GENERALIZATION
OF THE CONFORMAL CONIC PROJECTION AND THE DOUBLE CIRCULAR PROJECTION OF THE SPHERE

by

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I. — GENERALIZATION OF THE CONFORMAL CONIC PROJECTION :

(Translated from the French).

The conformal conic projection is used to such a great extent that it may not be without interest to study a conformal projection somewhat more general, of which it — as well as the projections with separate variables (Jung's projections) — are special cases, being also a projection of least distortion in the sense of Tissot, in permitting an adaptation to the shape of the country to be represented. This projection is well known, but some of its most interesting applications do not appear to have been used, at least in some special cases.

Let us assume here that a conformal projection of the terrestrial ellipsoid on the sphere has previously been made, an operation which is rendered easy and rapid by the use of the table in Special Publication N° 21, and therefore we shall hereafter concern ourselves only with the projection of the sphere, the radius of which is considered as unity.

If, on the terrestrial sphere, we select two points I and J, separated by the arc of the great circle IMJ having an angular value of $2C$, less than π , we may distinguish on this sphere between two systems of orthogonal circles, the first consisting of circles passing through the points I and J and the second formed by circles whose planes all intersect along the line of intersection of the two planes tangent to the sphere at I and J (see Fig. 1).

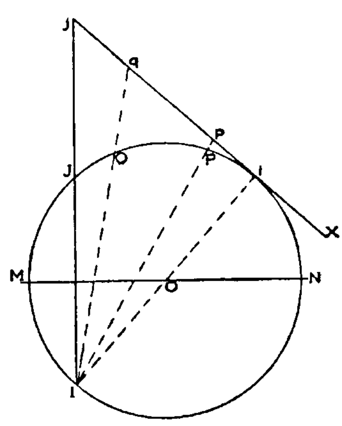


FIG. 1

Let us make a stereographic projection on the plane tangent to the point i, diametrically opposite to the point I. The projection of the first system of circles then becomes a pencil of straight lines concurrent at the point j, the projection of J, while that of the second system, being orthogonal to the first, can only be a group of circles having a common centre j.

Let us take the prolongation of ji as the "x" axis, and a perpendicular at i to the plane IJi as the "y" axis, and calling z the complex co-ordinate :

$$z = x + iy$$

of this stereographic projection, we propose to study the projection defined by the expression :

$$Z = a\,(z + 2 \operatorname{cotg} C)^{n},$$

in which Z represents the complex co-ordinate $X + iY$ in the system of co-ordinates having the same directions as those of z, but with the origin at j; a and n being real and positive quantities. This projection is conformal, being defined by the function of the complex variable of a conformal projection; and it conforms with the conformal conic projection if we substitute for C the value $\frac{\pi}{2}$. Like the latter, it is symmetrical only with respect to the axis of X, except in one special case.

The pencil of straight lines of the stereographic projection concurrent at j is then transformed into a pencil of straight lines concurrent at the same point j — the origin; but the angles which these lines make with each other are multiplied by n. There will therefore be an overlapping if n is greater than unity, but this does not interfere with the use of the projection for the representation of restricted areas.

The portion $JiNI$ of the great circle, a portion which is greater than π, is represented on the projection, regardless of the value of n, by the straight line ji and its prolongation beyond i. As for the portion IMJ, this may be considered as located in the plane which makes an angle π with that of the first portion. It will therefore be represented by a straight line making an angle n π with ji, and not by the prolongation of ij.

The system of concentric circles then becomes a system of circles having the same centre j, but the radii r will become ar^{n}.

If we substitute for Z the expression $\rho e^{i\omega}$, in which ρ represents the distance of a point on the new projection from the origin and ω the angle made by the axis of X with the straight line which joins it to the origin, then we have :

$$\rho = a\,[(x + 2 \operatorname{cotg} C)^2 + y^2]^{\frac{n}{2}}, \qquad tg\,\frac{\omega}{n} = \frac{y}{x + 2 \operatorname{cotg} C},$$

formulae which permit the deduction of ρ and ω from x and y.

We may calculate the scale m (linear modulus) of the projection by means of the formulae :

$$z = \left(\frac{Z}{a}\right)^{\frac{1}{n}} - 2 \operatorname{cotg} C \qquad \frac{dz}{dZ} = \frac{1}{an}\left(\frac{Z}{a}\right)^{\frac{1-n}{n}}$$

$$(1) \qquad m = an\left(\frac{\rho}{a}\right)^{\frac{n-1}{n}} \left[\frac{1}{4}\left(\frac{\rho}{a}\right)^{\frac{2}{n}} - \left(\frac{\rho}{a}\right)^{\frac{1}{n}} \operatorname{cotg} C \cos\frac{\omega}{n} + \frac{1}{\sin^2 C}\right].$$

This scale is not constant over the entire circumference concentric with the origin, as in the ordinary conical projections, but is a minimum along the axis of X. It is therefore necessary to study it in the different directions ω. Its derivatives are :

$$(2) \qquad \frac{\delta m}{\delta \rho} = \left(\frac{\rho}{a}\right)^{-\frac{1}{n}} \left[\frac{n+1}{4}\left(\frac{\rho}{a}\right)^{\frac{2}{n}} - n\left(\frac{\rho}{a}\right)^{\frac{1}{n}} \operatorname{cotg} C \cos\frac{\omega}{n} + \frac{n-1}{\sin^2 C}\right],$$

$$(3) \qquad \frac{\delta^2 m}{\delta \rho^2} = \frac{1}{an}\left(\frac{\rho}{a}\right)^{-\frac{n+1}{n}} \left[\frac{n+1}{4}\left(\frac{\rho}{a}\right)^{\frac{2}{n}} - \frac{n-1}{\sin^2 C}\right],$$

$$(4) \qquad \frac{\delta m}{\delta \omega} = a\,\frac{\rho}{a} \operatorname{cotg} C \sin\frac{\omega}{n},$$

$$(5) \qquad \frac{\delta^2 m}{\delta \omega^2} = \frac{a}{n}\,\frac{\rho}{a} \operatorname{cotg} C \cos\frac{\omega}{n},$$

$$(6) \qquad \frac{\delta^2 m}{\delta \rho\, \delta \omega} = \operatorname{cotg} C \sin\frac{\omega}{n}.$$

The derivative $\frac{\delta m}{\delta \rho}$ cannot vanish unless :

$$\sin^2\frac{\omega}{n} \leqslant \frac{1 - n^2 \sin^2 C}{n^2 \cos^2 C};$$

which requires that :

$$n \leqslant \frac{1}{\sin C}.$$

1°) $n > \frac{1}{\sin C}$. In this case the scale which is zero at the origin, increases continually with ρ for the same value of ω. Equation (3) shows that the curve which represents it along a straight line passing through the origin has a point of inflection, regardless of the orientation of this straight line, for the value of ρ given by :

$$\left(\frac{\rho}{a}\right)^{\frac{1}{n}} = 2\,\frac{\sqrt{n^2 - 1}}{(n+1)\sin C}.$$

The corresponding value of the scale (modulus) m is :

$$m = an \left(\frac{2}{n+1}\right)^n \frac{(n^2-1)^{\frac{n-1}{2}}}{\sin^{n+1} C} \left(n - \sqrt{n^2-1}\cos C \cos\frac{\omega}{n}\right).$$

It may be represented along any given radius by a curve of the form shown in Fig. 2.

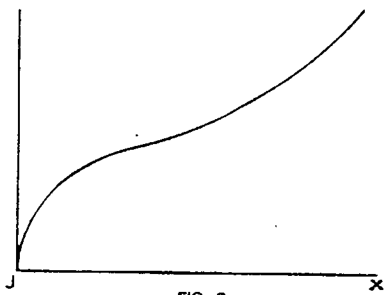


FIG. 2

For a given value of m, the isometric curve takes the shape of a spiral which approaches the point j when ω increases.

The absence of a minimum of m makes this projection generally less interesting.

1_a) $\mathbf{n = 2, C = \frac{\pi}{2}}$. Meanwhile if, for $C = \frac{\pi}{2}$, we give to n the value of 2, and if we place the points I and J on the equator and a radio-goniometric station on a meridian perpendicular to it, the line of azimuth relative to this station will be represented by a circle; a property which may sometimes be rather serviceable in practice.

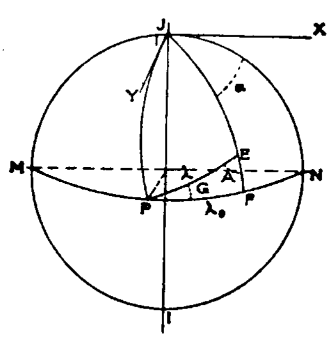


FIG. 3

In Fig. 3 let E be a point with co-latitude λ and longitude G, from which a bearing on the station F gives the azimuth A. The triangle JPE then shows the following relations :

$$\cos JE = \sin\lambda \sin G, \qquad \text{cotg}\ \alpha = tg\ \lambda \cos G.$$

Further, from the relation :

$$Z = a\, z^2,$$

we deduce :

$$X = a\,(x^2 - y^2), \qquad Y = 2\,a\,xy, \qquad X^2 + Y^2 = a^2\,(x^2 + y^2)^2,$$

$$tg\ \alpha = \frac{\sqrt{X^2 + Y^2} - X}{Y}.$$

We have therefore :

$$\operatorname{tang}^2 \frac{JE}{2} = \frac{\sqrt{X^2 + Y^2}}{4\,a}, \qquad \cos JE = \frac{4\,a - \sqrt{X^2 + Y^2}}{4\,a + \sqrt{X^2 + Y^2}},$$

$$\sin^2 JE = \frac{16\,a\sqrt{X^2 + Y^2}}{(4\,a + \sqrt{X^2 + Y^2})^2},$$

$$\sin^2 \lambda = \frac{\cos^2 JE + \operatorname{cotg}^2 \alpha}{1 + \operatorname{cotg}^2 \alpha}, \qquad \cos^2 \lambda = \frac{\sin^2 JE}{1 + \operatorname{cotg}^2 \alpha};$$

and consequently :

$$\operatorname{tang}^2 \lambda = \frac{Y^2 + (X + 4\,a)^2}{8\,a\,(\sqrt{X^2 + Y^2} - X)}, \qquad \frac{1}{\cos^2 \lambda} = \frac{(4\,a + \sqrt{X^2 + Y^2})^2}{8\,a\,(\sqrt{Y^2 + Y^2} - X)}.$$

The triangle PEF gives an equation which, multiplied by $\dfrac{\sin \lambda}{\cos^2 \lambda}$, may be written

$$\operatorname{cotg} \lambda_0 \operatorname{tang}^2 \lambda = \operatorname{cotg} A \frac{\sin G \sin \lambda}{\cos^2 \lambda} + \operatorname{tang} \lambda \cos G.$$

By substituting for the functions of λ and of G their values as given above, we obtain the equation of the circle :

$$X^2 + Y^2 + 8\,a \frac{\sin A}{\sin(\lambda_0 + A)}(X \cos \lambda_0 - Y \sin \lambda_0) = 16\,a^2 \frac{\sin(\lambda_0 - A)}{\sin(\lambda_0 + A)}.$$

2°) $\mathbf{n} \leqslant \dfrac{1}{\sin \mathbf{C}}$. Let us assume that n is less than $\dfrac{1}{\sin C}$ or is equal to it; we may then put :

$$n = \frac{\sin \mu}{\sin C},$$

and always take μ, like C, smaller than $\dfrac{\pi}{2}$ or equal to $\dfrac{\pi}{2}$.

The derivative $\dfrac{\delta m}{\delta \rho}$ (formula 2) can be made to vanish if we have :

$$\sin \frac{\omega}{n} \leqslant \frac{\operatorname{tang} C}{\operatorname{tang} \mu}.$$

If μ is smaller than C, this will always occur.

If μ is greater than C, this will not occur unless ω has a value lying between zero and a limiting value ω_1.

In the first case, n will be smaller than unity; in the second case it will be greater. If n is equal to unity, we then rediscover the stereographic projection.

When n is less than $\frac{1}{\sin C}$, the isometers have a double point in every case where ω is equal to $n\pi$ or is a multiple of $n\pi$ and where $\frac{\delta m}{\delta \rho}$ is zero.

2a) $\mu > C$. Let us first take the case where n is greater than unity but smaller than $\frac{1}{\sin C}$; that is, where μ is greater than C.

The derivative $\frac{\delta m}{\delta \rho}$ will vanish for two positive values of ρ when ω is smaller than ω_1. Let ρ_1 be the greater and ρ_2 the smaller. The first corresponds to a minimum of m, the second to a maximum.

The point of inflection is between the two, at the same distance from j, no matter what the value of ω.

$$\left(\frac{\rho_1}{a}\right)^{\frac{1}{n}} = \frac{2 \sin \mu \operatorname{cotg} C}{\sin \mu + \sin C} \left(\cos \frac{\omega}{n} + \sqrt{\operatorname{tg}^2 C \operatorname{cotg}^2 \mu - \sin^2 \frac{\omega}{n}}\right),$$

$$\left(\frac{\rho_2}{a}\right)^{\frac{1}{n}} = \frac{2 \sin \mu \operatorname{cotg} C}{\sin \mu + \sin C} \left(\cos \frac{\omega}{n} - \sqrt{\operatorname{tg}^2 C \operatorname{cotg}^2 \mu - \sin^2 \frac{\omega}{n}}\right).$$

On the axis of X, we shall call these values ρ_p and ρ_q:

$$\left(\frac{\rho_p}{a}\right)^{\frac{1}{n}} = \frac{2}{\sin C} \frac{\cos \frac{\mu + C}{2}}{\cos \frac{\mu - C}{2}}, \qquad \left(\frac{\rho_q}{a}\right)^{\frac{1}{n}} = \frac{2}{\sin C} \frac{\sin \frac{\mu - C}{2}}{\sin \frac{\mu + C}{2}}.$$

The corresponding values of m are then:

$$m_p = \frac{a \sin \mu}{\cos C + \cos \mu} \left(\frac{2 \cos \frac{\mu + C}{2}}{\sin C \cos \frac{\mu - C}{2}}\right)^n,$$

$$m_q = \frac{a \sin \mu}{\cos C - \cos \mu} \left(\frac{2 \sin \frac{\mu - C}{2}}{\sin C \sin \frac{\mu + C}{2}}\right)^n.$$

From this it follows :

$$(7) \qquad \frac{\rho_p}{m_p} = \frac{\cos C + \cos \mu}{\sin \mu}, \qquad \frac{\rho_q}{m_q} = \frac{\cos C - \cos \mu}{\sin \mu},$$

If we calculate the values of x on the stereographic projection for the radii vectores ρ_p and ρ_q, we shall obtain :

$$x_p = -2 \operatorname{tang} \frac{\mu - C}{2}, \qquad x_q = -2 \operatorname{cotg} \frac{\mu + C}{2}.$$

This shows that the points on the projection corresponding to the maximum and minimum of m are located on the sphere at the points P and Q, such that : (See Fig. 1).

$$NP = MQ = \mu .$$

The point P plays the same role as the point of contact of the cone tangent to the sphere in the conic projections where $C = \frac{\pi}{2}$.

It is better to make m_p equal to unity, which gives to a the value :

$$(8) \qquad a = \frac{\cos C + \cos \mu}{\sin \mu} \left(\frac{\sin C \cos \frac{\mu - C}{2}}{2 \cos \frac{\mu + C}{2}} \right)^n .$$

This is the assumption we shall make in the following for the purpose of simplification.

But we might also adopt for m_p a value of $1 - \varepsilon$, in order to reduce by half the amount of the error at the limits of the region represented.

The points where the maximum and minimum values of m occur along each radius vector lie on a closed curve, tangent to the radii vectores which correspond to their limiting values $\pm \omega_1$. Its equation is obtained by equating the portion in brackets in equation (2) to zero.

The values of m on the radius vector may be represented by a curve analogous to that of Fig. 2, if ω is greater than ω_1, and by that of Fig. 4 if ω is smaller than ω_1.

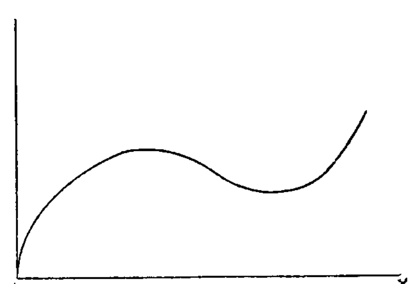


FIG. 4

The isometric curve for the same value of m intersects the radii vectores in three points or one point, depending on the values of m and of ω. In Fig. 5 we give an approximate illustration for the case in which n = 2 and $C = 25°$. The numerals inscribed opposite the curves indicate the values of m on the curves. The dotted curve is that of the maxima and minima of m.

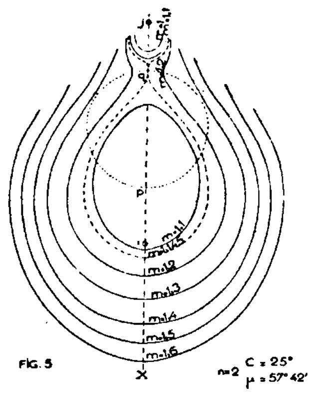


FIG. 5

au point p 1 m/m vaut 500 kilomètres

The ordinary conic projections are not employed except for values of n less than unity; we see that the generalized projection may be employed for values of n greater than unity; not for a region which contains the point J because around this point overlapping would occur to an inadmissible extent, but in order to represent the region extending around point P.

2b) $\mu < C$. Suppose that μ be smaller than C, and that therefore n is less than unity.

For every value of ω, the derivative $\frac{\delta m}{\delta \rho}$ will vanish for two values of ρ, the one positive, ρ_1 which corresponds to a minimum of m, and the other ρ_2 , which corresponds to a maximum, but which being negative cannot be taken into consideration; m is infinite at the point j and reaches a maximum at the point P. For every value of ω it may be represented by a curve analogous to that in Fig. 6.

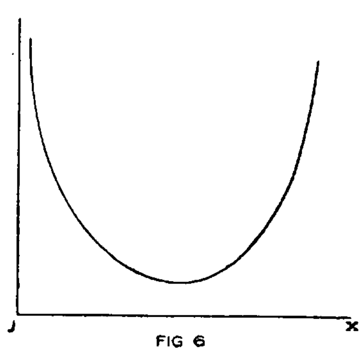


FIG 6

The isometers will be the curves represented in Fig. 7 by

$$n = \frac{1}{2} \text{ and } \mu = 25°.$$

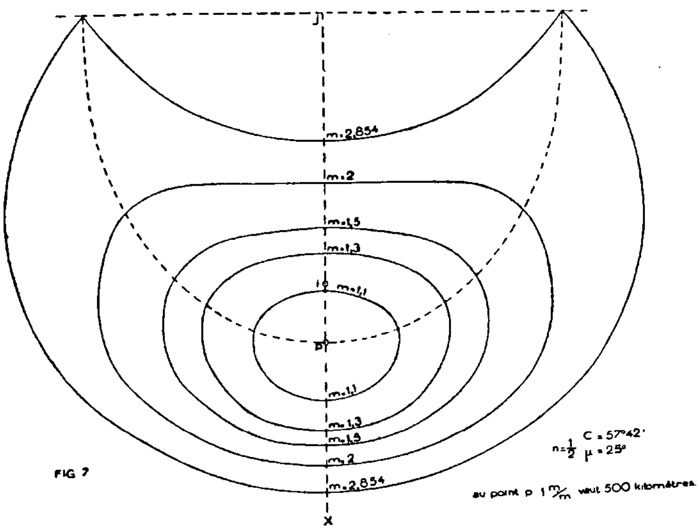


FIG 7

3) The generalized conic projection may be used in cases 2a and 2b as a *projection of minimum distortion,* because one can use the quantities μ and C in such a manner as to adapt it as well as possible to the shape of the region to be represented.

This is more clearly shown when we develop the expression for m given by equation (1) according to the powers of $\rho - \rho_p$ and of $\omega\, \rho_p$. This expansion is the following :

$$(9) \qquad 2\,(m - m_p)\, \rho_p \frac{\sin \mu}{\cos C} = (\rho - \rho_p)^2 \frac{\cos \mu}{\cos C} + \omega^2 \rho_p^{\,2} + \cdots\cdots$$

The isometer in the vicinity of point p is comparable to an ellipse in which the ratio of the axes is equal to :

$$\sqrt{\frac{\cos \mu}{\cos C}}\,.$$

It differs from it by the fact that one of these axes is circular instead of being rectilinear.

The curve is elongated in the direction of the axis of X if μ is greater than C; and in the direction of the arc in the opposite case.

By giving C in equation (9) the value of $\frac{\pi}{2}$, we come back to the two circumferences having their centre at j in the ordinary conic projection, for which n is less than unity.

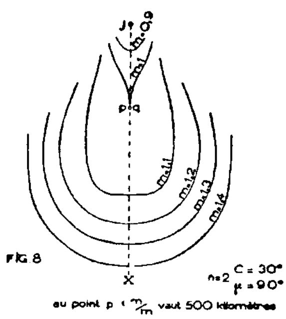
The shape of the region to be represented allows one to establish the relation :

$$\frac{\cos\ \mu}{\cos\ C},$$

and at the same time the position of the point P. It leads therefore to the selection of the great circle, the axis of X, for which the projection shall be symmetrical on each side.

4) $\mu = \frac{\pi}{2}$, $n = \frac{1}{\sin C}$. By giving μ a value of $\frac{\pi}{2}$, the two points P and Q in Fig. 1 then coincide. The isometer then shows an upturned point, such that the variations of m will be very slight in its vicinity.

The isometers are represented in Fig. 8 for $n = 2$.



4a) $\mu = \frac{\pi}{2}$, $C = 0$, $n = \infty$. The isometers in the preceding figure will be the more flattened out the smaller the value of C. In case C becomes equal to zero, the formula :

$$Z = a\,(z + 2 \operatorname{cotg} C)^n$$

retains in the meanwhile its significance, because if the part raised to the n^{th} power becomes infinite, the value of a becomes zero and the product tends to approach a limit while C tends towards zero. In fact we have according to formula (8) when : $\mu = \frac{\pi}{2}$:

$$a = \left(\frac{\sin C\,(1 + \sin C)}{2 \cos C}\right)^n \cos C\,;$$

from which :

$$Z = \cos C\,(1 + \sin C)^n\,(\tfrac{1}{2} z \operatorname{tg} C + 1)^n,$$

$$* \quad \operatorname{Log} Z = \operatorname{Log} \cos C + \frac{\operatorname{Log}\,(1 + \sin C)\,(\frac{1}{2} z \operatorname{tg} C + 1)}{\sin C}.$$

* The expression " Log " signifies the Naperian logarithm to the base e.

We shall obtain the value of the last term when C tends towards zero by taking the ratio of the derivatives of the numerator and the denominator. The ratio is equal to :

$$1 + \tfrac{1}{2}\zeta \cdot$$

from which :

$$(10) \qquad \operatorname{Log} Z = 1 + \tfrac{1}{2}\zeta, \qquad\qquad Z = e^{1 + \frac{1}{2}\zeta}.$$

The points I and J coincide with the point M. On the projection J is the origin, I is at infinity. The circles of the sphere of which the planes pass through the tangent to the great circle of symmetry at M are represented by straight lines passing through the origin; those whose planes pass through a perpendicular to M in the plane of symmetry are represented by a series of concentric circles. These two systems of circles are represented on the stereographic projection, taken as the point of departure, by the straight lines parallel respectively to the axis of x and the axis of y. We deduce therefore the formula (10) for these relations :

$$\frac{Y}{X} = \operatorname{tg} \frac{y}{2}, \qquad\qquad X^2 + Y^2 = e^{x+2}.$$

Point N is found on the axis of X at a distance from the origin equal to e and the united points P and Q at a distance l.

We have :

$$m = \rho\left[1 - \operatorname{Log}\rho + \tfrac{1}{2}(\operatorname{Log}\rho)^2 + \tfrac{1}{2}\omega^2\right],$$

$$2\frac{\delta m}{\delta \rho} = (\operatorname{Log}\rho)^2 + \omega^2.$$

The isometer $m = l$ has an upturned point on the axis of X at the point where $\rho = l$.

The isometric curves have the shape indicated in Fig. 9. In an extended region having a length of about 10,000 kilometres and a width of about 5,000 kilometres the scale varies less than 1/10.

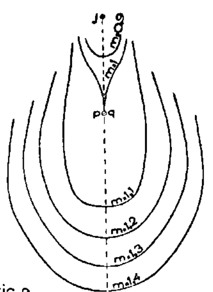


FIG. 9

n=∞ C=0 μ=90°

au point p 1 m/m vaut 500 kilomètres

5) **C** $= 0$, **n** $= \infty$. Let us examine the case where C becomes zero without μ becoming equal to $\frac{\pi}{2}$. A calculation analogous to the preceding gives us the projection :

$$Z = \operatorname{cotg} \frac{\mu}{2}\, e^{\sin \mu \left(\frac{1}{2} \zeta + \operatorname{tg} \frac{\mu}{2} \right)}.$$

We have :

$$m = \rho \ \operatorname{tang} \frac{\mu}{2} \left[1 - \operatorname{Log}\left(\rho \operatorname{tg} \frac{\mu}{2} \right) + \frac{\left(\operatorname{Log} \rho \operatorname{tg} \frac{\mu}{2} \right)^2}{4 \sin^2 \frac{\mu}{2}} + \frac{\omega^2}{4 \sin^2 \frac{\mu}{2}} \right]$$

$$2 \frac{\delta m}{\delta \rho} \sin \mu = \left(\operatorname{Log} \rho \operatorname{tg} \frac{\mu}{2} \right)^2 + \omega^2 + 2 \cos \mu \operatorname{Log} \rho \operatorname{tg} \frac{\mu}{2},$$

$$\rho \frac{\delta^2 m}{\delta \rho^2} \sin \mu = \cos \mu + \operatorname{Log} \rho \operatorname{tg} \frac{\mu}{2}.$$

In order that $\frac{\delta m}{\delta \rho}$ may cancel out, it is necessary that :

$$\omega \leqslant \cos \mu .$$

The isometers are analogous to those in Fig. 5, but more flattened out for the same value of μ. They approach, in the vicinity of p, to the ellipses in which the ratio of the axis is given by : $\sqrt{\frac{1}{\cos \mu}}$.

This projection may be used, like that which was studied under 2a) as a projection of least distortion.

The small circle passing through the tangent at M of the sphere in the plane of symmetry and whose plane makes an angle of α with this plane, is represented on the projection by a straight line passing through the origin making with the X axis an angle ω equal to $\sin \mu \operatorname{tang} \alpha$.

The small circle passing through the tangent M of the sphere perpendicular to the plane of symmetry and whose plane makes an angle β with the plane passing through MN and perpendicular to the plane of symmetry is represented on the projection by a circle having its centre at the origin and whose radius is :

$$\rho = \operatorname{cotg} \frac{\mu}{2}\, e^{\sin \mu \left(\operatorname{tg} \beta + \operatorname{tg} \frac{\mu}{2} \right)}.$$

If μ equals $\frac{\pi}{2}$, these values are :

$$\omega = \text{tang}\ \alpha\ , \qquad \rho = e^{1 + \text{tg}\ \beta}.$$

6) $\mu = 0\ ,\ n = 0$ *Projections of Jung.* We may also assume a value of zero for μ and consequently for n also; C having a value different from zero. The point P then coincides with the point N.

In order to study this projection, let us shift the origin to the point P; and let Z' represent the new coordinate $X' + iY'$.

$$Z' = Z - a\left(\frac{2}{\sin\ C}\right)^n,$$

or, by substituting for a its value deduced from (8) :

$$Z' = 2\ \cos^2 \frac{C}{2}\ \frac{(\frac{1}{2}\zeta \sin C + \cos C)^n - 1}{\sin\ \mu}.$$

When μ tends towards zero, the second factor tends towards a limit which we obtain by taking the ratio of the derivatives, which is equal to :

$$\frac{1}{\sin\ C}\ \text{Log}\ (\tfrac{1}{2}\zeta \sin C + \cos C)\ .$$

We have then :

$$(11) \qquad Z' = \text{cotg}\ \frac{C}{2}\ \text{Log}\ (\tfrac{1}{2}\zeta \sin C + \cos C),$$

from which we may deduce conversely :

$$(12) \qquad \zeta = \left(e^{Z'\ \text{tg}\ \frac{C}{2}} - \cos\ C\right)\frac{2}{\sin\ C}, \qquad \frac{d\zeta}{dZ'} = \frac{1}{\cos^2 \frac{C}{2}}\ e^{Z'\ \text{tg}\ \frac{C}{2}}.$$

and consequently :

$$m = \frac{e^{X'\ \text{tg}\ \frac{C}{2}} + e^{-X'\ \text{tg}\ \frac{C}{2}} - 2\ \cos\ C \cos\ (Y'\ \text{tg}\ \frac{C}{2})}{4\ \sin^2 \frac{C}{2}}.$$

We may also note here that this scale (linear modulus), the expression for which is symmetrical with respect to X' as well as to Y' and which shows the variables separated, is the same as that obtained for the projections of the *General Solution N° 2 of Laborde* or the projection of Jung (See Hydrographic Review Vol. X N° 1 of May 1933, page 83). We may in fact restore the equation (12) to the form given it on page 85 of Vol. X N° 1. For that purpose the stereographic projection is made from N in place of i. Its coordinate z' is related to the coordinate z by the formula :

$$z = \frac{z' + 2 \operatorname{tg} \frac{C}{2}}{1 - \frac{1}{2} z' \operatorname{tg} \frac{C}{2}} .$$

We deduce then from equation (12):

$$z' = 2 \operatorname{cotg} \frac{C}{2} \frac{e^{Z' \operatorname{tg} \frac{C}{2}} - 1}{e^{Z' \operatorname{tg} \frac{C}{2}} + 1} = -2\, i \operatorname{cotg} \frac{C}{2} \operatorname{tg} \left(\frac{i}{2} Z' \operatorname{tg} \frac{C}{2} \right)$$

$$= \frac{2}{i \operatorname{tg} \frac{C}{2}} \operatorname{tang} \left(\frac{i \operatorname{tg} \frac{C}{2}}{2} Z' \right) .$$

This is the general expression for the projections of Jung, by putting

$$A = i \operatorname{tg} \frac{C}{2} .$$

The isometers of this projection have a form analogous to those in Fig. 7, but they are symmetrical with respect to the axes of X' and Y'. In the vicinity of point P they approach the ellipses, the ratio of whose axes will be : $\sqrt{\cos C.}$

The points I and J are then transferred to infinity. The first system of circles of the sphere is represented by the straight lines parallel to the axis of X' at a distance of $\alpha \cot \frac{C}{2}$ if their plane makes an angle of α with the plane of symmetry. If γ is the arc of the great circle of symmetry which separates the point i from the common point of this great circle and a circle of the 2nd system, this last circle becomes transformed into a straight line parallel to the axis of Y' at a distance equal to :

$$\operatorname{cotg} \frac{C}{2} \operatorname{Log} \left(\cos C + \sin C \operatorname{tg} \frac{Y}{2} \right) .$$

7) $\mu = 0$, $C = \frac{\pi}{2}$, $n = 0$ *Mercator Projection.* The formulae of the preceding paragraph are still valid when, μ being zero, C becomes equal to $\frac{\pi}{2}$. The expression of the projection then becomes :

$$Z' = \operatorname{Log} \left(\tfrac{1}{2} z \right) .$$

This is the equation of a Mercator projection relative to the equator MN perpendicular to the plane of symmetry. If, in fact, we call L the latitude with respect to the equator, G the longitude with respect to the plane of symmetry, we have :

$$z = 2 \operatorname{tg}\left(\frac{\pi}{4} - \frac{L}{2}\right) e^{i G}, \qquad Z' = \operatorname{Log} \operatorname{tg}\left(\frac{\pi}{4} - \frac{L}{2}\right) + i\, G,$$

$$X' = \operatorname{Log} \operatorname{tang}\left(\frac{\pi}{4} - \frac{L}{2}\right), \qquad Y' = G.$$

8) *Choice of the quantities* μ **and** **C.** We have seen (3) that the shape of the region to be represented allows us to determine the ratio $\frac{\cos\,\mu}{\cos\,C}$, to choose the point P on the sphere as well as the great circle passing through this point which will be the circle of symmetry.

a) We have seen also that we may consider the isometer in the vicinity of the point P as symmetrical with respect to the arc of a circle of radius ρ_p. If the shape of the terrain lends itself especially to this representation, it will provide a value of ρ_p.

We have :

$$\rho_p = \frac{\cos\,\mu + \cos\,C}{\sin\,\mu}.$$

By adding to it :

$$\frac{\cos\,\mu}{\cos\,C} = t,$$

we adapt the projection in the best possible manner to the shape of the terrain by determining μ, C and N from the formulae :

$$\operatorname{tang}\,\mu = \frac{1 + t}{t\,\rho_p}, \qquad \operatorname{tang}\,C = \frac{\sqrt{t+1}}{\rho_p}\sqrt{t + 1 + \rho_p^2\,(t-1)},$$

$$\frac{1}{n} = \sqrt{1 + \rho_p^2\,\frac{t-1}{t+1}}.$$

If the shape of the terrain is rather more symmetrical with respect to the rectangular axes, we then employ the formulae of para. 6), Jung's projections, by making μ equal to zero and determining C in accordance with the eccentricity of the indicatrix ellipse.

b) Another relation which enables us to determine μ and C may also be obtained by simplifying the expressions for certain curves of the sphere.

Every circle of the sphere passing through I is represented by a straight line on the stereographic projection under consideration.

Let :

$$x \cos \alpha - y \sin \alpha = d$$

be the equation of such a line. From the general expression of the projection we deduce :

$$(13) \qquad \begin{aligned} x &= \left(\frac{\rho}{a}\right)^{\frac{1}{n}} \cos \frac{\omega}{n} - 2 \operatorname{cotg} C, \\ y &= \left(\frac{\rho}{a}\right)^{\frac{1}{n}} \sin \frac{\omega}{n}. \end{aligned}$$

The equation of any circle on the sphere passing through I will be :

$$(14) \qquad \left(\frac{\rho}{a}\right)^{\frac{1}{n}} \cos\left(\alpha + \frac{\omega}{n}\right) = 2 \cos \alpha \operatorname{cotg} C + d.$$

In the cases studied in paragraphs 4a to 5 where n is infinite, we see also that the equation for these circles is :

$$\rho = \operatorname{cotg} \frac{\mu}{2} e^{\frac{d + 2 \operatorname{tg} \frac{\mu}{2} \cos \alpha}{2 \sin \mu \cos \alpha}} e^{\omega \operatorname{tg} \alpha}.$$

In the case where n is zero (para. 6 and 7) we find :

$$e^{\rho' \operatorname{tg} \frac{C}{2} \cos \omega'} \cos\left(\alpha + \rho' \operatorname{tg} \frac{C}{2} \sin \omega'\right) = \cos C \cos \alpha + \frac{d}{2} \sin C.$$

If we have n = 2, equation (14) squared then gives us — substituting the coordinates X and Y for the polar coordinates — the following equation for the parabola whose axis makes an angle with the axis of X of twice that of the straight line :

$$(X \sin 2\alpha + Y \cos 2\alpha)^2 + 4a\,(2 \cos \alpha \operatorname{cotg} C + d)^2$$
$$[X \cos 2\alpha - Y \sin 2\alpha - a\,(2 \cos \alpha \operatorname{cotg} C + d)^2] = 0.$$

If n = $\frac{1}{2}$ we have an equilateral hyperbola, referred to its centre and whose asymptotes make the angles $\frac{\alpha}{2}$ with the axes :

$$X^2 - Y^2 - 2XY \operatorname{tg} \alpha = a^2 \frac{2 \cos \alpha \operatorname{cotg} C + d}{\cos \alpha}.$$

The values of n : 2 and $\frac{1}{2}$ thus permit us to represent all the circles passing through I by relatively simple curves; now we may always give to n one of these values. The shape of the region to be represented having imposed upon us the relation :

$$\frac{\cos \mu}{\cos C} = t ,$$

if $t > 1$, we take $n = \frac{1}{2}$, and we have then :

$$\operatorname{tg} C = 2 \sqrt{\frac{t^2 - 1}{3}} , \qquad \operatorname{tg} \mu = \sqrt{\frac{t^2 - 1}{3 t^2}} ;$$

if $t < 1$, we take $n = 2$; we have then :

$$\operatorname{tg} C = \sqrt{\frac{1 - t^2}{3}} , \qquad \operatorname{tg} \mu = 2 \sqrt{\frac{1 - t^2}{3 t^2}} .$$

Therefore, knowing C and μ, we may deduce from the position of the point P, the position of the points i, I, N, and M.

c) If the great circle of symmetry is the meridian of the point P, it may be advantageous to locate the point J at the pole.

In fact every circle passing through the point J has the following equation on the stereographic projection :

$$X^2 + Y^2 - 2\, ux - 2\, vy = 4 \operatorname{cotg} C\, (u + \operatorname{cotg} C) ,$$

an equation which becomes, according to formulae (13), the following easily calculated expression :

$$\frac{1}{2} \left(\frac{\rho}{a}\right)^{\frac{1}{n}} = (u + 2 \operatorname{cotg} C) \cos \frac{\omega}{n} + \imath \sin \frac{\omega}{n} .$$

If this circle is a great circle of the sphere we have :

$$u = \operatorname{tg} C - \operatorname{cotg} C ,$$

and its equation will be :

$$\frac{1}{2} \left(\frac{\rho}{a}\right)^{\frac{1}{n}} = \frac{2}{\sin 2\, C} \cos \frac{\omega}{n} + v \sin \frac{\omega}{n} .$$

If the point J is at the pole, a meridian of longitude G with respect to the circle of symmetry will have the following equation :

$$\left(\frac{\rho}{a}\right)^{\frac{1}{n}} = 4 \frac{\sin\left(G - \frac{\omega}{n}\right)}{\sin G \sin 2\, C} .$$

The point J is at a distance $\pi - 2C$ from the point i, the other pole at a distance of $2C$. The co-latitude of point P with reference to the elevated pole then gives us the smaller of the two numbers $C + \mu$ and $\pi - (C + \mu)$; this, added to the relation $\frac{\cos \mu}{\cos C} = t$ permits us to determine C and consequently μ.

d) Therefore we can obtain a simplification if we are able to take the points i and I as poles. On the stereographic projection the meridians are then straight lines passing through the origin and, according to equation (14), the meridians of longitude G will have as their equation :

$$\left(\frac{\rho}{a}\right)^{\frac{1}{n}} = 2\frac{\sin G \operatorname{cotg} C}{\sin\left(G - \frac{\omega}{n}\right)}$$

The position of the pole determines for us the value of $\mu - C$ and we deduce from it :

$$\operatorname{tg} \mu = \frac{1 - t \cos(\mu - C)}{t \sin(\mu - C)}, \qquad \operatorname{tg} C = \frac{\cos(\mu - C) - t}{\sin(\mu - C)},$$

$$n = \frac{1 - t\cos(\mu - C)}{\cos(\mu - C) - t}.$$

The parallels of co-latitude λ in this case will also have a more simplified expression, as follows :

$$\frac{1}{2}\left(\frac{\rho}{a}\right)^{\frac{1}{n}} = \operatorname{cotg} C \cos\frac{\omega}{n} \pm \sqrt{\operatorname{tg}^2\frac{\lambda}{2} - \operatorname{cotg}^2 C \sin^2\frac{\omega}{n}}.$$

II. DOUBLE CIRCULAR CONFORMAL PROJECTIONS.

Let us substitute in the equation of the conical conformal projections :

$$Z = a\,(z + 2 \operatorname{cotg} C)^{n}$$

for the coordinate z, derived from the view-point I, its value as a function of the coordinate z' calculated from the view-point M. We then have :

$$z = \frac{z' + 2 \operatorname{tg}\frac{C}{2}}{1 - \frac{1}{2} z' \operatorname{tg}\frac{C}{2}},$$

and, making the substitution :

$$Z = a \left(\frac{2}{\sin C}\right)^{n} \left(\frac{1 + \frac{1}{2} z' \operatorname{tg}\frac{C}{2}}{1 - \frac{1}{2} z' \operatorname{tg}\frac{C}{2}}\right)^{n}. \tag{15}$$

The projection Z has been established with respect to the view-point I. If, under these same conditions, we have established a projection ζ with respect to the view-point J, this projection will not differ from the preceding except for the change of z' to $-z'$ (in order to retain the same convention for the signs).

We have therefore :

$$Z\,\zeta = a^{2} \left(\frac{2}{\sin C}\right)^{2n}.$$

Thus, replacing the view-point I by J, is equivalent to effecting an inversion or transformation by means of reciprocal radii vectores. It is the same if, in the expression for Z we substitute for n the quantity — n. We obtain by means of these transformations a new projection which is of the same kind as the first and which does not differ except for the position of the view-point and the scale factor. Thus we may dispense with a study of the conical projections in which the exponent n becomes negative.

This transformation by inversion is made starting from the origin of the conical projection Z. Evidently it should replace the system of straight lines converging on the point representing J and representing the circles of the sphere which pass through I and J, by another bundle converging on the point which is going to represent I in the ζ projection. This point will also be the common centre of the circles representing the 2nd system of circles of the sphere. Thus we shall again obtain a conformal conic projection. This applies also to the projections studied above for which n equals zero or infinity, when they are transformed by inversion.

This would not be the same if the transformation were effected from a point which is not the origin of the projection Z. Let L be any point whatever in this projection, of which we shall call L the complex coordinate:

$$L = l\, e^{i\varphi} .$$

We shall study the projection defined by :

$$\zeta = \frac{K^2}{Z - L}$$

On this projection, the point L will be at infinity and the circle with the centre j and the radius l on which it is located will be represented by a straight line. The origin of the ζ projection will be represented by the point I; as for the point J it will have the value $-\frac{K^2}{L}$ for the coordinate. The expression for ζ contains the parameter $\frac{K^2}{a}$; in the following discussion we may therefore give a the value of unity.

The system of circles of the sphere, passing through I and J, will become a system of circles passing through the origin and through the point of the coordinate $-\frac{K^2}{L}$. The second system of circles of the sphere will become a system of circles orthogonal to the preceding and whose centres will consequently be located on the straight line joining the origin with the point of the coordinate $-\frac{K^2}{L}$. The projection ζ is therefore certainly a double circular conformal projection. There will be duplication if $n > 1$. Its coordinates R and α may be deduced from the following formulae for the coordinates ρ and ω of the projection Z :

$$R = \frac{K^2}{\sqrt{\rho^2 + l^2 - 2\, l\rho \cos(\omega - \varphi)}}, \qquad \operatorname{tang} \alpha = - \frac{\rho \sin \omega - l \sin \varphi}{\rho \cos \omega - l \cos \varphi} .$$

The scale M of the ζ projection can be deduced from that of m in the projection Z by the formula :

$$M = \frac{K^2\, m}{\rho_2 + l^2 - 2\, l\, \rho \cos(\omega - \varphi)} .$$

The denominator of the expression for M represents the square of the distance, on the projection Z, from the point in question to point L. M will pass through a maximum or a minimum for a given value of ω if we have :

$$\text{(16)} \qquad \frac{\delta m}{\delta \rho} = 2\, m \frac{\rho - l \cos(\omega - \varphi)}{\rho^2 + l^2 - 2\, l\rho \cos(\omega - \varphi)} .$$

The second member is negative between the point j and the foot of the perpendicular dropped from L on the radius vector in question in the Z projection. It is thereafter positive. This formula allows us to determine whether there will be a maximum or minimum of M in a specified region.

One circle of the sphere of each of the two systems considered passes through the point L. The transformation by inversion then transforms them into two rectangular straight lines which meet at the point L' with the coordinates :

$$\zeta_{L'} = -\frac{K^2}{2\,L}\,.$$

If we shift the origin of coordinates to this point the expression for the projection then becomes, (by taking as axes the two rectangular straight lines of which we have just spoken) :

$$\zeta' = -\left(\zeta + \frac{K^2}{2\,L}\right)e^{\varphi i} = \frac{-K^2}{2\,l}\,\frac{Z+L}{Z-L} = \xi' + i\,\eta'\,.$$

All of the circles representing the circles of the sphere of the 1st system will have their centres on the axis of η' ; all those which represent the circles of the second system on the axis ξ'.

At the points representing J and I, with the coordinates $\pm\frac{K^2}{2\,l}$ the circles of the first system make between them an angle equal to n times that which they make on the sphere. We find their equations as follows :

The coordinates ξ' and η' deduced from the expression of ζ' are :

$$\xi' = \frac{-K^2}{2\,l}\,\frac{\rho^2 - l^2}{\rho^2 + l^2 - 2\,l\rho\cos(\omega-\varphi)}$$

$$\eta' = +\frac{K^2}{2\,l}\,\frac{2\,l\rho\sin(\omega-\varphi)}{\rho^2 + l^2 - 2\,l\rho\cos(\omega-\varphi)}\,.$$

We deduce from the first equation :

$$\cos(\omega-\varphi) = \frac{\xi'(\rho^2+l^2) + \dfrac{K^2}{2\,l}(\rho^2 - l^2)}{2\rho\,l\,\xi'}\,,$$

Then from the two equations :

$$\xi'^2 + \eta'^2 = \frac{K^4}{4\,l^2}\,\frac{\rho^2 + l^2 + 2\rho\,l\cos(\omega-\varphi)}{\rho^2 + l^2 - 2\rho\,l\cos(\omega-\varphi)}\,.$$

By substituting for cos $(\omega - \varphi)$ in this equation its value given above, we shall obtain the equation of the circles of the second kind, corresponding to a constant value of ρ :

$$\left(\xi' + \frac{K^2}{2\,l}\,\frac{\rho^2 + l^2}{\rho^2 - l^2}\right)^2 + \eta'^2 = \left(\frac{K^2\,\rho}{l^2 - \rho^2}\right)^2.$$

The radius of these circles, which all have their centres on the axis of ξ', as we have seen, is therefore :

$$\frac{K^2\,\rho}{l^2 - \rho^2}.$$

In the same manner we find the equation for the circles of the first kind :

$$\xi'^2 + \left[\eta' \pm \frac{K^2}{2\,l}\,\operatorname{cotg}\,(\omega - \varphi)\right]^2 = \left(\frac{K^2}{2\,l\,\sin\,(\omega - \varphi)}\right)^2.$$

We only make use of the portion of the circles of the first kind which lies on the same side of the straight line IJ. All of the sphere will therefore be covered when ω varies from zero to $n\,\pi$.

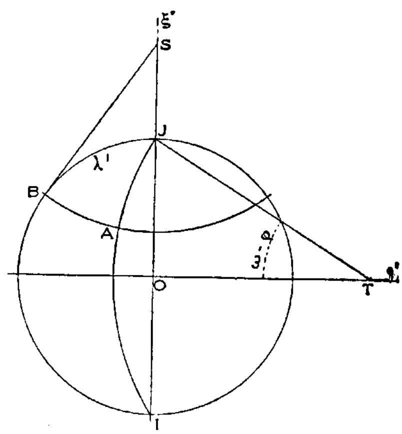


FIG. 10

If T and S are the centres of the two circles of the first kind and second kind corresponding to the point A, IJ is equal to $\frac{K^2}{l}$ (See fig. 10), the angle JTO to $\omega - \varphi$. If we call λ' the arc JB, we have :

$$O\,B = S\,B\ \operatorname{cotg}\,\lambda'\ ;$$

whence :

$$\operatorname{tg}\,\lambda' = \frac{2\,l\,\rho}{l^2 - \rho^2}$$

and :

$$\operatorname{tg} \frac{\lambda'}{2} = \frac{\rho}{l} .$$

The circle of diameter IJ corresponds to a value of $\omega - \varphi$ equal to $\pm \frac{\pi}{2}$; that is, to a straight line of the pencil of the projection Z perpendicular to jL. The straight line OT corresponds to a value of ρ equal to l.

1°) $\mathbf{n = 2, C = \frac{\pi}{2}}$. We have seen that, in this case, by taking the points I and J diametrically opposite, on the equator, the projection Z represented the line of azimuth by a circle passing through the pole. The transformation by inversion, obtained by taking the point L at the pole, will transform this circle into a straight line and we then return to the Littrow projection. Thus we have :

$$Z = z^2, \qquad L = -4, l = 4, \qquad \zeta' = \frac{K^2}{8} \frac{1 - \frac{1}{4} z^2}{1 + \frac{1}{4} z^2} .$$

Instead of using the stereographic projection z from J, let us employ the stereographic projection z' at the pole. We shall have :

$$z = 2\, i \,\frac{1 - \frac{1}{2} z'}{1 + \frac{1}{2} z'} , \qquad \zeta' = \frac{K^2}{8} \left(\frac{1}{z'} + \frac{1}{4} z' \right) .$$

If λ is the colatitude of a point; G its longitude with respect to the meridian parallel to the plane tangent at I,

$$z' = 2 \operatorname{tg} \frac{\lambda}{2} (\sin G + i \cos G) ;$$

whence :

$$\zeta' = \frac{K^2}{8} \frac{\sin G - i \cos G \cos \lambda}{\sin \lambda} ,$$

$$\xi' = \frac{K^2}{8} \frac{\sin G}{\sin \lambda} , \qquad \eta' = -\frac{K^2}{8} \cos G \operatorname{cotg} \lambda .$$

These are essentially the formulae for the Littrow projection, which consequently may be considered as a spherical projection.

All the great circles of the sphere passing through the points on the equator at longitude 90° and 270° are represented by circles; and the same is true of all the small circles orthogonal to them.

2) $\mathbf{n = 1}$. *Stereographic Projection.* The Z projection is a stereographic projection, which, as such, represents the circles on the sphere by

circles or straight lines. A transformation by inversion will still result in the circles of the sphere being represented by circles and straight lines; this will again be a stereographic projection in which the view-point, instead of being at I is transposed to L. It is easy to demonstrate this with the formulae. Thus, if L is on the sphere, distant from point i by the arc λ_0, making an angle of ψ_0 with the plane $MJNI$, we have :

$$L - Z = 2 \operatorname{tg} \lambda_0 \, e^{i\psi_0} - z .$$

Let us turn the axes of z in such a manner that it will cause the axis of x' to pass through the point L', diametrically opposite to L. We shall have :

$$z = - z' \, e^{i\psi_0} .$$

If we now transfer the origin of coordinates to L', we have :

$$z' = \frac{z'' + 2 \cot g \frac{\lambda_0}{2}}{1 - \frac{1}{2} z'' \cot g \frac{\lambda_0}{2}} ,$$

Consequently :

$$L - Z = \frac{4 \, e^{i\psi_0} .}{\sin \lambda_0 \left(1 - \frac{1}{2} z'' \cot g \frac{\lambda_0}{2} \right)} .$$

We have therefore :

$$\zeta = \frac{K^2}{4} \cos^2 \frac{\lambda_0}{2} \, e^{- i\psi_0} \left(2 \operatorname{tg} \frac{\lambda_0}{2} - z'' \right) ,$$

which is clearly the stereographic projection z'' referred to the origin I, with a change of scale.

3) *Symmetrical Projection.* The formula for the ζ' projection contains five arbitrary parameters. It will be advantageous to eliminate some of them in order to make the projection symmetrical with respect to the arc of the great circle JNI of the sphere. For this, it is necessary to locate the point L on the circle, and we shall assume for its coordinates the value of $- l$. In this manner, this point, which is displaced to infinity on the ζ' projection, will, if n is greater than unity, be in a region where there is duplication, where the projection is not utilised in consequence and, if n is less than unity, entirely outside of the representation of the sphere, which is all at a finite distance.

In these conditions the origin of the ζ' projection will be at the point L' with the coordinate $Z = l$; and we shall have :

$$\zeta' = \frac{-K^2}{2\,l}\,\frac{Z - l}{Z + l}\,.$$

The necessary condition for the modulus M to pass through a minimum or a maximum on the circle of symmetry will then become, according to the equation (16) :

$$\frac{\delta\, m}{\delta\, \rho} = \frac{2\, m}{\rho + l};$$

which gives the equation :

$$(17) \qquad \left(\rho^{\frac{2}{n}} - \frac{4}{\sin^2 C}\right)(\rho + l) = 4\, m\, \rho^{\frac{1-n}{n}}\,(\rho - l)\,.$$

We see that $\rho - l$ and $\rho - \left(\frac{2}{\sin\, C}\right)^n$ should be of the same sign in order that this equality should be possible. The point of the coordinate :

$$\rho = \left(\frac{2}{\sin\, C}\right)^n$$

is the point N of the sphere; the scale M cannot have either a maximum or a minimum between the points N and L'.

Let P (Fig. 1) be a point on the great circle of symmetry for which the scale M presents a minimum. Again let us call μ the arc NP, reckoned positively from N towards J and negatively from N towards I. The value ρ_p which corresponds to the point P is :

$$\rho_p = \left(\frac{2\cos\frac{\mu + C}{2}}{\sin C\cos\frac{\mu - C}{2}}\right)^n.$$

We have therefore :

$$m_p = n\;\rho_p\,\frac{\sin\, C}{\cos\, C + \cos\, \mu}\,.$$

Substituting these values in equation (17) we find that the condition in which the point P shall correspond to a maximum or to a minimum of M is given by :

$$l = \frac{n\, \sin\, C + \sin\, \mu}{n\, \sin\, C - \sin\, \mu}\,\rho_p\,.$$

The position of the point L' is fixed by this relation, which shows us also that, the quantity l having to be positive, we must have :

$$\text{(18)} \qquad n^2 > \left(\frac{\sin \mu}{\sin C}\right)^2 \quad (*)$$

Finally, we determine the value of K^2 by the condition that the scale M shall be equal to unity at the point P, which requires that :

$$K^2 = 4\, n \sin C \frac{\cos C + \cos \mu}{(n \sin C - \sin \mu)^2} \rho_p .$$

The parameters μ, n, and C then permit us to make of this projection a projection of minimum deformation in a region of limited extent around point P.

For this purpose, we shall expand the value of M in the vicinity of the point P. We shall obtain :

$$M = 1 + \left(\frac{\rho - \rho_p}{2\, m_p}\right)^2 \left[1 + \frac{(1 - n^2) \sin^2 C}{(\cos C + \cos \mu)^2}\right]$$

$$+ \left(\frac{\rho_p\, \omega}{2\, m_p}\right)^2 \left[1 + \frac{(n^2 - 1) \sin^2 C}{(\cos C + \cos \mu)^2}\right] + \cdots\cdots$$

On the other hand, if we move the origin of ζ' to the point corresponding to P, we shall have :

$$\text{(19)} \qquad \zeta'' = \frac{-K^2}{\rho_p + l} \frac{Z - \rho_p}{Z + l} = -2 \frac{\cos C + \cos \mu}{n \sin C - \sin \mu} \frac{Z - \rho_p}{Z + l}$$

$$= -2 \frac{(\cos C + \cos \mu)\,(Z - \rho_p)}{n \sin C\,(z + \rho_p) - \sin \mu\,(z - \rho_p)} .$$

From this we deduce the principal values of ξ'' and η'' :

$$\xi'' = \frac{\rho_p - \rho}{m_p}, \qquad \eta'' = -\,\omega \frac{\rho_p}{m_p};$$

and consequently :

$$M = 1 + \frac{\xi''^2}{4}\left[1 + \frac{(1 - n^2) \sin^2 C}{(\cos C + \cos \mu)^2}\right] + \frac{\eta''^2}{4}\left[1 + \frac{(n^2 - 1) \sin^2 C}{(\cos C + \cos \mu)^2}\right] + \cdots$$

(*) This inequality shows that for the same values of C and μ, the value of the parameter n of the double circular projections will always be greater than that of the same parameter in the conformal conic projections.

The major axis of the ellipse is oriented along the axis of η'' if n is smaller than unity, and follows that of ξ'' if n is greater than unity.

But the modulus will not be a minimum at P except when the factors of ξ''^2 and of η''^2 are positive.

For the factor of η''^2, this imposes on us the condition :

$$n^2 \sin^2 C > \sin^2 \mu - 2 \cos C (\cos C + \cos \mu) ,$$

which will always be verified if the inequality (18) is verified.

For the factor of ξ''^2 we have

$$n^2 \sin^2 C < \sin^2 \mu + 2 \cos \mu (\cos C + \cos \mu) . \tag{20}$$

μ cannot vary in absolute value except between the limits of zero and $\pi - C$. The term $\cos C + \cos \mu$ is therefore always $\geqslant 0$.
It is necessary that $\cos \mu$ also be $\geqslant 0$, and consequently that μ be smaller in absolute value than $\frac{\pi}{2}$.

NOTE :— The inequalities (18) and (20) may also be written :

$$\cos^2 \mu > 1 - n^2 \sin^2 C ; \tag{18 a}$$

$$\left(\frac{\cos \mu + \cos C}{\sin C}\right)^2 > n^2 - 1 . \tag{20 a}$$

α) We see therefore that if n is less than 1, the inequality (20a) is always verified; it will suffice therefore if the inequality (18) is also verified; and we see that μ should be, in absolute value, less than C and less than a value of μ_0 such that :

$$\sin \mu_0 = n \sin C .$$

If n is greater than 1, let us put :

$$n^2 = \frac{1}{\sin^2 C} + \varepsilon .$$

where ε is either a positive or a negative quantity :

β) Where ε is positive, that is, when n is greater than $\frac{1}{\sin C}$, the inequality (18a) is always verified; the inequality (20a) gives the condition :

$$\cos \mu > \sqrt{n^2 - 1} \sin C - \cos C ,$$

which is possible only when n is less than $\frac{1}{\sin \frac{C}{2}}$.

In that case we should have :

$$\frac{1}{\sin C} < n < \frac{1}{\sin \frac{C}{2}};$$

and μ should be smaller in absolute value than a certain value μ_0 such as :

$$\cos \mu_0 = \sqrt{n^2 - 1} \sin C - \cos C.$$

If n has a value of $\frac{1}{\sin C}$, μ may have any value at all comprised between the values of $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

If n has a value of $\frac{1}{\sin \frac{C}{2}}$, μ will necessarily be zero.

γ) When ε is negative, the inequality (20a) is always verified, because μ is smaller than $\frac{\pi}{2}$ in absolute value. There remains only the inequality (18) then to be satisfied; which requires that the absolute value of μ should be less than C, or should not exceed C by more than a certain quantity.

We see therefore that there is no point at which this double circular projection shows a minimum deformation if n is greater than $\frac{1}{\sin \frac{C}{2}}$.

δ) In accordance with the inequalities (18) and (20) we may, calling q any angle whatever comprised between O and $\frac{\pi}{2}$, write :

$$n^2 \sin^2 C = \sin^2 \mu + 2 \sin q \cos \mu (\cos C + \cos \mu);$$

and we readily see that the distance $L'P$, on this projection, is given by the expression :

$$L' P = \frac{1}{\sin q} \operatorname{tang} \mu.$$

4°) $\mathbf{n}^2 = \frac{\sin^2 \mu}{\sin^2 \mathbf{C}}$. *Conic Projection.* In this case if μ is > 0, l becomes infinite; if μ is < 0, l is zero.

In the first case we find again the conformal conic projection :

$$\zeta'' = \frac{\cos C + \cos \mu}{\sin \mu} \left(1 - \frac{Z}{\rho_p}\right),$$

and in the second case :

$$\zeta'' = \frac{\cos C + \cos \mu}{\sin \mu}\left(1 - \frac{\rho_p}{Z}\right),$$

an expression which, according to what has been stated above, also represents a conformal conic projection.

5°) $\mathbf{n^2 \sin^2 C = \sin^2 \mu + 2 \cos \mu (\cos C + \cos \mu)}$. The coefficient of ξ''^2 in the development of M will be zero; the expression for the scale reduces to :

$$M = 1 + \frac{\eta''^2}{2} + \ldots\ldots$$

The isometer, in the vicinity of the point P is formed by two straight lines parallel to the axis of ξ'' ; the variation of the scale y is as slow as possible on the circle of symmetry.

6) $\mu = 0$. The circle of the second order which passes through the point L' was represented on the preceding projections by a straight line parallel to the axis of η ; but the projection was not symmetrical with regard to that axis. It will become so if we move the point L' to the point N and if the minimum of the scale occurs there.

We then have, taking into account formula (15):

$$l = \rho_p = \left(\frac{2}{\sin C}\right)^n,$$

$$\zeta' = \frac{2}{n} \operatorname{cotg} \frac{C}{2} \frac{\left(1 - \frac{1}{2}\zeta' \operatorname{tg} \frac{C}{2}\right)^n - \left(1 + \frac{1}{2}\zeta' \operatorname{tg} \frac{C}{2}\right)^n}{\left(1 - \frac{1}{2}\zeta' \operatorname{tg} \frac{C}{2}\right)^n + \left(1 + \frac{1}{2}\zeta' \operatorname{tg} \frac{C}{2}\right)^n}$$

Z' being the coordinate of the stereographic projection with respect to the view-point M.

The inequalities (18) and (20) become :

$$0 < n < \frac{1}{\sin \frac{C}{2}}$$

The ratio of these axes of the indicatrix ellipse is :

$$1 - 2 \frac{(1 - n^2) \sin^2 \frac{C}{2}}{1 - n^2 \sin^2 \frac{C}{2}}.$$

The parameters n and C may be chosen such that the projection will become a projection of minimum deformation.

7) $n = 0$, $\mu = 0$ *Jung's Projections.* It is easy to prove that we can bring this back to the projections of Jung. In fact the expression for ζ' becomes at the limit :

$$\zeta' = - \operatorname{cotg} \frac{C}{2} \operatorname{Log} \frac{1 + \frac{1}{2} z' \operatorname{tg} \frac{C}{2}}{1 - \frac{1}{2} z' \operatorname{tg} \frac{C}{2}} .$$

If we substitute for z' its value as a function of the stereographic coordinate z'', corresponding to the view-point J, and consequently with the sign reversed :

$$z' = \frac{z'' - 2 \operatorname{tg} \frac{C}{2}}{1 + \frac{1}{2} z'' \operatorname{tg} \frac{C}{2}} .$$

We shall have :

$$\zeta' = \operatorname{cotg} \frac{C}{2} \operatorname{Log} \left(\tfrac{1}{2} z'' \sin C + \cos C \right),$$

which is, notwithstanding the scale factor, the expression obtained in (6 — first part).

8) $C = \frac{\pi}{2}$. The symmetrical projection, which is given by the expression, 19) still contains three arbitrary parameters; a single one will be determined by the condition of minimum deformation. We shall obtain considerable simplification by adopting for C the value of $\frac{\pi}{2}$.

The expressions for ζ'', ρ_p and M become :

$$\zeta'' = - 2 \frac{\cos \mu}{n - \sin \mu} \frac{z^n - \rho_p}{z^n + l} = - \frac{2 \cos \mu \, (z^n - \rho_p)}{n \, (z^n + \rho_p) - \sin \mu \, (z^n - \rho_p)} ,$$

$$\rho_p = \left(\frac{2 \cos \mu}{1 + \sin \mu} \right)^n ,$$

$$M = 1 + \frac{\xi''^2}{4} \left(1 + \frac{1 - n^2}{\cos^2 \mu} \right) + \frac{\eta''^2}{4} \left(1 + \frac{n^2 - 1}{\cos^2 \mu} \right) .$$

This becomes particularly useful if we can locate the points I and J at the terrestrial poles, because then the meridians and the parallels will be represented by circles. The plane of symmetry will be that which contains the poles and the point L'. If we assign to n a value less than unity, the entire sphere can be represented without duplication. The value of n^2 must be further comprised between $\sin^2\mu$ and $1 + \cos^2\mu$; n cannot therefore have a greater value than $\sqrt{2}$, which it will only attain when μ is zero.