables us to deduce, from the development of $A - Z$ in powers of G, the following equation in which the angles are expressed in radians.

$$A - Z = \frac{d}{2} \sin Z \sin L + \frac{d^2}{12} \sin 2Z \,(1 - 3 \cos^2 L) - \frac{d^3}{48} \sin Z \sin L$$

$$[\sin^2 Z \cos^2 L + (11 \cos^2 L - \sin^2 L)\,(2 \cos^2 Z - \sin^2 Z)] \;;$$

or, introducing L_0 :

$$A - Z = \frac{d}{2} \sin Z \sin L_0 + \frac{d^2}{12} \sin 2Z.$$

Expressing the angles in sexagesimal degrees, we get approximately

$$(A - Z)^\circ = \frac{d^\circ}{2} \sin Z \sin L_0 + 0.145 \left(\frac{d^\circ}{10}\right)^2 \sin 2\,Z.$$

If the station Q is the transmitter, we take the intersection of the loxodrome through Q in the azimuth $\pi - A$ with the circle of centre Q and radius d; through this point we draw the straight line of azimuth in the direction $Z + \varphi$. (See Fig. 6).

$$\varphi = 2\,(A - Z) + \frac{d^2}{12} \sin 2\,Z \;\; (1 - 3 \cos^2 L).$$

It will, further, suffice if we take φ as being equal to $2\,(A - Z)$, for the straight line of azimuth need only be used for a short distance from the point M. If we have not a pair of compasses handy, for describing the circle, it may be replaced by a perpendicular to MQ drawn through the D.R. position M. The introduction, in the first term, of the D.R. latitude L_0 is only legitimate if this D.R. value differs sufficiently little from that which will be furnished by drawing QM, not to cause an error of an order greater than the value of the third term which we neglect.

If the ship M is the transmitter and receives from the station Q an indication of the bearing ω, we calculate :

$$(A - \omega)^\circ = \frac{d^\circ}{2} \sin \omega \sin L + 0.145 \left(\frac{d^\circ}{10}\right)^2 \sin 2\,\omega;$$

we take the intersection of the same circle with the loxodrome drawn through Q in the direction A. (See Fig. 7).

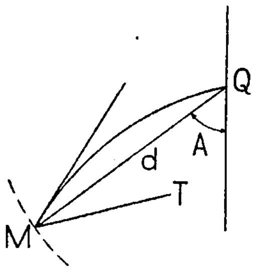

Fig. 6.

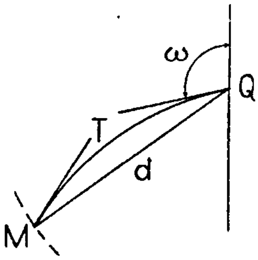

Fig. 7.

(b) *When* A *is less than* 45° *or more than* 135°, we can also take the intersection of the straight line QM with the D.R. parallel. To do this, it is necessary to obtain $A-Z$ by a development in powers of $L-L_0$, a quantity which we shall call λ. This development is as follows :

$$A-Z=\frac{\lambda}{2}\tan Z\tan L-\frac{\lambda^2}{12}\tan Z\,(4+\tan^2 L-3\tan^2 Z\tan^2 L)\ ;$$

or, introducing the latitude L_0 in the first term, which does not here call for the reservation which we had to make in case (a), and taking

$$P=\frac{\lambda}{2}\tan Z\tan\left(L-\frac{2}{3}\lambda\right),$$

we get

$$A-Z=P+2\frac{P^2}{\sin 2Z};$$

or, approximately, expressing the angles in sexagesimal degrees :

$$(A-Z)^{\circ}=P^{\circ}+\left(\frac{P^{\circ}}{10}\right)^2\frac{7}{2\sin 2Z}\cdot$$

We have here :

$$\varphi=2(A-Z)+\frac{\lambda^2}{6}\tan Z\,(\tan^2 L-2)\ ;$$

but it will be sufficient to take the value as $2\,(A-Z)$ as in (a).

If the ship M is the transmitter and receives from the station Q an indication of the bearing ω, we shall calculate in the same way, taking

$$P=\frac{\lambda}{2}\tan\omega\tan\left(L_0-\frac{2}{3}\lambda\right)$$

$$(A-\omega)^{\circ}=P^{\circ}+\left(\frac{P^{\circ}}{10}\right)^2\frac{7}{2\sin 2\omega}.$$

(c) *If* A *lies between* 45° *and* 135°, we shall be able to calculate $A-Z$ by a development in powers of the estimated value of G ; the addition of the observed value of Z will give A, the loxodromic azimuth.

If the station Q is the transmitter, we draw the straight line QM through the point Q on the chart in a direction $\pi - A$, as far as its intersection with the D.R. meridian; then through this point M we draw a straight line MT making with this meridian an angle $A + (A - Z)$, which is the straight line of azimuth we are seeking.

For the point of intersection M to be well determined, and for the second term of the development to remain small, A must lie between 45° and 135°. We can judge sufficiently well that this is so from the value of A deduced from the D.R. position, and even (for the development is only applicable to small values of $A - Z$) from the observed value Z.

In this case the formula, expressing the arcs in radians, is

$$A - Z = \frac{G}{2} \sin L - \frac{G^2}{12} \cot Z (1 + 3 \cos^2 L) + \frac{G^3}{24} \sin L \, (1 + \cot^2 Z + 4 \cos^2 L).$$

The first term is very quickly calculated with a simple table of natural values of sin L. The coefficient of the second term is a function of Z and of L. It can be simplified and diminished by a dodge reducing it to cot Z. To do this, we introduce into the first term the estimated latitude L_0; this gives rise to the reservation which we had to make in case (a).

The formula then takes a simpler form than in cases (a) and (b); it becomes

$$A - Z = \frac{G}{2} \sin \frac{L + L_0}{2} - \frac{G^2}{12} \cot Z \; ;$$

or approximately, expressing the angles in sexagesimal degrees:

$$(A - Z)^\circ = \frac{G^\circ}{2} \sin \frac{L + L_0}{2} - \left(\frac{G^\circ}{10}\right)^2 \frac{\cot Z}{7}$$

It is easy in this form to draw up tables rendering any calculation unnecessary.

In drawing the straight line of azimuth in the direction $A + (A - Z)$, we adopt, for the angle which we called φ above, the value $2(A - Z)$. In reality we have

$$\varphi = G \sin L_0 = 2(A - Z) + \frac{G^2}{6} \cot Z \; (1 - 3 \cos^2 L) \; ;$$

but, as the straight line of azimuth need only be employed for a short distance from the point M, the term in G^2 may be neglected.

If the ship M is the transmitter, and receives from the station Q the indication of the bearing ω, the calculation will be similar (see Fig. 7):

$$(A - \omega)^\circ = \frac{G^\circ}{2} \sin \frac{L + L_0}{2} - \left(\frac{G^\circ}{10}\right)^2 \frac{\cot \omega}{7}$$

Through the point Q we draw the straight line QM in the azimuth A as far as its intersection with the D.R. meridian; then through the point M we draw a straight line MT in the azimuth $\pi - A - (A - \omega)$.

This method also must only be applied if A lies between 45° and 135°.

III.

(a) The calculation of $A - Z$ by method *II* (c) requires fairly extensive tables; they can be very much reduced if, instead of calculating $A-Z$, we calculate the length $QR = \delta$, which fixes the position of the point R, the point of intersection of a loxodrome drawn through M in the azimuth Z with the meridian of Q (see Fig. 8).

In the triangle MRQ we have

$$\delta = d \ \frac{\sin (A-Z)}{\sin Z},$$

on condition that δ is measured like d along the longitude scale. Following the development of $A-Z$ we shall have:

$$\delta = \frac{d^2}{2} \sin L + \frac{d^3}{6} \cos Z \,(1 - 3 \cos^2 L) - \frac{d^4}{48} \sin L \,[\sin^2 Z + (11 \cos^2 L - \sin^2 L)\,(2 \cos^2 Z - \sin^2 Z)]$$

or, introducing L_0 and with the same reservations as in *II* (a),

$$\delta = \frac{d^2}{2} \sin L_0 + \frac{d^3}{6} \cos Z - \frac{d^4}{24} \sin L \ [1 + 2 \ \cos^2 Z \ (5 \cos^2 L - \sin^2 L)] \ ;$$

or, expressing the angles in sexagesimal degrees, we get approximately:

$$\delta^o = \frac{7}{8} \left(\frac{d^o}{10}\right)^2 \sin L_0 + 0.05 \left(\frac{d^o}{10}\right)^3 \cos Z.$$

This expression, which it is very easy to calculate with a table of natural trigonometrical ratios, has, like the development of $A-Z$ given in *II* (c), the advantage that the coefficient of each term depends on one quantity only. Below we give two tables enabling δ^o to be calculated by a simple addition.

In the same way δ could be expressed as a function of G or of λ, or we could calculate, as a function of G, d or λ, the distance of the point Q from the point of intersection of the straight line MR and the parallel of Q. But the coefficients of the terms would then be functions of L (or L_0) and of Z; we should lose the advantage of the simplicity of the former formula, which has the further advantage of suiting all values of the azimuth.

When the station Q is the transmitter, δ will have the sign of L_0; we plot it from Q. From the point R thus determined, we draw a loxodrome of azimuth $\pi - Z$ as far as its intersection with the circle of centre Q and radius d, a circle which may be replaced by a perpendicular to MQ drawn through the D.R. position. The straight line of azimuth MT will be symmetrical with MR with respect to MQ.

If the ship M is the transmitter and receives from the station Q the indication of the bearing ω, we calculate

$$\delta^o = \frac{\ }{8} \left(\frac{d^o}{10}\right)^2 \sin L + 0.05 \left(\frac{d^o}{10}\right)^3 \cos \omega$$

but we plot it on the meridian of Q in the direction opposite to its sign (see Fig. 9).

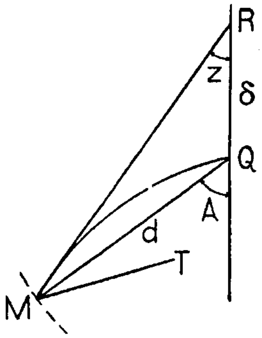

Fig. 8.

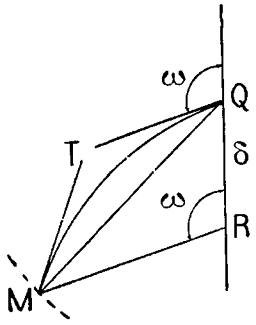

Fig. 9

Through R, we draw a loxodrome of azimuth ω as far as its intersection with the circle of centre Q and radius d (or with the perpendicular to MQ through the D.R. position). The locus of the point M is the great circle MQ, which we shall replace by its tangent MT which is symmetrical with MR with respect to MQ.

(b) *When* Z *is less than* 45° *or more than* 135°, we can also determine a point on the straight line MR in the following manner. Plot along the meridian, starting from Q, a length $QB = v$.

$$v = \delta \cos^2 Z.$$

Take C, the intersection of the parallel of B with the straight line QC in the azimuth Z; then on the meridian of C take the point D on the straight line QD in the azimuth $\frac{\pi}{2} - Z$. Through the point D draw the straight line DM of azimuth $\pi - Z$ as far as its intersection with the D.R. parallel of the ship (see Fig. 10).

The development of v as a function of λ is:

$$v \cos L = \frac{\lambda^2}{2} \tan (L - \frac{2}{3} \lambda) + \frac{\lambda^3}{2} \tan^2 L \tan^2 Z.$$

$v \cos L$ is none other than the arc QB measured along the *latitude* scale, taking B at the latitude $L + v \cos L$.

The first term is easy to make up into tables; the second is not quite so simple, but nevertheless enables a fairly rapid calculation to be made.

The quantity v is always smaller than the quantity δ.

With the angles in radians, the formula will be:

$$v^0 \cos L = \frac{7}{8} \left(\frac{\lambda^0}{10}\right)^2 \tan (L - \frac{2}{3} \lambda) + 0.15 \left(\frac{\lambda^0}{10}\right)^3 \tan^2 L \tan^2 Z.$$

If the ship M is the transmitter and receives from the station Q the indication of the bearing ω, we calculate, taking $\lambda = L_0 - L$:

$$v^0 \cos L_0 = \frac{7}{8} \left(\frac{\lambda^0}{10}\right)^2 \tan (L_0 - \frac{2}{3} \lambda) + 0.15 \left(\frac{\lambda^0}{10}\right)^3 \tan^2 L_0 \tan^2 \omega.$$

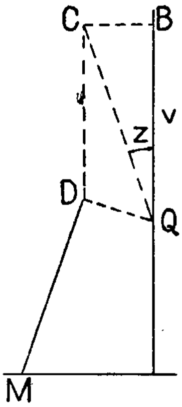

Fig. 10.

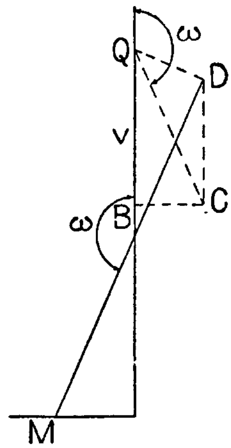

Fig. 11.

We take $QB = v$, measured on the longitude scale, and plotted in the opposite direction to its sign. We deduce the points C and D as before; from the point D we draw a straight line in the azimuth ω as far as its intersection at M with the D.R. parallel. (See Figure 11).

(c) *When* Z *lies between* 45° *and* 135°, an expression can be used which is as simple as that given in (a), by utilising the estimated difference of longitude G; this dispenses with the measurement of the length which we have called d, and has the further advantage of not using the D.R. latitude when the bearing is taken from the ship.

Let us draw through Q a straight line of azimuth $\frac{\pi}{2} - Z$; it meets the straight line RM at a point D; let B be the point on the meridian of Q which has the same latitude as D. Let us denote the quantity QB by u (see Fig. 12).

$$u = \delta \sin^2 Z = d \sin Z \sin (A - Z).$$

Let us replace $d \sin Z$ by its value as a function of G, and let us replace $A - Z$ by the development as a function of G; we get

$$u = \frac{G^2}{2} \sin L - \frac{G^3}{3} \cot Z.$$

u will be measured, like G and d, on the longitude scale.

Expressing the angles in sexagesimal degrees, we shall obtain approximately:

$$(u^o) = \frac{7}{8}\left(\frac{G^o}{10}\right)^2 \sin L - 0.1 \left(\frac{G^o}{10}\right)^3 \cot Z.$$

For the calculation of the first term we can use table I, replacing d by G and L_0 by L (latitude of the point whose bearing is taken). This term is smaller than in case (a); but the second term is greater in absolute value than in case (a); in this case table II *a* is applicable.

Having calculated u, we obtain the point B; and, on the straight line QD drawn in the azimuth $\frac{\pi}{2} - Z$, we take D at the same latitude as B. Through D we draw a straight line of azimuth $\pi - Z$ as far as its intersection with the D.R. meridian of M.

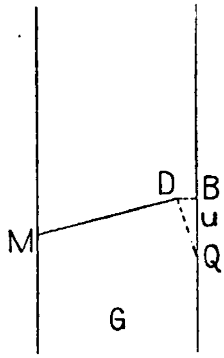

Fig. 12.

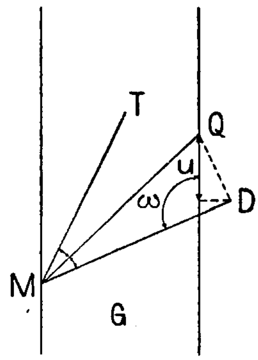

Fig. 13.

If the ship M is the transmitter and receives from the station Q the indication of the bearing ω, we calculate

$$(u)^o = \frac{7}{8}\left(\frac{G^o}{10}\right)^2 \sin L_0 - 0.1 \left(\frac{G^o}{10}\right)^3 \cot \omega \ ;$$

u will be plotted along the meridian of Q in the sense opposite to its sign, and we take the point of intersection at D of the parallel of B and the straight line QD of azimuth $\frac{\pi}{2} + \omega$ (see Fig. 13). Through the point D we draw a straight line of azimuth ω as far as its intersection at M with the D.R. meridian. ω must lie between 45^o and 135^o.

Note. — The quantities A and Z, differing from each other by a small number of degrees in the expressions given in *II* and *III*, are affected roughly in the same way by the flattening of the earth; it is thus unnecessary in this case to allow for the quantity ε of which we spoke further back.

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The following tables enable δ or u to be calculated by a simple addition.

Table I gives in degrees and fractions of a degree of *longitude* the value of the first term as a function of d, expressed in degrees of *longitude*, and of L_0 or L, the latitude of the point where the measurement of the azimuth is taken. This term has the same sign as the latitude.

Table II gives in degrees and fractions of a degree of *longitude* the value of the second term as a function of d and of the observed azimuth. This term has the same sign as $\cos Z$; it must be added algebraically to the first. Its sign is shown in the Z column

For every value of d which does not exceed 20°, the value of the terms neglected in the development of δ does not exceed 1'; for a value of d equal to 25°, it may be of the order of 5' at the most.

Table I also gives the first term of u, replacing d by G and taking for the latitude that of the point whose bearing is taken.

Table II*a* gives the value of the second term of u as a fun^ction of G and of the observed azimuth. It must be added algebraically to the first; its sign is shown in the Z column.

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TABLE I. $\frac{d^2}{2}\sin L_0$. (Sign of L_0, latitude of the observing station).

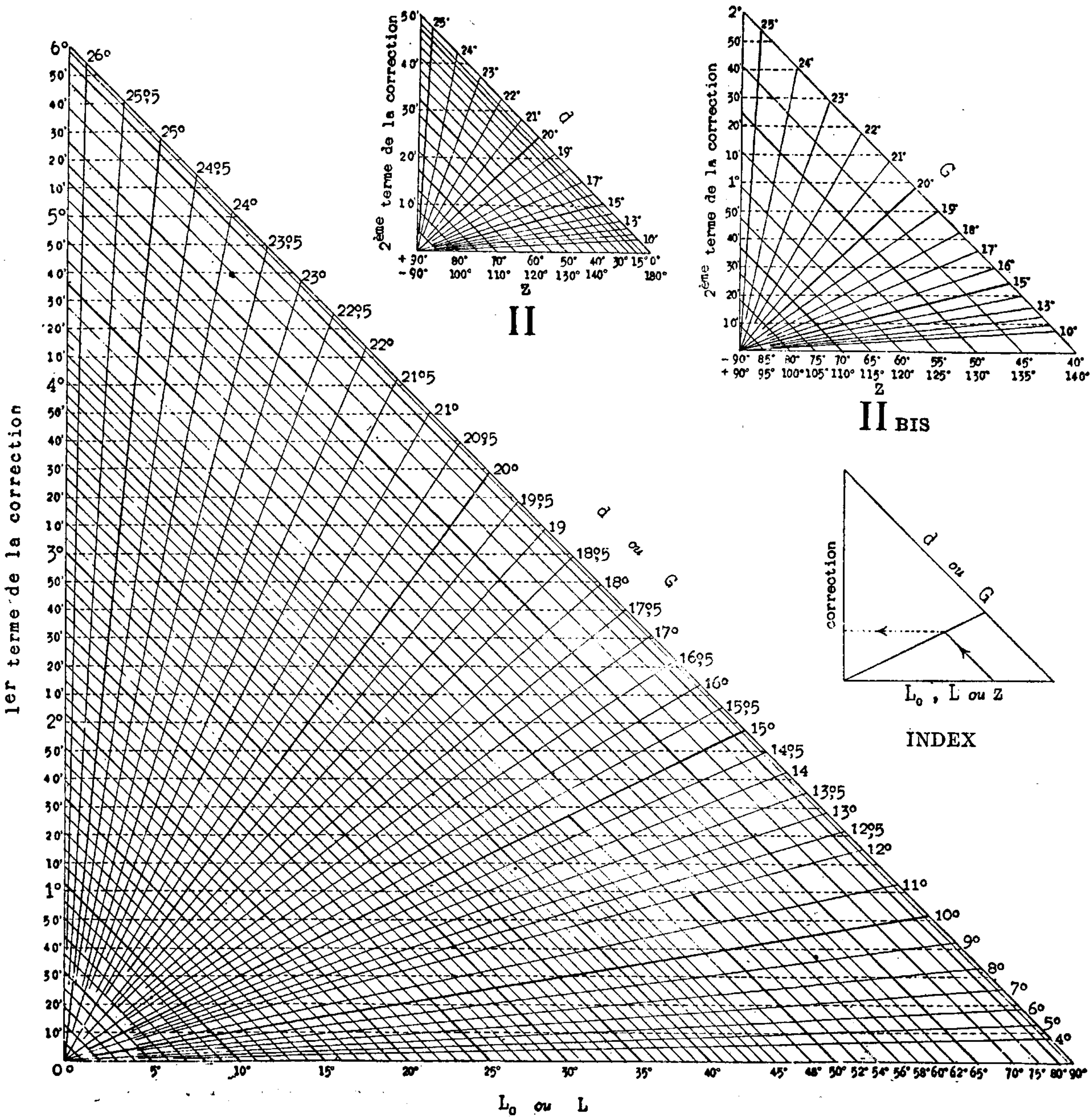
L^0 \ d	1°	2°	3°	4°	5°	6°	7°	8°	9°	10°	11°	12°	13°	14°	5°	16°	17°	18°	19°	20°	21°	22°	23°	24°	25°
0°	0°,00	0°,00	0°,00	0°,00	0°,00	0°,00	0°,00	0°,00	0°,00	0°,00	0°,00	0°,00	0°,00	0°,00	0°,00	0°,00	0°,00	0°,00	0°,00	0°,00	0°,00	0°,00	0°,00	0°,00	0°,00
5°	0 ,00	0 ,00	0 ,01	0 ,01	0 ,02	0 ,03	0 ,04	0 ,05	0 ,06	0 ,08	0 ,09	0 ,11	0 ,13	0 ,15	0 ,17	0 ,19	0 ,22	0 ,25	0 ,27	0 ,30	0 ,34	0 ,37	0 ,40	0 ,44	0 ,48
10	0 ,00	0 ,01	0 ,01	0 ,02	0 ,04	0 ,05	0 ,07	0 ,10	0 ,12	0 ,15	0 ,18	0 ,22	0 ,26	0 ,30	0 ,34	0 ,39	0 ,44	0 ,49	0 ,55	0 ,61	0 ,67	0 ,73	0 ,80	0 ,87	0 ,95
15	0 ,00	0 ,01	0 ,02	0 ,04	0 ,06	0 ,08	0 ,11	0 ,14	0 ,18	0 ,23	0 ,27	0 ,32	0 ,38	0 ,44	0 ,51	0 ,58	0 ,65	0 ,73	0 ,82	0 ,90	1 ,00	1 ,09	1 ,19	1 ,30	1 ,81
20	0 ,00	0 ,01	0 ,03	0 ,05	0 ,07	0 ,11	0 ,15	0 ,19	0 ,24	0 ,30	0 ,36	0 ,43	0 ,50	0 ,59	0 ,67	0 ,76	0 ,86	0 ,97	1 ,08	1 ,19	1 ,32	1 ,44	1 ,58	1 ,72	1 ,47
25	0 ,00	0 ,01	0 ,03	0 ,06	0 ,09	0 ,13	0 ,18	0 ,24	0 ,30	0 ,37	0 ,45	0 ,53	0 ,62	0 ,72	0 ,83	0 ,94	1 ,07	1 ,19	1 ,33	1 ,48	1 ,63	1 ,79	1 ,95	2 ,12	2 ,31
30	0 ,00	0 ,02	0 ,04	0 ,07	0 ,11	0 ,16	0 ,21	0 ,28	0 ,35	0 ,44	0 ,53	0 ,63	0 ,74	0 ,86	0 ,98	1 ,12	1 ,26	1 ,41	1 ,58	1 ,75	1 ,92	2 ,11	2 ,31	2 ,51	2 ,73
35	0 ,01	0 ,02	0 ,05	0 ,08	0 ,13	0 ,18	0 ,25	0 ,32	0 ,41	0 ,50	0 ,61	0 ,72	0 ,85	0 ,98	1 ,13	1 ,28	1 ,45	1 ,62	1 ,81	2 ,00	2 ,21	2 ,42	2 ,65	2 ,88	3 ,13
40	0 ,01	0 ,02	0 ,05	0 ,09	0 ,14	0 ,20	0 ,27	0 ,36	0 ,45	0 ,56	0 ,68	0 ,81	0 ,95	1 ,10	1 ,26	1 ,44	1 ,62	1 ,82	2 ,02	2 ,24	2 ,48	2 ,71	2 ,97	3 ,23	3 ,51
45	0 ,01	0 ,02	0 ,06	0 ,10	0 ,15	0 ,22	0 ,30	0 ,39	0 ,50	0 ,62	0 ,75	0 ,89	1 ,04	1 ,21	1 ,39	1 ,58	1 ,78	2 ,00	2 ,23	2 ,47	2 ,72	2 ,99	3 ,26	3 ,55	3 ,86
50	0 ,01	0 ,03	0 ,06	0 ,11	0 ,16	0 ,24	0 ,33	0 ,43	0 ,54	0 ,67	0 ,81	0 ,96	1 ,13	1 ,31	1 ,50	1 ,71	1 ,93	2 ,17	2 ,41	2 ,67	2 ,95	3 ,24	3 ,54	3 ,85	4 ,[illegible]8
55	0 ,01	0 ,03	0 ,06	0 ,11	0 ,18	0 ,26	0 ,35	0 ,46	0 ,58	0 ,71	0 ,86	1 ,03	1 ,21	1 ,40	1 ,60	1 ,83	2 ,07	2 ,32	2 ,58	2 ,86	3 ,15	3 ,46	3 ,78	4 ,12	4 ,47
60	0 ,01	0 ,03	0 ,07	0 ,12	0 ,19	0 ,27	0 ,37	0 ,48	0 ,61	0 ,76	0 ,91	1 ,09	1 ,28	1 ,48	1 ,70	1 ,93	2 ,18	2 ,45	2 ,73	3 ,02	3 ,33	3 ,66	4 ,00	4 ,35	4 ,72
65	0 ,01	0 ,03	0 ,07	0 ,13	0 ,20	0 ,28	0 ,39	0 ,51	0 ,64	0 ,79	0 ,96	1 ,14	1 ,34	1 ,55	1 ,78	2 ,02	2 ,29	2 ,56	2 ,86	3 ,16	3 ,49	3 ,83	4 ,18	4 ,56	4 ,90
70	0 ,01	0 ,03	0 ,07	0 ,13	0 ,21	0 ,30	0 ,40	0 ,52	0 ,66	0 ,82	0 ,99	1 ,18	1 ,39	1 ,61	1 ,85	2 ,10	2 ,37	2 ,66	2 ,96	3 ,28	3 ,62	3 ,97	4 ,34	4 ,72	5 ,13
75	0 ,01	0 ,03	0 ,08	0 ,13	0 ,21	0 ,30	0 ,41	0 ,54	0 ,68	0 ,84	1 ,02	1 ,21	1 ,42	1 ,65	1 ,90	2 ,16	2 ,44	2 ,73	3 ,04	3 ,37	3 ,72	4 ,08	4 ,46	4 ,86	5 ,27
80	0 ,01	0 ,03	0 ,08	0 ,14	0 ,21	0 ,31	0 ,42	0 ,55	0 ,70	0 ,86	1 ,04	1 ,24	1 ,45	1 ,68	1 ,93	2 ,20	2 ,48	2 ,78	3 ,10	3 ,44	3 ,79	4 ,16	4 ,55	4 ,95	5 ,37
85	0 ,01	0 ,03	0 ,08	0 ,14	0 ,22	0 ,31	0 ,43	0 ,56	0 ,70	0 ,87	1 ,05	1 ,25	1 ,47	1 ,70	1 ,96	2 ,23	2 ,51	2 ,82	3 ,14	3 ,48	3 ,83	4 ,21	4 ,60	5 ,01	5 ,43

TABLE II. $\frac{d^3}{6}\cos Z$. (Sign of $\cos Z$, azimuth Z reckoned from o to 180° from North towards East or West).

Z \ d	5°	7°	8°	9°	10°	11°	12°	13°	14°	15°	16°	17°	18°	19°	20°	21°	22°	23°	24°	25°	d / Z
+																					—
0°	0°,01	0°,02	0°,03	0°,04	0°,05	0°,07	0°,09	0°,11	0°,14	0°,17	0°,21	0°,25	0°,30	0°,35	0°,41	0°,47	0°,54	0°,62	0°,70	0°,79	180°
10	0 ,01	0 ,02	0 ,03	0 ,04	0 ,05	0 ,07	0 ,09	0 ,11	0 ,14	0 ,17	0 ,20	0 ,25	0 ,29	0 ,34	0 ,40	0 ,46	0 ,53	0 ,61	0 ,69	0 ,78	170
20	0 ,01	0 ,02	0 ,02	0 ,03	0 ,05	0 ,06	0 ,08	0 ,10	0 ,13	0 ,16	0 ,20	0 ,23	0 ,28	0 ,33	0 ,38	0 ,44	0 ,51	0 ,58	0 ,66	0 ,75	160
30	0 ,01	0 ,02	0 ,02	0 ,03	0 ,04	0 ,06	0 ,08	0 ,10	0 ,12	0 ,15	0 ,18	0 ,22	0 ,26	0 ,30	0 ,35	0 ,41	0 ,47	0 ,53	0 ,61	0 ,69	150
40	0 ,00	0 ,01	0 ,02	0 ,03	0 ,04	0 ,05	0 ,07	0 ,09	0 ,11	0 ,13	0 ,16	0 ,19	0 ,23	0 ,27	0 ,31	0 ,36	0 ,41	0 ,47	0 ,54	0 ,61	140
50	0 ,00	0 ,01	0 ,02	0 ,02	0 ,03	0 ,04	0 ,06	0 ,07	0 ,09	0 ,11	0 ,13	0 ,16	0 ,19	0 ,22	0 ,26	0 ,30	0 ,35	0 ,40	0 ,45	0 ,51	130
60	0 ,00	0 ,01	0 ,01	0 ,02	0 ,03	0 ,03	0 ,04	0 ,06	0 ,07	0 ,09	0 ,10	0 ,12	0 ,15	0 ,17	0 ,20	0 ,24	0 ,27	0 ,31	0 ,35	0 ,40	120
70	0 ,00	0 ,01	0 ,01	0 ,01	0 ,02	0 ,02	0 ,03	0 ,04	0 ,05	0 ,06	0 ,07	0 ,09	0 ,10	0 ,12	0 ,14	0 ,16	0 ,18	0 ,21	0 ,24	0 ,27	110
80	0 ,00	0 ,00	0 ,00	0 ,01	0 ,01	0 ,01	0 ,02	0 ,02	0 ,02	0 ,03	0 ,04	0 ,04	0 ,05	0 ,06	0 ,07	0 ,08	0 ,09	0 ,11	0 ,12	0 ,14	100
85	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 .01	0 ,01	0 ,01	0 ,01	0 ,01	0 ,02	0 ,02	0 ,03	0 ,03	0 ,04	0 ,04	0 05	0 ,05	0 ,06	0 ,07	95
90	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	90

TABLE II*a*.- $\frac{G^3}{3}$ cot Z.

Z \ G	5°	7°	8°	9°	10°	11°	12°	13°	14°	15°	16°	17°	18°	19°	20°	21°	22°	23°	24°	25°	G / Z
—																					+
40°	0°,02	0°,04	0°,06	0°,09	0°,12	0°,16	0°,21	0°,27	0°,33	0°,41	0°,50	0°,59	0°,71	0°,83	0°,97	1°,12	1°,29	1°,47	1°,67	1°,89	140°
45	0 ,01	0 ,03	0 ,05	0 ,07	0 ,10	0 ,14	0 ,18	0 ,22	0 ,28	0 ,34	0 ,42	0 ,50	0 ,59	0 ,70	0 ,81	0 ,94	1 ,08	1 ,24	1 ,40	1 ,59	135
50	0 ,01	0 ,03	0 ,04	0 ,06	0 ,09	0 ,11	0 ,15	0 ,19	0 ,23	0 ,29	0 ,35	0 ,42	0 ,50	0 ,58	0 ,68	0 ,79	0 ,91	1 ,04	1 ,18	1 ,33	130
55	0 ,01	0 ,02	0 ,04	0 ,05	0 ,07	0 ,09	0 ,12	0 ,16	0 ,20	0 ,24	0 ,29	0 ,35	0 ,41	0 ,49	0 ,56	0 ,66	0 ,76	0 ,87	0 ,98	1 ,11	125
60	0 ,01	0 ,02	0 ,03	0 ,04	0 ,06	0 ,08	0 ,10	0 ,13	0 ,16	0 ,20	0 ,24	0 ,29	0 ,34	0 ,40	0 ,47	0 ,54	0 ,62	0 ,71	0 ,81	0 ,92	120
70	0 ,00	0 ,01	0 ,02	0 ,03	0 ,04	0 ,05	0 ,06	0 ,08	0 ,10	0 ,12	0 ,15	0 ,18	0 ,22	0 ,25	0 ,30	0 ,34	0 ,39	0 ,45	0 ,51	0 ,58	110
80	0 ,00	0 ,01	0 ,01	0 ,01	0 ,02	0 ,02	0 ,03	0 ,04	0 ,05	0 ,06	0 ,07	0 ,09	0 ,10	0 ,12	0 ,14	0 ,17	0 ,19	0 ,22	0 ,25	0 ,28	100
85	0 ,00	0 ,00	0 ,00	0 ,01	0 ,01	0 ,01	0 ,02	0 ,02	0 ,03	0 ,03	0 ,04	0 ,05	0 ,05	0 ,06	0 ,07	0 ,09	0 ,10	0 ,11	0 ,13	0 ,14	95
90	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	0 ,00	90



I

Diagrams for finding δ or u by Methods III (a) and III (c)
(See Figs. 8 and 12).

Diagram I is common to both methods. Enter it with the latitude L_0 of the station taking the bearing to find δ, or with the latitude L of the station whose bearing is taken to find u. Starting from this point, follow the sloping line to the point where it meets the line joining the zero point to the hypotenuse at the proper value of d (for finding δ) or of G (for finding u); [d is the distance between the transmitting and receiving stations, G is the difference of longitude of these stations, in both cases measured on the longitude scale of a MERCATOR chart.] Starting from this point of intersection, follow the horizontal line to the left and read the number of degrees and minutes on the vertical scale.

Diagram II gives the second term of δ. Enter it with the value Z of the observed azimuth angle, then follow the same directions as before. The number of minutes read on the vertical scale is to be added algebraically to the number read on diagram I, and this gives the value of δ measured on the longitude scale.

Diagram II *bis* similarly gives the second term of u, but using G instead of d.