STEREOGRAPHIC PROJECTIONS

by

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The stereographic projection, or the conformal perspective, of a limited region of the sphere about the point of origin, is a perspective on the plane tangent to the origin (1) taken from the point of view located on the sphere at the second point of intersection with the normal, the same being also the diameter passing through the origin. It possesses some remarkable properties of which the following are the most important:—it is conformal; the geodetics passing through the origin, which are at the same time normal sections, are here represented by concurrent straight lines ; all of the circles on the surface are here transformed into circles ; the curves of equal linear distortion (indicatrices or isometres in the sense of Tissot) are here rigorously circles on the plane of projection.

This projection is one of the simplest and often the most advantageous for use as a working projection and for the calculation of the rectangular co-ordinates : but the problem of the projection should be treated on the terrestrial ellipsoid and not on the sphere. None of the projections on the ellipsoid possessing all of the properties above-mentioned, it is necessary to make a choice which is necessarily somewhat arbitrary, and to adopt a projection which will retain one or several of these characteristics and which will approach more or less closely to the stereographic projection of the sphere. Several authors have therefore adopted different projections of the ellipsoid which they have called stereographic projections.

We wish here to review the principal, and to compare them amongst themselves : we shall treat those in particular which have the important property of being conformal ; but none of the other characteristics can be exactly obtained at the same time.

I

In particular, no perspective of the ellipsoid will furnish a projection which is strictly conformal. That which approaches most closely to conformality for the restricted region in the vicinity of the origin would have its point of view on the normal to the ellipsoid at the origin, not at its second point of intersection with the surface, but at a distance from the

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(1) For the purpose of diminishing the linear distortion at the periphery, one may project on to a secant plane parallel to the tangent plane. This does not alter anything in the theory and, to avoid complicating the discussion, we shall not speak further of it.

origin equal to twice the radius of total curvature $\sqrt{N_0 \rho_0}$ at that point : the geometric mean of the radii of curvature ρ_0 and N_0 of the meridian and of the normal section which is perpendicular to it. The sections normal to the origin are here represented by converging straight lines, the meridians and the parallels by conic sections.

II

In like manner, no conformal projection of the ellipsoid may represent the geodetics passing through the origin, by concurrent straight lines.

If one plots on the ellipsoid all of the geodetics passing through the origin and if one lays off on each of them, from the same point, an equal distance σ, the locus of their extremities is a curve normal to the geodetics to which Gauss has given the name "geodetic circle" (2). By changing the value of σ, we obtain thus on the ellipsoid a system of geodetics and lines which are orthogonal to them. If it were possible to make this grid correspond to that which, on the plane, is formed by the concurrent straight lines and the concentric circles, we should have a representation which would have important characteristics in common with the stereographic projection of the sphere ; but the correspondence cannot be realized. This projection has been proposed (3), seemingly without its impossibility having been taken into consideration. The geodetics which form one of the families in this grid have in fact a constant geodetic curvature equal to zero ; but in order that this grid should be isometric, it would be necessary that the other family which is orthogonal to it, should also have a constant curvature ; but this is not the case for the geodetic circles (4). It has seemed interesting to us, however, to demonstrate directly this impossibility.

At the point M let $d\sigma$ represent the element of length of a geodetic line passing through the origin, and $\triangle$ the element of the geodetic circle which is normal to it at the same point. Let us take the point of origin as the axis of co-ordinates, the normal to the ellipsoid at this point as the axis of z, the tangent to the meridian as the axis of x and the perpendiculars to xz as the axis of y. We then express the co-ordinates of the point M as a function of the length σ by taking first an axis of x_1 tangent to the geodetic line at the origin and the perpendicular axis y_1 The following formulae are then readily deduced from those of E. Fichot given in his article :— "Sur la réduction au sphéroïde terrestre des données fournies par les opérations de la triangulation" : (On the reduction to the terrestrial spheroid of the data furnished by the triangulation operations) : *Annales*

(2) « Disquisitiones generales circa superficies curvas » Art. 15 and 16.
(3) « Die stereographische Abbildung des Erdellipsoïds » by O. Eggert : Zeitschrift für Vermessungswesen, March 1936, p. 164.
(4). See « Théorie générale des surfaces » by Darboux, III[e] Partie, livre VI, chapter VII.

Hydrographiques, 1907, p. 55 *et seq.* We shall retain the greater part of the terms of the 4th order in σ.

$$\begin{cases} x_1 = \sigma - \frac{1}{6} r_o^2 \sigma^3 - \frac{1}{8} r_o \lambda_o \sigma^4 \\ y_1 = \quad - \frac{1}{6} r_o s_o \sigma^3 - \frac{r_o \mu_o + 2 s_o \lambda_o}{24} \sigma^4 \\ z = \frac{1}{2} r_o \sigma_o^2 + \frac{1}{6} \lambda_o \sigma^3 + \frac{a_o - 4 r_o^3 - 4 r_o s_o^2}{24} \sigma^4 \end{cases}$$

$$\begin{cases} x_1^2 = \sigma^2 - \frac{r_o^2}{3} \sigma^4 - \frac{r_o \lambda_o}{4} \sigma^5 \\ y_1^2 = \frac{r_o s_o^2}{36} \sigma^6 \end{cases} \qquad \begin{cases} x = x_1 \cos \omega - y_1 \sin \omega \\ y = y_1 \cos \omega + x_1 \sin \omega \end{cases}$$

$$\frac{\Delta^2}{d\omega^2} = \left(\frac{d}{d\omega}\right)^2 + \left(\frac{dy}{d\omega}\right)^2 + \left(\frac{dz}{d\omega}\right)^2 = \left(\frac{dx_1}{d\omega}\right)^2 + \left(\frac{dy_1}{d\omega}\right)^2 + \left(\frac{dz}{d\omega}\right)^2 + x_1^2 + y_1^2 + 2\left(x_1 \frac{dy_1}{d\omega} - y_1 \frac{dx_1}{d\omega}\right)$$

The coefficients r_o, s_o, λ_o, μ_o and a_o are the partial second, third and fourth derivatives of z with respect to x_1 and y_1 ; ω is the azimuth of the geodetic. We shall call L_o the latitude of the point of origin : e (5) the eccentricity. We shall also utilize the following equations :

$$r_o = \frac{\sin^2 \omega}{N_o} + \frac{\cos^2 \omega}{\rho_o} \qquad s_o = \left(\frac{1}{N_o} - \frac{1}{\rho_o}\right) \sin \omega \cos \omega$$

$$\lambda_o = -3 \frac{r_o s_o}{\sin \omega} \operatorname{tg} L_o \qquad \mu_o = \left(\frac{r_o}{\cos \omega} - 2 \frac{s_o}{\sin \omega}\right) s_o \operatorname{tg} L_o$$

and we find, finally :

$$\Delta^2 = \left(\sigma^2 - \frac{1}{3} \frac{\sigma^4}{N_o \rho_o} - \frac{2}{3} \frac{e^2}{1-e^2} \sin L_o \cos L_o \cos \omega \frac{\sigma^5}{N_o^2 \rho_o} + \dots\right) d\omega^2$$

$$\Delta = \left(1 - \frac{1}{6} \frac{\sigma^2}{N_o \rho_o} - \frac{1}{3} \frac{e^2}{1-e^2} \sin L_o \cos L_o \cos \omega \frac{\sigma^3}{N_o^2 \rho_o}\right) \sigma \, d\omega$$

If l is a length which, on projection, corresponds to σ, the condition of conformality is given by the proportion :

$$\frac{dl}{l} = \frac{d\sigma}{\sigma} \left(1 + \frac{1}{6} \frac{\sigma^2}{N_o \rho_o} + \frac{1}{3} \frac{e^2}{1-e^2} \sin L_o \cos L_o \cos \omega \frac{\sigma^3}{N_o^2 \rho_o}\right),$$

of which the integration furnishes the concordant relation between l and σ ; but this relation will express l as a function of ω, such that the transformation of the geodetic circle will not be a circle and will not be normal to the concurrent straight lines which represent the geodetics. It is therefore impossible to establish a conformal projection on this basis, unless by neglecting the terms in $e^2\sigma^3$ in the preceding equation, that is, in operating in a sphere of radius $\sqrt{N_o \rho_o}$. It is, however, necessary to make exception of the case in which the origin is at the pole ; all of the meridians, which are then also geodetics, will be represented by straight lines. The term in $e^2\sigma^3$ cancels out also when the origin is on the equator ; but by pushing the development still further, it will be found that there remain in this case some terms of a higher order containing ω.

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(5) e will be employed for the eccentricity of the terrestrial ellipsoid; e to designate the number which serves as basis for the neaperian logarithms.

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page 27 : third formula read :

$$\left(\frac{\delta X_1}{\delta \omega}\right)^2 + \left(\frac{\delta Y_1}{\delta \omega}\right)^2 + \left(\frac{\delta Z}{\delta \omega}\right)^2 + X_1 \frac{\delta Y_1}{\delta S} - Y_1 \frac{\delta X_1}{\delta S} + \frac{\delta S}{\delta \omega} = 0$$

page 37 : instead of **VIII** read **VII**

III

We obtain an analogous result if, in place of considering the geodetics, we try to make the grid containing the normal sections passing through the origin and their orthogonal curves correspond to the plain grid.

Let X, Y, Z be the co-ordinates of the point M of a straight section of length S referred to the same axis as above, X_1 an auxiliary axis tangent to the geodetic coming from the origin and passing through the point M. This axis X_1 makes an angle ω with the axis of the X and a very small angle δ with the normal section passing through M. The formulae of E. Fichot in the work cited above permit us to calculate the co-ordinates X_1, Y_1, Z and the angle δ as a function of the length S. Since, however, δ does not differ from σ, except after the 5th degree, the expressions for X_1, Y_1, and Z are the same as those which were given for x_1, y_1, z, by replacing there σ by S. The expression of δ is :

$$\delta = -\frac{1}{6} r_o s_o S^2 - \frac{1}{24}\left(r_o \mu_o + 2 s_o \lambda_o\right) S^3.$$

The condition of perpendicularity of the normal section and its orthogonal curve will be :—

$$\left(\frac{\delta X}{\delta \omega} d\omega + \frac{\delta X}{\delta S} dS\right)\frac{\delta X}{\delta S} + \left(\frac{\delta Y}{\delta \omega} d\omega + \frac{\delta Y}{\delta S} dS\right)\frac{\delta Y}{\delta S} + \left(\frac{\delta Z}{\delta \omega} d\omega + \frac{\delta Z}{\delta S} dS\right)\frac{\delta Z}{\delta S} = 0,$$

or, by passing to the axes X_1, Y_1 :—

$$\left(\frac{\delta X_1}{\delta \omega}\right)^2 + \left(\frac{\delta Y_1}{\delta \omega}\right)^2 + \left(\frac{\delta Z}{\delta \omega}\right) + X_1 \frac{\delta Y_1}{\delta S} - X_1 \frac{\delta X_1}{\delta S} + \frac{\delta S}{\delta \omega} = 0$$

The square $\triangle^2$ of the element of the orthogonal curve will have a value, taking into consideration the condition of perpendicularity, of:

$$\Delta^2 = \Sigma\left(\frac{\delta X}{\delta \omega} d\omega + \frac{\delta X}{\delta S} dS\right)\frac{\delta X}{\delta \omega} d\omega = \Sigma\left(\frac{\delta X}{\delta \omega}\right)^2 d\omega^2 + \Sigma \frac{\delta X}{\delta \omega}\frac{\delta X}{\delta S} dS\, d\omega\,;$$

or, passing to the axes X_1, Y_1 :

$$\frac{\Delta^2}{\delta\omega^2} = X_1^2 + Y_1^2 + \Sigma\left(\frac{\delta X_1}{\delta \omega}\right)^2 + 2\left(X_1 \frac{\delta Y_1}{\delta \omega} - Y_1 \frac{\delta X_1}{\delta \omega}\right)$$
$$- \Sigma\left(\frac{\delta X_1}{\delta \omega}\right)^2 \Sigma \frac{\delta X_1}{\delta \omega}\frac{\delta X_1}{\delta S} - \left(X_1 \frac{\delta Y_1}{\delta S} - Y_1 \frac{\delta X_1}{\delta S}\right) \Sigma\left(\frac{\delta X_1}{\delta \omega}\right)^2$$
$$- \left(X_1 \frac{\delta Y_1}{\delta S} - Y_1 \frac{\delta X_1}{\delta S}\right) \Sigma \frac{\delta X_1}{\delta \omega}\frac{\delta X_1}{\delta S} - \left(X_1 \frac{\delta Y_1}{\delta S} - Y_1 \frac{\delta X_1}{\delta S}\right)^2$$

The terms of the second line are at least of the 7th degree and those of the third of the 6th degree. Therefore, by stopping at the terms of the 5th order, we shall have the same value for $\triangle^2$ as for the geodetics :

$$\Delta^2 = \left(S^2 - \frac{1}{3}\frac{S^4}{N_o \rho_o} - \frac{2}{3}\frac{e^2}{1-e^2} \sin L_o \cos L_o \cos\omega \frac{S^5}{N_o^2 \rho_o}\right) d\omega^2.$$

Unless the term in S^5 is neglected, it will not be possible to obtain a conformal projection where the normal sections passing through the point of origin will be represented by concurrent straight lines. Since O. Eggert has proposed to use this projection as a conformal stereographic projection, it has appeared necessary to demonstrate this : (see note 3).

The same remarks hold true as in the preceding section, when the origin is at the pole or the equator.

IV

It is also impossible to obtain a conformal projection of the ellipsoid in which the isometres shall consist of rigorous concentric circles at the origin.

The geodetic curvatures Γ and γ of the concurrent straight lines and the curves of which they are the transformations are actually joined by the relation (6).

$$\Gamma = \frac{1}{m} (\gamma + m'_n)$$

m being the linear modulus and m'_n its derivative normal to the radius-vector.

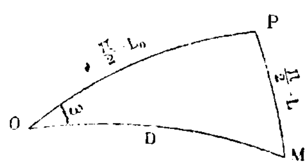
If the isometres are circles, m'_n is zero ; but the same holds for Γ and it is therefore necessary that γ should be zero, that is, that the curves represented by the concurrent straight lines should be geodetics, which, as we have seen, is impossible.

V

We know that if one has established a conformal projection of a surface on a plane, every continuous function of the complex co-ordinate of this projection may be taken as the complex co-ordinate of a new projection which, automatically, will be strictly conformal. One well-known conformal projection of the ellipsoid is the Mercator projection. In the case of a spherical surface we shall establish the function which will permit us to pass from the Mercator projection to the stereographic projection of the sphere.

A function of the same form, applied to the complex co-ordinate of the Mercator projection of the ellipsoid, will define the complex co-ordinates of a projection which will be strictly conformal and which we might conveniently call stereographic. There is here a general method for passing from a conformal projection of the sphere to a conformal projection of the ellipsoid, without having to write for the latter projection the equations of conformality.

We shall take as a unit the equatorial radius of the ellipsoid, and we shall express the arcs of longitude, latitude and meridional parts in radians, in order to avoid the necessity in the equations of multiplying them by *sin* 1" or *sin* 1.'



(6) See: « Traité des Projections de Cartes Géographiques » by L. Driencourt and J. Laborde, Part. 4, pp. 35-36.

On a sphere of unit radius, let O be the point of origin chosen (See : Fig. 1) with latitude L_0 and let M be a point in latitude L and with longitude G, reckoned from the meridian of origin. Let D and ω be the length and azimuth of the great circle OM; l and l_0 the meridional parts for the points M and O on the sphere.

Between the latitudes and the meridional parts we have the following relations :

$$\cos L_0 = \frac{2\, e^{l_0}}{e^{2l_0} + 1}, \qquad \cos L = \frac{2\, e^{l}}{e^{2l} + 1},$$

$$\sin L_0 = \frac{e^{2l_0} - 1}{e^{2l_0} + 1}, \qquad \sin L = \frac{e^{2l} - 1}{e^{2l} + 1}.$$

If the axis of the x of the stereographic projection is tangent to the meridian OP at O and points towards the north, the complex stereographic co-ordinate will have a value :

$$z = 2\, e^{i\omega}\, \mathrm{tg.}\, \frac{D}{2}.$$

The triangle OPM gives us :

$$\sin \omega = \frac{\cos L \sin G}{\sin D}, \quad \cos \omega = \frac{\sin L \cos L_0 - \sin L_0 \cos L \cos G}{\sin D}, \quad \mathrm{tg}\, \frac{D}{2} = \frac{\sin D}{1 + \cos D},$$

$$1 + \cos D = 1 + \sin L \sin L_0 + \cos L \cos L_0 \cos G\,;$$

from whence we obtain :

$$z = 2 \cos L_0 \frac{\sin L - \mathrm{tg}\, L_0 \cos L \cos G + i\, \dfrac{\cos L \sin G}{\cos L_0}}{1 + \sin L \sin L_0 + \cos L \cos L_0 \cos G}.$$

By substituting for the latitudes the above expressions for the meridional parts and using the formulae :

$$\sin G = \frac{e^{iG} - e^{-iG}}{2i}, \qquad \cos G = \frac{e^{iG} + e^{-iG}}{2};$$

we obtain an expression for z which contains in the numerator and in the denominator the common factor ;

$$1 + e^{l + l_0 - iG}$$

which can be made to cancel out. Calling ζ the complex co-ordinate of the Mercator projection of the sphere :

$$\zeta = l - l_0 + i\, G,$$

we have finally the relation sought which defines the stereographic projection of the sphere :

$$z = 2\, e^{l_0} \frac{e^{\zeta} - 1}{e^{2l_0} e^{\zeta} + 1},$$

and which is a simple function of e^{ζ} (7).

(7) If the origin were at the pole, this expression for z becomes : $z = \dfrac{-2}{e^{l + iG}}$.

A similar argument to that which we have given for the general case would show that if the origin is at the pole, α cannot but be equal to unity, and that one would have to adopt for z the expression:

$$Z = \frac{-2\, N_0}{\left(\dfrac{1+e}{1-e}\right)^{-\frac{e}{2}} e^{l + iG}}.$$

(See: Hydrographic Review, Vol. VI, N° 1, page 84; there is also given a table for rays of the parallels between 30° and 90° with the accuracy of 0.5 m).

We shall agree to call the following expression the stereographic projection of the ellipsoid :

$$Z = 2\,p\,\frac{e^{\alpha\zeta} - 1}{q\,e^{\alpha\zeta} + 1},$$

in which p, q and α are the constants to be determined and ζ the complex co-ordinate of the Mercator projection of the ellipsoid :

$$\zeta = l - l_0 + i\,G = v + i\,G,$$

by calling now l and l_0 the meridional parts on the ellipsoid and not on the sphere, v their difference, $l - l_0$.

This projection will be conformal no matter what values are adopted for p, q and α.

As a primary condition we shall specify that the scale (linear modulus) *m at the origin,* shall be unity and a minimum.

To calculate *m* let us give G an increase of d G.

$$\frac{d\,Z}{d\,G} = 2\,i\,p\,\alpha\,e^{\alpha\zeta}\,\frac{q + 1}{(q\,e^{\alpha\zeta} + 1)^2}.$$

Multiplying by the conjugate expression we have :

$$\frac{\sqrt{d\,X^2 + d\,Y^2}}{d\,G} = 2\,p\,\alpha\,e^{\alpha v}\,\frac{q + 1}{q^2\,e^{2\alpha v} + 2q\,e^{\alpha v}\cos\alpha\,G + 1}.$$

On the ellipsoid the increase is equal to :—

$$N \cos L\,d\,G.$$

We have therefore :

$$m = \frac{2\,p\,\alpha}{N\cos L}\,e^{\alpha v}\,\frac{q + 1}{q^2\,e^{2\alpha v} + 2q\,e^{\alpha v}\cos\alpha\,G + 1}.$$

In order that *m* may be equal to unity at the origin, it is necessary that :

$$2\,p\,\alpha = (q + 1)\,N_0 \cos L_0.$$

The derivative of *m* with respect to L will be zero at the origin if we annul the quantity :

$$\alpha(q - 1)\,N_0 \cos L_0\,\frac{d\,l}{d\,L} + (q + 1)\left(\frac{d\,N}{d\,L}\cos L_0 - N_0 \sin L_0\right).$$

We know that :

$$N = (1 - e^2 \sin^2 L)^{-\frac{1}{2}},\quad \frac{d\,N}{d\,L} = N\,e^2\,\frac{\sin L\,\cos L}{1 - e^2 \sin^2 L},\quad \rho = (1 - e^2)\,(1 - e^2 \sin^2 L)^{-\frac{3}{2}},$$

$$e^{2l} = \frac{1 + \sin L}{1 - \sin L}\left(\frac{1 - e \sin L}{1 + e \sin L}\right)^e,\qquad \frac{d\,l}{d\,L} = \frac{\rho}{N\,\cos\,L}.$$

The quantity to be annulled becomes :

$$\alpha\,(q - 1) = (q + 1)\sin L_0.$$

These two conditions permit us to obtain p and q as functions of α

$$p = \frac{N_0\,\cos L_0}{\alpha - \sin L_0},\qquad q = \frac{\alpha + \sin L_0}{\alpha - \sin L_0},\qquad q - 1 = \frac{-2\sin L_0}{\alpha - \sin L_0},\qquad q + 1 = \frac{2\,\alpha}{\alpha - \sin L_0}.$$

The expression for the complex co-ordinate of the stereographic projection of the ellipsoid then becomes :

$$(1) \qquad Z = 2\ N_o \cos L_o \frac{e^{\alpha\zeta} - 1}{(\alpha + \sin L_o)\ e^{\alpha\zeta} + \alpha - \sin L_o},$$

in which there remains a single arbitrary constant α.

The expressions for the co-ordinates are (8) :

$$\begin{cases} X = 2\ N_o \cos L_o \dfrac{(\alpha + \sin L_o)\ e^{2\alpha v} - 2\ \sin L_o\ e^{\alpha v} \cos \alpha G - (\alpha - \sin L_o)}{(\alpha + \sin L_o)^2\ e^{2\alpha v} + 2\ (\alpha^2 - \sin^2 L_o)\ e^{\alpha v} \cos \alpha G + (\alpha - \sin L_o)^2}, \\ Y = \dfrac{4\ \alpha\ N_o \cos L_o\ e^{\alpha v} \sin \alpha G}{(\alpha + \sin L_o)^2\ e^{2\alpha v} + 2\ (\alpha^2 - \sin^2 L_o)\ e^{\alpha v} \cos \alpha G + (\alpha - \sin L_o)^2} \end{cases}$$

Multiplying on the other hand the expression (1) by the conjugate expression we obtain, retaining the coefficients p and q :

$$X^2 + Y^2 = 4\ p^2 \frac{e^{2\alpha v} - 2\ e^{\alpha v} \cos \alpha\ G + 1}{q^2\ e^{2\alpha v} + 2q\ e^{\alpha v} \cos \alpha\ G + 1};$$

from which we deduce :

$$2\ e^{\alpha v} \cos\ \alpha\ G = \frac{[4\ p^2 - q^2\ (X^2 + Y^2)]\ e^{2\alpha v} + 4\ p^2 - (X^2 + Y^2)}{4\ p^2 + q\ (X^2 + Y^2)}.$$

By means of these formulae we obtain then, depending upon whether we eliminate v and αG, or only $e^{\alpha v} \cos \alpha G$ the two following expressions for *m* :

$$m = \frac{N_o \cos L_o}{N \cos L} \sqrt{1 + \frac{X}{p} + \frac{X^2 + Y^2}{4\ p\ 2}} \quad \sqrt{1 - q\frac{X}{p} + q^2 \frac{X^2 + Y^2}{4\ p\ 2}},$$

or :

$$m = \frac{N_o \cos L_o}{N \cos L} \frac{1 + q}{1 + q\ e^{2\alpha v}}\ e^{\alpha v} \left(1 + q\ \frac{X^2 + Y^2}{4\ p^2}\right).$$

The factors $\dfrac{N_o \cos L_o}{N \cos L}$ and $\dfrac{N_o \cos L_o}{N \cos L} \dfrac{(1 + q)\ e^{\alpha v}}{1 + q\ e^{2\alpha v}}$ are equal to unity at the origin and may be developed according to the powers of v. We then have :

$$\frac{N_o \cos L_o}{N \cos L} = 1 + v \sin L_o + \frac{v^2}{2}(1 + \eta) + \frac{v^3}{6} \sin L_o\ (1 - 3\ \eta - 4\ \eta\ e'^2 \cos^2 L_o) + \frac{v^4}{24}$$
$$[(1 + \eta)\ (1 - 3\ \eta - 4\ \eta\ e'^2 \cos^2 L_o) + 12\ \eta \sin^2 L_o\ (1 + 3\ e'^2 \cos^2 L_o + 2\ \eta\ e'^2)]$$

by putting :

$$e'^2 = \frac{e^2}{1 - e^2}, \qquad \eta = e'^2 \cos^4 L_o.$$

(8) These formulae are easier to calculate provided we use a table of hyperbolic functions:-

$$\begin{cases} X = 2\ N_o \cos L_o \dfrac{\alpha \sinh \alpha v + \sin L_o\ (\cosh \alpha v - \cos \alpha\ G)}{(\alpha^2 + \sin^2 L_o) \cosh \alpha v + 2\ \alpha \sin L_o \sinh \alpha v + (\alpha^2 - \sin^2 L_o) \cos \alpha\ G}, \\ Y = 2\ N_o \cos L_o \dfrac{\alpha \sin \alpha\ G}{(\alpha^2 + \sin^2 L_o) \cosh \alpha v + 2\ \alpha \sin L_o \sinh \alpha v + (\alpha^2 - \sin^2 L_o) \cos \alpha\ G}. \end{cases}$$

The factor $\frac{N_o \cos L_o}{N \cos L} \frac{1 + q) e^{\alpha v}}{1 + q e^{2\alpha v}}$ has for development :

$$1 + \frac{v^2}{2} (1 + \eta - \alpha^2) - \frac{v^3}{3} \sin L_o \left[1 + \eta - \alpha^2 + 2 \eta (1 + e'^2 \cos^2 L_o)\right] + \ldots$$

It does not differ from unity but by terms having each the factor e^2, if α^2 does not itself differ from unity but by a factor of e^2. It is actually equal to unity no matter what the value of L if the eccentricity is zero and if α^2 is equal to unity.

The last development shows that the value of m may be reduced considerably by adopting for α^2 the value $1 + \eta$. If, further, the principal value of v is $\frac{X}{N_o \cos L_o}$ we shall then have for m the value :

$$m = \left(1 + \frac{X^2 + Y^2}{4 N_o \rho_o}\right) \left(1 - \frac{e'^2}{3} \sin 2 L_o \frac{X^3}{N_o^2 \rho_o} + \ldots\right). \tag{2}$$

The indicatrix is therefore a circle with terms of the third order nearly, and all of the terms higher than the second order are then multiplied by the square of the eccentricity.

If one adopts for α^2 the value 1, we have an indicatrix of the circle somewhat different :

$$m = \left(1 + \frac{X^2 + Y^2}{4 N_o^2}\right) \left(1 + \frac{e'^2 \cos^2 L_o}{2 N_o^2} X^2 + \ldots\right).$$

VI

A very general procedure and useful in practice in obtaining a conformal projection of the terrestrial ellipsoid on a plane, consists in establishing first a conformal projection of the ellipsoid on a sphere, then employing whatever plane of projection one desires of this sphere: in the present case the stereographic projection. It may be agreed to call the result a stereographic projection of the ellipsoid and this procedure is generally called the "double projection".

It is much simpler in this case to choose two poles on the auxiliary sphere and to make the meridians and parallels of the sphere correspond to the meridians and parallels of the ellipsoid. These in turn will be represented by circles on the plane of projection.

Let R be the radius of curvature of the auxiliary sphere and φ_o the latitude on this sphere at the point of origin of latitude L_o on the ellipsoid. We shall adopt a correspondence of meridians such that the longitude of a point on the sphere, reckoned from the point of origin, shall be equal to a corresponding point on the ellipsoid mutiplied by a constant factor α.

In order that the scale may be the same at the origin on the two surfaces, it is necessary that:

$$\alpha R \cos \varphi_o = N_o \cos L_o ;$$

and in order that the projection may be conformal, it is necessary that :

$$\frac{d \varphi}{d L} = \alpha \frac{\rho \cos \varphi}{N \cos L}.$$

we should obtain the latitude of any point whatever on the sphere by integrating this equation ; but it should be noted that if we make a Mercator projection on the auxiliary sphere, we must obtain for two corresponding points the same co-ordinates with respect to the origin as we should if we established the Mercator projection directly from the ellipsoid.

If therefore λ and λ_0 are the meridional parts for the latitudes φ and φ_0 on the sphere, l and l_0 the meridional parts of the latitudes L and L_0, on the ellipsoid, the two values of x to be equated are :

$$\frac{\lambda - \lambda_0}{2\pi} R \cos \varphi_0 \quad \text{and} \quad \frac{l - l_0}{2\pi} N_0 \cos L_0.$$

We have therefore :

$$\lambda - \lambda_0 = \frac{N_0 \cos L_0}{R \cos \varphi_0} (l - l_0) = \alpha (l - l_0).$$

The ratio of scales (or linear modulus) calculated on the parallel will be :

$$K = \alpha \frac{R \cos \varphi}{N \cos L}.$$

In order that it shall be a minimum at the origin, it is necessary that :—

$$\alpha = \frac{\sin L_0}{\sin \varphi_0};$$

from which :

$$\operatorname{tg} \varphi_0 = \frac{R}{N_0} \operatorname{tg} L_0.$$

A preliminary solution of the problem has been obtained by Gauss by taking α equal to unity, from which :

$$\varphi_0 = L_0, \qquad R = N_0.$$

The auxiliary sphere has a radius N_0 and is tangent to the ellipsoid all along the parallel of origin.

The preceding equations become :

$$\lambda - \lambda_0 = l - l_0 = v$$

and

$$K = \frac{N_0 \cos \varphi}{N \cos L},$$

an expression which near the origin will have the form :

$$1 + \frac{v^2}{2} e'^2 \cos^4 L_0 + \text{- - - -}$$

The linear distortion will be reduced to an order lower than this if the second derivative of K is zero at the origin. We shall then have :

$$\alpha^2 \cos^2 \varphi_0 = \frac{N_0}{\rho_0} \cos^2 L_0,$$

and

$$R = \sqrt{N_0 \rho_0}.$$

The auxiliary sphere will osculate the ellipsoid all along the parallel of origin. This is the second solution of Gauss.

φ_0 will be given by :

$$\operatorname{tg} \varphi_0 = \sqrt{\frac{\rho_0}{N_0}} \operatorname{tg} L_0,$$

and α by :

$$\alpha = \frac{\sin L_0}{\sin \varphi_0} = \sqrt{1 + \left(\frac{N_0}{\rho_0} - 1\right) \cos^2 L_0} = \sqrt{1 + e'^2 \cos^4 L_0} = \sqrt{1 + \eta}.$$

The development of K in the vicinity of the origin will be :

$$K = 1 - \frac{v^3}{3} e'^2 \frac{N_0}{\rho_0} \sin 2 L_0 \cos^3 L_0 + \text{----}$$

(The following terms of the development will all have e'^2 as factor ; this was, besides, the case in the first solution).

This procedure by double projection terminates in exactly the same results as that which we have employed in V. In fact, the co-ordinate Z of the stereographic projection of the auxiliary sphere will be :

$$Z = 2 R \cos \varphi_0 \frac{e^{\alpha\zeta} - 1}{(1 + \sin \varphi_0) e^{\alpha\zeta} + 1 - \sin \varphi_0};$$

from which, by substituting for R and φ_0 their values, we have exactly the same expression which we have already found in V :

$$(1) \qquad Z = 2 N_0 \cos L_0 \frac{e^{\alpha\zeta} - 1}{(\alpha + \sin L_0) e^{\alpha\zeta} + \alpha - \sin L_0}.$$

The scale (linear modulus) on the plane will be :

$$m = K\left(1 + \frac{X^2 + Y^2}{4 R^2}\right),$$

a value which will become, when we adopt for R the value $\sqrt{N_0 \zeta_0}$:

$$(2) \qquad m = \left(1 + \frac{X^2 + Y^2}{4 N_0 \rho_0}\right)\left(1 - \frac{e'^2}{3} \sin 2 L_0 \frac{X^3}{N_0^2 \rho_0} + \text{----}\right),$$

by neglecting, in the second factor, the terms of a higher order than $e'^2 X^3$.

For this stereographic projection, what is, on the ellipsoid, the family of orthogonal curves which corresponds to the family of the plane surface formed by the concurrent straight lines at the origin and the concentric circles ? We have seen in II and III that it cannot contain either the geodetics or the normal sections concurrent at the origin.

The geodetic (9) passing through the origin and the point M, situated at a distance D from the origin in the bearing ω on the plane, makes at the origin with the direction ω an angle c which is given by the formula :

$$c = \frac{1}{2} \Gamma_3 D + \frac{1}{72} \Gamma_0'' D^3.$$

Γ_3 is the curvature at the first third of the radius OM, and is given by the formula :

$$\Gamma = \frac{1}{m}\left(m'_x \sin \omega - m'_y \cos \omega\right).$$

(9) See: « Traité des Projections des Cartes Géographiques » by L. Driencourt and J. Laborde, Part. 4, pp. 69-72.

We have found that m is equal to the product :

$$K\left(1+\frac{X^2+Y^2}{4R^2}\right).$$

We have therefore :

$$\begin{cases} m'_x=\left(1+\dfrac{X^2+Y^2}{4R^2}\right)K'_x+\dfrac{KX}{2R^2}, \\ m'_y=\left(1+\dfrac{X^2+Y^2}{4R^2}\right)K'_y+\dfrac{KY}{2R^2}; \end{cases}$$

from which :

$$m'_x\sin\omega-m'_y\cos\omega=\left(1+\frac{X^2+Y^2}{4R^2}\right)\left(K'_x\sin\omega-K'_y\cos\omega\right)$$
$$+\frac{K}{2R^2}\left(X\sin\omega-Y\cos\omega\right).$$

but the last term is zero, the point M lying on the bearing ω.

We have therefore :

$$\Gamma=\frac{K'_x\sin\omega-K'_y\cos\omega}{K}.$$

The development of the expression for K may be written :

$$K=1+\frac{v^2}{2}\frac{\rho_o}{N_o}\left(1-\frac{N_o\rho_o}{R^2}\right)-\frac{v^3}{3}e'^2\frac{\rho_o^2}{N_o^2}\sin 2L_o;$$

on the other hand, by neglecting the 4th order :

$$\begin{cases} v^2=\dfrac{X^2}{N_o^2\cos^2 L_o}+\dfrac{X(X^2-Y^2)}{N_o^3\cos^3 L_o}\sin L_o, \\ v^3=\dfrac{X^3}{N_o^3\cos^3 L_o}. \end{cases}$$

Therefore :

$$K=1+\frac{X^2}{2N_o^2\cos^2 L_o}\frac{\rho_o}{N_o}\left(1-\frac{N_o\rho_o}{R^2}\right)+\frac{X(X^2-Y^2)}{2N_o^3\cos^3 L_o}\frac{\rho_o}{N_o}\sin L_o\left(1-\frac{N_o\rho_o}{R^2}\right)$$
$$-\frac{X^3}{3N_o^3\cos^3 L_o}\frac{\rho_o^2}{N_o^2}e'^2\sin 2L_o;$$

$$K'_x=\frac{X}{N_o^2\cos^2 L_o}\frac{\rho_o}{N_o}\left(1-\frac{N_o\rho_o}{R^2}\right)+\frac{3X^2-Y^2}{2N_o^3\cos^3 L_o}\frac{\rho_o}{N_o}\sin L_o\left(1-\frac{N_o\rho_o}{R^2}\right)$$
$$-\frac{X^2}{N_o^3\cos^3 L_o}\frac{\rho_o^2}{N_o^2}e'^2\sin 2L_o;$$

$$K'_y=-\frac{XY}{N_o^3\cos^3 L_o}\frac{\rho_o}{N_o}\sin L_o\left(1-\frac{N_o\rho_o}{R^2}\right).$$

By substituting for X the term D cos ω, and for Y the term D sin ω, we have, by neglecting the third order :

$$\Gamma=\frac{D\rho_o\sin\omega}{N_o^3\cos^2 L_o}\left[\left(1-\frac{N_o\rho_o}{R^2}\right)\left(\cos\omega+\frac{D}{N_o}\operatorname{tg}L_o\frac{6\cos^2\omega-1}{2}\right)\right.$$
$$\left.-\frac{2D\rho_o}{N_o^2}e'^2\sin L_o\cos^2\omega\right].$$

Multiplication by $1/K$ will give terms in D^3 which we do not take into consideration. This expression represents therefore Γ at the distance D. Γ_o'' is equal to double the coefficient of D^2.

Therefore :

$$\Gamma_o''=\frac{\rho_o\sin\omega}{N_o^4\cos^2 L_o}\left[\left(1-\frac{N_o\rho_o}{R^2}\right)\operatorname{tg}L_o(6\cos^3\omega-1)-\frac{4\rho_o}{N_o}e'^2\sin L_o\cos^2\omega\right].$$

We have then :

$$c = \frac{D^2 \rho_0 \sin \omega}{6 N_0^3 \cos^2 L_0} \left[\left(1 - \frac{N_0 \rho_0}{R^2}\right)\left(\cos \omega + \frac{D}{N_0} \operatorname{tg} L_0 \frac{6 \cos^2 \omega - 1}{4}\right) - \frac{D \rho_0}{N_0^2} e'^2 \sin L_0 \cos^2 \omega\right].$$

The angle c will be of the second order in D if R = N_0. Its principal value will be, replacing the unity by the equatorial radius a and expressing the angle in seconds :

$$e^2 \frac{\rho_0}{N_0} \frac{D^2}{12 a^2} \frac{\sin 2 \omega}{\sin 1''};$$

a value which, like the distance which we shall calculate in the following, is almost independent of the latitude ; it is of the order of three-hundredths of a second for D = 100 kilometres, and of seven-tenths of a second for D = 500 kilometres, with $\omega = \pm 45$ or $\pm 135°$.

The angle c will be of the third order in D if R = $\sqrt{N_0 \rho_0}$. Its principal value will then be :

$$- e^2 \frac{\rho_0}{N_0} \frac{D^3}{6 a^2 N_0} \frac{\sin L_0}{\cos^2 L_0} \frac{\sin \omega \cos^2 \omega}{\sin 1''}.$$

The distance of this geodetic to the radius vector, to the point where it departs the most, is equal, when R = N_0, and by neglecting the terms in D^4, to :

$$e^2 \frac{\rho_0}{N_0} \frac{D^3}{32 a^2} \sin 2 \omega,$$

a quantity which is of the order of 5 millimetres if D is equal to 100 kilometres and ω equal to 45°; and which would be even less than 1.4 m. if D were equal to 640 kilometres.

If R = $\sqrt{N_0 \rho_0}$ this distance will be :

$$- e^2 \frac{\rho_0}{N_0} \frac{D^4}{32 a^2 N_0} \frac{\sin L_0}{\cos^2 L_0} \sin \omega \cos^2 \omega.$$

It will not reach 7 centimetres if D is equal to 640 kilometres, ω and L_0 being 45°.

We may state therefore that in these stereographic projections, the projection of the geodetic passing through the origin differs very little from a straight line, and that the grid formed by the geodetics proceeding from the origin and their orthogonal curves may be considered without appreciable error as represented on the projection by the straight lines concurrent at the origin and the concentric circles.

We know that the normal section makes with the geodetic an angle for which the expression, by neglecting the terms in e^4 and D^3 may be written :

$$\delta = e^2 \frac{D^2}{12 a^2} \cos^2 L_0 \frac{\sin 2 \omega}{\sin 1''}.$$

The angle of its projection with the radius vector will be then, if R is equal to N_0 :

$$e^2 \frac{D^2}{12 a^2} \sin^2 L_0 \frac{\sin 2 \omega}{\sin 1''}.$$

The transformation of the normal section is therefore comprised between the transformation of the geodetic and the radius vector, if we adopt for R the value N_0 and make it approach closer to the latter than the geodetic.

But if we take for R the value $\sqrt{N_0 \rho_0}$, the geodetic is much closer to the radius vector than to the normal section.

VIII

The expressions found for Z, and consequently for X and Y, in the two preceding chapters require quite long computations, even if we use the intermediary of the auxiliary sphere. But the expression found for Z may easily be developed according to ascending powers of ζ; and, no matter at which term the progression is stopped, the projection thus defined will always be rigorously conformal. It will therefore be useful, in order to diminish the distortions and to render the calculation easier, if the indicatrix, limited in any case to the second order, can, further, be circular.

If we employ the complete series in the development of Z, the modulus m does not differ from $1 + \frac{X^2 + Y^2}{4 R^2}$ but by the terms of a higher degree than the second, all having e^2 as the factor (10). If we now stop the development of Z at the terms of the n^{th} degree inclusive, the calculation of dX and dY should not be pushed beyond the $n - 1^{th}$ degree; nothing will be changed in the development of m up to the $n - 1^{th}$ degree inclusive, but the following terms will no longer have e^2 as a factor. If we now develop Z up to the terms of degree 3 inclusive, the indicatrix, limited to the second order, will still be a true circle ; but the term of the third order, not having e^2 as a factor, will have a value which cannot be neglected. If we develop Z to the 4th order inclusive, the term of the indicatrix of the 3rd order will contain e^2 as factor; that of the 4th order, not containing it, will be comparable in value to the preceding. It would be too complicated to consider the 1st term which does not contain e^2 as a factor. It is therefore necessary to push the development of Z more or less, depending upon the degree of accuracy desired for m, or, rather, depending upon whether one wishes to employ the projection at a greater or lesser distance from the origin ; but this distance may be as great as desired on the condition that we employ a sufficient number of terms in the development of Z. We shall give below the calculation on the development of Z up to the term of the 4th order inclusive : (11)

$$(3) \qquad \frac{Z}{N_0 \cos L_0} = \zeta - \frac{\zeta^2}{2} \sin L_0 + \frac{\zeta^3}{12} A + \frac{\zeta^4}{24} B \sin L_0 \; ;$$

(10) Taking, naturally, α^2 equal to the increased unit of any factor of e^2.

(11) The following term will be:— $+ \frac{\zeta^5}{120} \left(\alpha^4 - \frac{15}{2} \alpha^2 \sin^2 L_0 + \frac{15}{2} \sin^4 L_0 \right)$.

with :

$$A = 3\sin^2 L_o - \alpha^2, \qquad B = 2\alpha^2 - 3\sin^2 L_o,$$

$$\frac{dY + i\,dX}{i\,N_o \cos L_o\, dG} = 1 - \zeta \sin L_o + \frac{\zeta^2}{4} A + \frac{\zeta^3}{6} B \sin L_o,$$

$$= 1 - v \sin L_o + A\,\frac{v^2 - G^2}{4} + B\,v\,\frac{v^2 - 3G^2}{6} \sin L_o - i\,G \sin L_o \left(1 - \frac{A}{2\sin L_o} v - B\,\frac{3v^2 - G^2}{6}\right),$$

$$= P - i\,G \sin L_o\, Q.$$

multiplying by the conjugate expression we have :

$$\frac{dX^2 + dY^2}{N_o^2 \cos^2 L_o\, dG^2} = P^2 \left(1 + G^2 \sin^2 L_o \frac{Q^2}{P^2}\right);$$

from which, by neglecting the 4th order :

$$m = \frac{\sqrt{dX^2 + dY^2}}{N_o \cos L_o\, dG}\,\frac{N_o \cos L_o}{N \cos L} = \frac{N_o \cos L_o}{N \cos L}\left(P + G^2 \sin^2 L_o \frac{Q^2}{2P}\right).$$

By substituting for P and Q as well as for A and B, their values, we find for the term in parenthesis the value :

$$1 - v \sin L_o + \frac{v^2}{4}(3\sin^2 L_o - \alpha^2) + \frac{G^2}{4}(\alpha^2 - \sin^2 L_o) + \frac{v^3}{6}(2\alpha^2 - 3\sin^2 L_o)\sin L_o + \frac{v\,G^2}{2}(\sin^2 L_o - \alpha^2)\sin L_o.$$

We have given above (V) the development of $\frac{N_o \cos L_o}{N \cos L}$.

By taking the product we have :

$$m = 1 + \frac{v^2 + G^2}{4}(\alpha^2 - \sin^2 L_o) + \frac{v^2}{2}(1 + \eta - \alpha^2) + v^3 \sin L_o \frac{\alpha^2 - 1 - \eta - 3\cos^2 L_o - 11\eta - 8\eta e'^2 \cos^2 L_o}{12} + v\,G^2 \sin L_o \frac{\sin^2 L_o - \alpha^2}{4}.$$

For the rest, from equation (3) we deduce :

$$(4)\quad \begin{cases} \dfrac{X}{N_o \cos L_o} = v - \dfrac{v^2 - G^2}{2}\sin L_o + v(v^2 - 3G^2)\dfrac{3\sin^2 L_o - \alpha^2}{12} + (v^4 - 6v^2G^2 + G^4)\dfrac{2\alpha^2 - 3\sin^2 L_o}{24}\sin L_o, \\[2ex] \dfrac{Y}{N_o \cos L_o} = G - v\,G \sin L_o + G(3v^2 - G^2)\dfrac{3\sin^2 L_o - \alpha^2}{12} + v\,G(v^2 - G^2)\dfrac{2\alpha^2 - 3\sin^2 L_o}{6}\sin L_o. \end{cases}$$

These expressions furnish us inversely with the values of v and G which we shall push to the terms of the 3rd order :

$$\begin{cases} v = \dfrac{X}{N_o \cos L_o} + \dfrac{X^2 - Y^2}{2N_o^2 \cos^2 L_o}\sin L_o + X\,\dfrac{X^2 - 3Y^2}{12 N_o^3 \cos^3 L_o}(3\sin^2 L_o + \alpha^2), \\[2ex] G = \dfrac{Y}{N_o \cos L_o} + \dfrac{XY}{N_o^2 \cos^2 L_o}\sin L_o + Y\,\dfrac{3X^2 - Y^2}{12 N_o^3 \cos^3 L_o}(3\sin^2 L_o + \alpha^2). \end{cases}$$

Substituting these values in the expression for *m* we find :

$$m = 1 + \frac{X^2 + Y^2}{4\,N_o^2\,\cos^2 L_o}(\alpha^2 - \sin^2 L_o) + \frac{X^2}{2\,N_o^2\,\cos^2 L_o}(1 + \eta - \alpha^2)$$
$$+ \frac{X^3 \sin L_o}{6\,N_o^3\,\cos^3 L_o}(1 + \eta - \alpha^2 - 4\,\eta - 4\,\eta\,e'^2 \cos^2 L_o) + \frac{X\,Y^2}{2\,N_o^3\,\cos^3 L_o}(\alpha^2 - 1 - \eta)\sin L_o.$$

If α^2 is equal to $1 + \eta$, the expression for the linear scale becomes, noting that :

$$1 + \frac{\eta}{\cos^2 L_o} = \frac{N_o}{\rho_o},$$

$$m = 1 + \frac{X^2 + Y^2}{4\,N_o\,\rho_o} - e'^2\,\frac{X^3}{3\,N_o^2\,\rho_o}\sin 2L_o. \qquad (5)$$

This is really the expression (2) which we have found in chapters V and VI ; but the term of the 4th degree will no longer be the same and will not have e^2 as a factor. In order that it should be the same it would have been necessary to continue the development of the formulae (3) and (4) to the 5th degree.

If we had taken α^2 equal to unity, we should have obtained the following value for *m* :

$$m = 1 + \frac{X^2 + Y^2}{4\,N_o^2} + e'^2\,\frac{X^2 \cos^2 L_o}{2\,N_o^2} + e'^2\,\frac{X^3}{12\,N_o^3}\left(1 - 4\,\frac{N_o}{\rho_o}\right)\sin 2L_o - e'^2\,\frac{X\,Y^2}{4\,N_o^3}\sin 2\,L_o;$$

and the equations (4) would have become :

$$\left\{\begin{aligned} \frac{X}{N_o \cos L_o} &= v - \frac{v^2 - G^2}{2}\sin L_o + v\,(v^2 - 3\,G^2)\,\frac{3\sin^2 L_o - 1}{12} \\ &\quad + (v^4 - 6\,v^2 G^2 + G^4)\,\frac{2 - 3\sin^2 L_o}{24}\sin L_o, \\ \frac{Y}{N_o \cos L_o} &= G - v\,G \sin L_o + G\,(3\,v^2 - G^2)\,\frac{3\sin^2 L_o - 1}{12} \\ &\quad + v\,G\,(v^2 - G^2)\,\frac{2 - 3\sin^2 L_o}{6}\sin L_o. \end{aligned}\right.$$

The expressions (4) are easily calculated if we take v from reasonably accurate tables of meridional parts. Those of the International Hydrographic Bureau (Special Publication N° 21) suffice to guarantee an accuracy to the centimetre on the X co-ordinate and an even greater accuracy on the Y. We note that in these formulae v and G are expressed as parts of the radius. They should be multiplied by *sin* 1' or *sin* 1'' depending upon whether they are minutes or seconds.

The calculation of the formulae (4) is more rapid when we put :

$$v^2 + G^2 = d^2, \qquad \frac{G}{v} = \mathrm{tg}\ \delta.$$

We then have ;

$$\zeta = d\,e^{i\delta}\ ;$$

and the formula (3) becomes :

$$\frac{Z}{N_o \cos L_o} = d\,e^{i\delta} - \frac{d^2}{2}\,e^{2i\delta}\sin L_o + \frac{d^3}{12}\,e^{3i\delta}\,A + \frac{d^4}{24}\,e^{4i\delta}\,B\,\sin L_o.$$

We then deduce :

$$\left\{ \begin{aligned} \frac{X}{N_o \cos L_o} &= d \cos \delta - \frac{d^2}{2} \sin L_o \cos 2\delta + \frac{d^3}{12} A \cos 3\delta + \frac{d^4}{24} B \sin L_o \cos 4\delta, \\ \frac{Y}{N_o \cos L_o} &= d \sin \delta - \frac{d^2}{2} \sin L_o \sin 2\delta + \frac{d^3}{12} A \sin 3\delta + \frac{d^4}{24} B \sin L_o \sin 4\delta; \end{aligned} \right.$$

and, the same way, by putting

$$X^2 + Y^2 = \rho^2, \quad \frac{Y}{X} = \text{tg}\ \omega,$$

we obtain, not going beyond the 4th order :

$$\zeta = \frac{Z}{N_o \cos L_o} + \frac{Z^2 \sin L_o}{2 N_o^2 \cos^2 L_o} + \frac{Z^3 \left(\sin^2 L_o + \frac{\alpha^2}{3}\right)}{4 N_o^3 \cos^3 L_o} + \frac{Z^4 (\sin^2 L_o + \alpha^2) \sin L_o}{8 N_o^4 \cos^4 L_o};$$

and, as a result the following values of v and G which may be utilized for calculating the geographic coordinates of a point of which we know the coordinates X and Y :

$$l - l_o = \frac{\rho \cos \omega}{N_o \cos L_o} + \frac{\rho^2 \sin L_o \cos 2\omega}{2 N_o^2 \cos^2 L_o} + \frac{\rho^3 \left(\sin^2 L_o + \frac{\alpha^2}{3}\right) \cos 3\omega}{4 N_o^3 \cos^3 L_o} + \frac{\rho^4 (\sin^2 L_o + \alpha^2) \sin L_o \cos 4\omega}{8 N_o^4 \cos^4 L_o},$$

$$G = \frac{\rho \sin \omega}{N_o \cos L_o} + \frac{\rho^2 \sin L_o \sin 2\omega}{2 N_o^2 \cos^2 L_o} + \frac{\rho^3 \left(\sin^2 L_o + \frac{\alpha^2}{3}\right) \sin 3\omega}{4 N_o^3 \cos^3 L_o} + \frac{\rho^4 (\sin^2 L_o + \alpha^2) \sin L_o \sin 4\omega}{8 N_o^4 \cos^4 L_o}.$$

VIII

The development studied in the preceding paragraph obviates the necessity of making a first projection upon the auxiliary sphere of radius R. If, however, we prefer to make this projection first, we shall then perhaps profit by using no longer the Mercator co-ordinates but those of Gauss which present a double symmetry. Let us call z the complex co-ordinate of the Gauss projection of a sphere of radius equal to unity (12). We know that according to I. JUNG (13) the stereographic co-ordinate Z may be written :

(6) $$Z = 2 R \,\text{tg}\, \frac{z}{2}.$$

The value of Z thus calculated provides exactly the same result as that which we have obtained in Chapter VI by means of the Mercator co-ordinates ζ.

We have :

$$X = 4 R e^y \frac{\sin x}{e^{2y} + 2 e^y \cos x + 1}, \qquad Y = 2 R \frac{e^{2y} - 1}{e^{2y} + 2 e^y \cos x + 1}.$$

(12) This projection is also called:— conformal inverse cylindrical projection, conformal traverse cylindrical projection, Mercator inverse projection, or Lambert conformal cylindrical projection. Vol. VI, N° 1 of Hydrographic Review contains a table giving its coordinates in minutes with two decimals. between latitudes 60° and 90°, enabling it to be used for the polar regions.

(13) See: Hydrographic Review, Vol. x, N° 1, pp. 84-85.

The calculation is made easier if, in place of the co-ordinates x, y of Gauss, we employ the co-ordinates, x, y' of Cassini Soldner, with which the calculation is more rapid and which offers the advantage of using only right-angled triangles. We have:

$$X = 2\,R\,\frac{\sin x \cos y'}{1 + \cos x \cos y'}, \qquad Y = 2\,R\,\frac{\sin y'}{1 + \cos x \cos y'}.$$

But the use of the Gauss co-ordinates is found to be more interesting if we make use of a limited development of expression (6). This development contains only the terms of the odd degrees, such that with only two terms we have the same advantages for *m* as with the four terms employed in chapter VII.

$$\left\{ \begin{array}{l} \dfrac{Z}{R} = z + \dfrac{z^3}{12}; \\ \dfrac{X}{R} = x + x\,\dfrac{x^2 - 3\,y^2}{12}, \qquad \dfrac{Y}{R} = y + y\,\dfrac{3\,x^2 - y^2}{12}. \end{array} \right.$$

We find again for *m* the value (5) by neglecting the term of the 4th degree, which has not e^2 as a factor. We might utilize this projection still further away from the origin, by taking the development:

$$\frac{Z}{R} = z + \frac{z^3}{12} + \frac{z^5}{120}.$$

However that may be, it does not appear to us that this method of procedure presents any advantage over that indicated in Chapter VII, which provides exactly the same degree of accuracy, with transformation formulae which are slightly longer, but which allow us to economise in the rather long calculations of the projection of the ellipsoid on the sphere and the Gauss co-ordinates on the sphere. The addition of the terms of the development of a higher degree does not mean a great increase in the work because these terms are rather small, and it allows one to utilize the projection as far away from the origin as one desires.

IX

It might appear more interesting, instead of having recourse to the meridional parts, to establish a stereographic projection of the ellipsoid directly by means of a development according to the increasing powers of the differences in latitude and longitude, even though we may have to use a more complicated expression. It is possible to do so, but the expression will no longer be rigorously conformal; there will always be a slight error in conformality. This error, however, may become quite negligible provided v is calculated with sufficient accuracy.

It will suffice to express v by the development according to increasing powers of $L - L_0 = V$; which amounts to the calculation of increasing meridional parts.

This is the expression for v, stopping at the 4th degree:

$$v = \frac{V\,\rho_0}{N_0 \cos L_0}\left[1 + \frac{V}{2}\,tg\,L_0 + \frac{V^2}{6}\,(1 + 2\,tg^2 L_0) + \frac{V^3}{24}\,tg\,L_0\,(5 + 6\,tg^2 L_0)\right]$$
$$+ e'^2 V^2 \frac{\rho_0^2 \sin L_0}{N_0^2}\left\{1 + \frac{V}{3}\,(1 + 5\,tg^2 L_0 - 4\,\frac{\rho_0}{N_0}\,tg^2 L_0)\,cotg\,L_0 + \frac{V^2}{12}\right.$$
$$\left.\left[2 + 3\,tg^2 L_0 + 12\,e'^2\,\frac{\rho_0}{N_0}\,(1 + \sin^2 L_0) - 24\,e'^2\,\frac{\rho_0^2}{N_0^2}\,\sin^2 L_0\right]\right\}$$

We then substitute the value of v, v^2, v^3, and v^4 in the expressions (4). The coefficients of these formulae are to be calculated once for all for the latitude of origin ; further, one need not be excessively troubled about their complexity ; it will, however, be much quicker to use a table of meridional parts and apply the formulae of Chapter VII.

At the prime meridian the value of X, which we shall call X_0, will be (by giving to α^2 the value of $1 + \eta$ and by neglecting the terms of a higher order than V^4 and of $e'^2\ V^3$) :

$$\frac{X_0}{\rho_0} = V + \frac{V^3}{12} + \frac{3e'^2 V^2}{4}\frac{\rho_0}{N_0}\sin 2L_0 + \frac{e'^2\ V^3}{12}\frac{\rho_0}{N_0}\left(5 + 19\sin^2 L_0 - 30\frac{\rho_0}{N_0}\sin^2 L_0\right).$$

It should be noted that the terms which are independent of the eccentricity are, as it should be, the first terms of the development of $2\ \mathrm{tg}\ \frac{L - L_0}{2}$.

X

The stereographic projection of M. Roussilhe :— Ingénieur hydrographe en chef Roussilhe presented to the Congress of Geodesy and Geophysics held in Rome in 1922, a conformal stereographic projection which has since been adopted, under the name of "quasi-stereographic projection", by the Polish Military Geographical Institute for the establishment of the maps of that country (14). This projection is based on the following definitions :—

The central meridian having been taken as the axis of the X, the law of representation for the points on this meridian will be given by the equation :

$$X_0 = s + t_2 s^2 + t_3 s^3 + t_4 s^4 + \dots,$$

in which S represents the distance from the point of origin, measured on the ellipsoid along the meridian, and t the constants which characterise the nature of the projection adopted.

Calling u the distance of a point to the central meridian, measured along the parallel of this point, λ the functions of s which will be developed in accordance with the powers of this quantity, and taking into consideration the symmetry with respect to the central meridian, the co-ordinates of any point whatever may be written :

$$\begin{cases} X = X_0 + \lambda_2 u^2 + \lambda_4 u^4 + \lambda_6 u^6 + \dots, \\ Y = \lambda_1 u_1 + \lambda_3 u^3 + \lambda_5 u^5 + \lambda_7 u^7 + \dots. \end{cases}$$

The equations of conformality on the ellipsoid furnish the determination of the coefficients λ by means of the series developed according to the powers of s continued as far as desired to obtain the necessary accuracy of conformality. The number of the coefficients λ, which may be as great as desired, will in practice not exceed seven.

(14) See: Travaux de la Section de Géodésie de l'Union Géodésique et Géophysique Internationale, Rome 1922, Tome I; Madrid 1924, Tome IV; Prague 1927, Tome VI; Lisbonne 1933, Tome XII.
See also: Hydrographic Review, 1930, Vol. VII, N° 1, p. 31.

This mode of projection of the ellipsoid is very general and has the advantage of not requiring a double projection ; but the calculations are rather long if great accuracy is required.

In the case of the stereographic projection, the author uses for X_0 the development of the function :

$$(7) \qquad 2\sqrt{N_0\rho_0}\ \mathrm{tg}\ \frac{s}{2\sqrt{N_0\rho_0}} = s + \frac{s^3}{12\,N_0\rho_0} + \frac{s^5}{120\,N_0^2\rho_0^2} + \frac{17\,s^7}{20160\,N_0^3\,\rho_0^3}.$$

by analogy with the stereographic projection of a sphere of radius R, of which the representation of the prime meridian is given by :

$$X_0 = 2\,R\ \mathrm{tg}\,\frac{s}{2\,R},$$

an expression in which the arc s is to be measured on the sphere.

The quasi-stereographic projection has the particular character of using, instead of the latitudes or the meridional parts, the length of the arc of the corresponding prime meridian and, in place of the longitude difference, the length u of the arc of the parallel. From these quantities very accurate tables have been computed for the international ellipsoid and for all latitudes.

If we compare this projection with that which we have indicated in VII we note first that the principle of the double projection leads us to introduce in formula (7), not the arc s of the meridian of the ellipsoid, but rather the arc s_1 of the meridian when it has been transformed by conformal projection to a sphere of radius $\sqrt{N_0\,\rho_0}$. This transformation is given by the formula :

$$ds_1 = K\,ds,$$

K being the scale found in VI :

$$K = \alpha\,\frac{R\cos\varphi}{N\cos L}.$$

If in this expression we make $\alpha^2 = 1 + \eta$, $R = \sqrt{N_0\,\rho_0}$, and if we develop according to powers of s taking into consideration the relations :

$$\frac{dL}{ds} = \frac{1}{\rho},\quad \frac{d\varphi}{ds} = \frac{\alpha\cos\varphi}{N\cos L},\quad \frac{dN}{ds} = e'^2\sin L\cos L,\quad \frac{d\rho}{ds} = 3\,\frac{\rho}{N}\,e'^2\sin L\cos L,$$

$$\frac{d\,N\cos L}{ds} = -\sin L,\qquad \frac{d\frac{N}{\rho}}{ds} = -\frac{3\,e'^2}{\rho}\sin L\cos L,$$

we obtain :

$$K = 1 - e'^2 s^3\,\frac{\sin 2\,L_0}{3\,N_0^2\,\rho_0} - e'^2 s^4\,\frac{\cos^2 L_0}{6\,N_0^3\,\rho_0}\left[1 + e'^2\,(1 - 7\sin^2 L_0)\right].$$

Consequently we have by integrating :

$$s_1 = s - e'^2 s^4\,\frac{\sin 2\,L_0}{12\,N_0^2\,\rho_0} - e'^2 s^5\,\frac{\cos^2 L_0}{30\,N_0^3\,\rho_0}\left[1 + e'^2\,(1 - 7\sin^2 L_0)\right].$$

By substituting this value of s in equation (7) we find an expression for the projection of the prime meridian which with the same degree of approximation should not differ from that which is furnished by equation (3) except by the use the variable s in place of v. We obtain thus, by limiting ourselves to the 5th degree, the expression (8) :

$$(8) \qquad X_o = s + \frac{s^3}{12\, N_o\, \rho_o} + \frac{s^5}{120\, N_o^2\, \rho_o^2} - e'^2 s^4 \frac{\sin 2\, L_o}{12\, N_o^2\, \rho_o} - e'^2 s^5 \frac{\cos^2 L_o}{30\, N_o^3\, \rho_o} \left[1 + e'^2 (1 - 7 \sin^2 L_o)\right];$$

which does not differ from expression (7) except from the terms in s^4 on, and even by the terms having e'^2 as a factor.

We know that a conformal projection is completely defined if we give the representation of a prime meridian on a straight line. The two projections defined by (7) and (8) are thus slightly different ; but they are both conformal if the coefficients λ are sufficiently developed, and that regardless of the number of terms employed in the developments (7) and (8).

In order to push the comparison further, we shall give the expressions for v and G by development according to powers of s and u:

$$\begin{cases} v\text{-}N_o \cos L_o = s + \dfrac{s^2}{2} \dfrac{\mathrm{tg}\, L_o}{N_o} + \dfrac{s^3}{3} \dfrac{M}{N_o^2} + \dfrac{s^4}{4} \mathrm{tg}\, L_o \dfrac{P}{N_o^3} + \dfrac{s^5}{5} \dfrac{Q}{N_o^4}, \qquad (15) \\ G\, N_o \cos L_o = u + us \dfrac{\mathrm{tg}\, L_o}{N_o} + us^2 \dfrac{M}{N_o^2} + us^3\, \mathrm{tg}\, L_o \dfrac{P}{N_o^3} + us^4 \dfrac{Q}{N_o^4}; \end{cases}$$

by putting :

$$\begin{cases} M = \mathrm{tg}^2 L_o + \dfrac{N_o}{2\, \rho_o}, \qquad P = \mathrm{tg}^2 L_o + \dfrac{3}{2} \dfrac{N_o}{\rho_o} - \dfrac{2}{3} \dfrac{N_o^2}{\rho_o^2}, \\ Q = \mathrm{tg}^4 L_o + 3 \dfrac{N_o}{\rho_o} \mathrm{tg}^2 L_o + \dfrac{N_o^2}{\rho_o^2}\left(\dfrac{3}{8} - \dfrac{17}{6} \mathrm{tg}^2 L_o\right) + \dfrac{N_o^3}{\rho_o^3}\left(\mathrm{tg}^2 L_o - \dfrac{1}{6}\right). \end{cases}$$

By substituting the expressions for v and G in equations (4), we have the expressions (16) for X and Y in a form analogous to that which we used for the quasi-stereographic projection, and we find that they do not differ except from the terms of the 4th degree on, and by the quantities having e'^2, as a factor. We do not give the calculation, which is rather lengthy, the coefficients λ offering in both cases an equal complexity.

The developments X and Y of the quasi-stereographic projection being prolonged to at least the 4th order, the scale (linear modulus) will have its term of the 3rd order multiplied by e'^2.

Its expression is :

$$m = 1 + \frac{X^2 + Y^2}{4\, N_o\, \rho_o} - e'^2 \frac{X\, Y^2}{N_o^2\, \rho_o} \sin 2 L_o.$$

It differs from the expression (5) found in VII because the terms of the development of X and Y are different in the two cases, and its last term depends on X and Y instead of depending solely on X. The terms

(15) This table enables the differences v of meridional parts to be computed quickly and with great accuracy by using accurate tables of the s values. It has the advantage over the direct computation of meridional parts, of using greatly inferior numbers when the origin is in a comparatively high latitude.

(16) These expressions enable the very accurate tables, just mentioned, of the s and u values to be used instead of tables of meridional parts for which tables as accurate as those, for the whole world, have not as yet been published. (See: Hydrographic Review, Vol. IV, N° 1, p.. 226).

in *m*, of a higher degree than 3, will have the factor e'^2 up to the term of the 6th degree inclusive if the developments of X and Y are pushed to the 7th degree.

CONCLUSION.

We have indicated three kinds of conformal stereographic projections of the ellipsoid, which differ slightly one from another.

The first, in V and VI described under two different forms terminating in identical results, is theoretically the most perfect, because its scale does not differ from $1 + \frac{X^2 + Y^2}{4 R^2}$ but by terms of a higher degree than the 2nd, all having e'^2 as a factor; this makes the corrections smaller and their calculation more precise.

The second, in VII and VIII, is a rigorously conformal projection ; no matter how many terms are employed in the development of X and Y. If one stops at the terms of the *n*th degree inclusive, the scale will be similar to the preceding up to the n — 1th degree of its development ; but the succeeding terms will not have e'^2 as a factor and the corrections which result from the first of these cannot be neglected if the development of the co-ordinate comprises too small a number of terms, or if the projection is employed at too great a distance from the origin. However, on the condition of making the number of terms of the development proportional to the distance from the origin at which the projection is to be employed, these projections are no less perfect than the preceding and are easier to calculate.

It should be carefully noted that, while remaining perfectly conformal, these projections yield results slightly different depending on whether one uses more or fewer terms in the development.

The third kind, in IX and X, comprises the projections which are practically conformal provided the developments are pushed quite far. In addition, the computation of the constant coefficients is much longer : but this calculation is made once and for all for the entire projection ; and one has the advantage of using :— in IX, only the differences in latitude and longitude referred to the origin ; and in X only the arcs which are given in the tables with all the accuracy which can be desired. The scale shows the same properties as those of the projections in VII and VIII.

If we wish to utilise this type of projection the simplest would be perhaps to adopt for the representation of the prime meridian the expression :

$$\frac{X_0}{\rho_0} = V + \frac{V^3}{12} + \frac{V^5}{120} + \frac{3}{4} e'^2 V^2 \frac{\rho_0}{N_0} \sin 2 L_0 + \frac{e'^2 V^3}{12} \frac{\rho_0^2}{N_0^2} \left[\frac{N_0}{\rho_0} (5 + 19 \sin^2 L_0) - 30 \sin^2 L_0\right],$$

and to determine the coefficients X and Y by means of the equations of conformality. In this manner one avoids the use of any kind of table.

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