THE ELLIPSE OF ERRORS IN THE DETERMINATION OF GEODETIC POINTS

by

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Dr. O. EGGERT has just brought out an eighth edition of the first volume of the "Handbuch der Vermessungskunde" of W. JORDAN. This work contains a very thorough treatment of the theory of errors and that of compensations by the method of least squares. The new edition includes some complementary studies, in particular with regard to the *ellipse of errors* (Fehlerellipse) the use of which, although old, seems destined to become more generally prevalent. It has occurred to us that the readers of this Review might be interested in a short account of this question, which furnishes a means of characterizing the accuracy of the position of a geodetic point on the different azimuths.

a) A geodetic point is generally determined by a superabundance of bearings and arcs containing the angles. Each one of these gives rise to a linear equation and the entire group of equations, treated by the method of least squares, is reduced to two linear equations which give the position of the point.

Let O be the point chosen. It corresponds to the bearings differing from the readings which should have been obtained, in the absence of error, by the unknown quantities : ε_1, ε_2, ... ε_n. We may assume, if their number is large enough, that their quadratic mean :

$$(1) \qquad m = \sqrt{\frac{[\varepsilon\varepsilon]}{n}}$$

is equal to that which we obtain by treating the deviations of the observed angles with those which result from the selected position O. The quantities ε are actually the deviations with the mean values ; they should follow the same law if their number is sufficiently large.

Each correction ε leads to replacing the adopted line passing through O by a parallel line whose equation, taking O as the origin of coordinates, will be :

$$ax + by = \varepsilon$$

By substituting the values of ε in formula (1) we obtain the equation for the ellipse of errors :

$$\Sigma\ (ax + by)^2 = nm^2$$

or, with the notation of GAUSS, generally employed in the theory of least squares :

$$(2) \qquad [aa]\, x^2 + [bb]\, y^2 + 2\, [ab]\, xy = nm^2.$$

This ellipse has its centre at the point O. The direction φ of its minor axis will be given by the formula :

$$\sin 2\,\varphi = \frac{2\,[ab]}{\sqrt{W}}, \qquad \cos 2\,\varphi = \frac{[aa] - [bb]}{\sqrt{W}}$$

in which the radical always has a positive sign and where :

$$W = ([aa] - [bb])^2 + 4\,([ab])^2$$

The squares of the lengths of the axes are :

$$\frac{nm^2}{2}\,\frac{[aa] + [bb] \pm \sqrt{W}}{[aa]\,[bb] - [ab]^2}$$

We see that the uncertainty of position of the point O is not the same for all azimuths ; it is a maximum in the direction of the major axis of the ellipse of errors, a minimum along the minor axis. The orientation of the axes and their relation does not depend upon the value assumed for m ; it depends solely upon the directions and the distances of the known points utilized.

The surface of the ellipse is :

$$\frac{\pi\, nm^2}{\sqrt{[aa]\,[bb] - [ab]^2}}$$

In order to characterize the accuracy of the point we may use either the length of the major axis of the ellipse or the surface of the ellipse, which are quantities independent of the orientation of the coordinates, but which are not necessarily minima at the same time.

The ellipse will only become a circle and the uncertainty of the position of the point O be the same in all directions provided :

$$[aa] = [bb] \quad \text{and} \quad [ab] = O$$

The radius of the circle is then :

$$\frac{m\sqrt{n}}{\sqrt{[aa]}}$$

The quantities $[aa]$, $[bb]$ and $[ab]$ are those which have been calculated in the determination of the point O.

No account has been taken here of the different weights which might be attached to the sights, but their introduction does not give rise to any difficulty.

The ellipse of errors may then be considered as the locus of the points having an equal probability, and it can be demonstrated that the probability of a point lying in the interior of the ellipse will be :

$$1 - e^{-\frac{1}{2}} = 0.39348$$

It is called the *mean ellipse of errors.*

Sometimes we need to consider the *probable ellipse of errors.* This is an ellipse homothetic to the preceding, such that there is a probability of ½ of the points lying within the ellipse. This is equivalent to employing for the data of the ellipse the value of m^2 multiplied by twice the value of the Naperian logarithm of 2, i.e. by 1.3863.

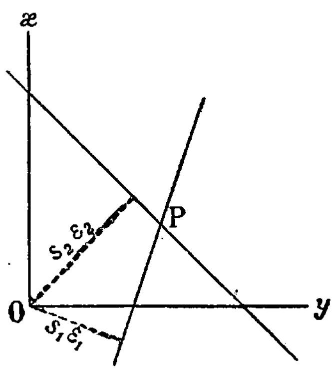
Finally ANDRAE has employed an ellipse obtained by replacing m^2 by $2m^2$. The probability of a point lying within the ellipse of ANDRAE is then

$$1 - e^{-1} = 0.63212$$

b) Although it is not legitimate to consider a quadratic mean such as (1) as subject to the laws of the theory of probability, when there is only a small number of values for ε, it appears to us however that it might be rather interesting to apply the above formulae to the two following cases, which are the simplest that one could possibly encounter.

1) Assume that the point O can only be determined by bearings from two known points located at distances s_1 and s_2 from the target. Let α_1 and α_2 be the mean bearings which have been utilised. They differ from the true bearings, which are moreover unknown, by the quantities ε_1 and ε_2, whose quadratic mean is :

$$m = \sqrt{\frac{\varepsilon_1^2 + \varepsilon_2^2}{2}}$$



Assume that m is equal to the quadratic mean of the deviations of the bearings taken with respect to their mean values. If O is the point determined by the selected bearings, let us take this point as the origin of coordinates. The corrections ε_1 and ε_2 to these bearings lead us to adopt instead of the point O another point P, located at the intersection of the two straight lines whose equations are :

$$x \frac{\sin \alpha_1}{s_1} - y \frac{\cos \alpha_1}{s_1} = \varepsilon_1$$

(3)

$$x \frac{\sin \alpha_2}{s_2} - y \frac{\cos \alpha_2}{s_2} = \varepsilon_2$$

If we assume that m remains constant, the unknown quantities ε_1 and ε_2 are such that the point P must be located on an ellipse whose equation is :

$$\left(x \frac{\sin \alpha_1}{s_1} - y \frac{\cos \alpha_1}{s_1} \right)^2 + \left(x \frac{\sin \alpha_2}{s_2} - y \frac{\cos \alpha_2}{s_2} \right)^2 = 2m^2$$

This ellipse has its centre at O ; the directions α_1 and α_2 are those of the two conjugate diameters, the extremities of which are obtained by taking their intersections with the parallels drawn at distance $\pm m \sqrt{2}$.

The axes of the ellipse may be readily deduced, either by a well-known geometrical construction, or by calculation. The direction φ of the minor axis is given by the formulae :

$$\sin 2\varphi = - \frac{s_2^2 \sin 2\alpha_1 + s_1^2 \sin 2\alpha_2}{\sqrt{(s_1^2 - s_2^2)^2 + 4 s_1^2 s_2^2 \cos^2 (\alpha_1 - \alpha_2)}}$$

$$\cos 2\varphi = - \frac{s_2^2 \cos 2\alpha_1 + s_1^2 \cos 2\alpha_2}{\sqrt{(s_1^2 - s_2^2)^2 + 4 s_1^2 s_2^2 \cos^2 (\alpha_1 - \alpha_2)}}$$

The squares of the axes have the following values :

$$m^2 \frac{s_1^2 + s_2^2 \pm \sqrt{(s_1^2 - s_2^2)^2 + 4 s_1^2 s_2^2 \cos^2 (\alpha_1 - \alpha_2)}}{\sin^2 (\alpha_1 - \alpha_2)}$$

These lengths are entirely independent of the orientation of the axes. They can only become equal provided that :

$$s_1 = s_2 \quad \text{and} \quad \alpha_1 - \alpha_2 = 90^\circ$$

The radius of the circle then becomes : $ms\sqrt{2}$.

The area of the ellipse of errors is :

$$\frac{2 \pi m^2 s_1 s_2}{\sin (\alpha_1 - \alpha_2)}$$

A consideration of this ellipse discloses the well-known fact that the most favourable conditions are realized when the two lines of sight intersect at right angles (resulting in the minimum surface of the ellipse) and that the two points sighted are equi-distant.

2) Let us consider now the case where the point O is determined by bearings taken on two points and by the arc containing the angle measured between these two points. We then have:

$$m = \sqrt{\frac{\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2}{3}}$$

and we shall assume that this quadratic mean is known.

To equations (3) it is necessary to add the equation derived from the arc containing the angle:

$$x \frac{s_1 \sin \alpha_2 - s_2 \sin \alpha_1}{s_1 \; s_2} - y \frac{s^1 \cos \alpha_2 - s_2 \cos \alpha_1}{s_1 \; s_2} = \varepsilon_3$$

We then have:

$$W = 3 \left(\frac{1}{s_1^2} - \frac{1}{s_2^2}\right)^2 + \left(\frac{1}{s_1^2} + \frac{1}{s_2^2} - \frac{4 \cos (\alpha_1 - \alpha_2)}{s_1 \; s_2}\right)^2$$

$$\sin 2 \varphi = - \frac{\dfrac{\sin 2 \alpha_1}{s_1^2} + \dfrac{\sin 2 \alpha_2}{s_2^2} - \dfrac{\sin (\alpha_1 + \alpha_2)}{s_1 \; s_2}}{\frac{1}{2} \sqrt{W}}$$

$$\cos 2 \varphi = - \frac{\dfrac{\cos 2 \alpha_1}{s_1^2} + \dfrac{\cos 2 \alpha_2}{s_2^2} - \dfrac{\cos (\alpha_1 + \alpha_2)}{s_1 \; s_2}}{\frac{1}{2} \sqrt{W}}$$

The squares of the lengths of the axes of the ellipse will be:

$$m^2 \frac{[s_1^2 + s_2^2 - s_1 s_2 \cos (\alpha_1 - \alpha_2)] \pm \frac{1}{2} s_1^2 s_2^2 \sqrt{W}}{\sin^2 (\alpha_1 - \alpha_2)}$$

The surface of the ellipse is:

$$\frac{\pi m^2 s_1 s_2 \sqrt{3}}{\sin (\alpha_1 - \alpha_2)}$$

It will be a minimum if the two lines of sight intersect at 90°.

The ellipse becomes a circle if:

$$s_1 = s_2 \qquad \alpha_1 - \alpha_2 = \pm 60^\circ$$

that is, if the triangle formed by the point O and the two known points is an equilateral triangle.

The radius of the circle is $ms\sqrt{2}$, as in case 1); but its surface is greater than that of the ellipse corresponding to the isoceles rectangular triangle at O.

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