MEASUREMENTS WITH A NEW TYPE OF LEVELLING INSTRUMENT WITH DOUBLE IMAGE AND INCLINABLE LINE OF SIGHT FOR HILLY COUNTRY

By :

DIPL.-ING. WALTER RIEMANN, SCIENT. ASSISTANT.

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With the ordinary method of geometrical levelling, differences of height are determined exclusively from horizontal sightings. The maximum height difference hereby manageable in one setting up of the instrument, when a 4-metre long levelling staff is used, is 4 metres. For greater differences of height, either much shorter, or in certain particular cases, disproportionately longer sightings have to be chosen instead of the normal 40-metre one; this necessity reduces considerably the efficiency of the instrument and is a disavantage for the accuracy of the work. In order to obviate these limitations, the tilting screw was developed into a gradient screw, which enables inclined lines of sight, with corresponding readings for the calculation of the horizontal pointings, to be used for levelling work in steep country. But mechanical inadequacies of various kinds minimize to a great extent the accuracy of such levellings, so that they could not gain admission in practice.

By means of the new levelling instrument for hilly country (Fig. 1), evolved by Prof Dr.-Ing. A. Berroth (Aix-la-Chapelle) and produced by the firm of Otto Fennel Söhne (Cassel), it is now possible to perform "steeply inclined" levellings also according to a rigorous theoretical method (i.e. levellings with inclined line of sight).

The principle involved is that of the double-image range finding. Whenever a "level up" with inclined line of sight is desirable, the distance has, of necessity, a rôle to play. *The basic idea underlying the instrument rests thus on the peculiarity of measuring distances, in making them available as differences of height.* The guiding principle followed in the construction was to simplify as far practicable the sequence of the calculations, and to make them similar to those of ordinary levelling. The advantages of "steeply inclined" levellings in hilly country are obvious: extended field of measurement resulting in fewer points of change and thereby increased speed of operation. The instrument was first tested by the writer; here are his findings.

1. *The Principle of the Levelling Instrument for hilly Country.*

The determination of differences of height with the levelling instrument for hilly country is effected by clean levelling-staff readings, both with horizontal line of sight and line of sight inclined to the horizontal by the constant angle α. By a special selection of this angle, the use of normal 40-metre sighting distances and of a 4-metre staff, differences of height up to 12 m. are manageable in one setting up of the instrument. The inclined sighting is obtained with the aid of a tilting screw, by centering one of the two levels mounted sideways on the telescope and inclined to the line of sight by the angle $+\alpha$ or $-\alpha$ respectively. The difference in height between two ground points is computed as follows (Fig. 2):

$$\Delta h = R - V = (r + r') - (v + v')$$

whereby V and v', in the case of the figure, are negative. The readings r and v correspond to the readings in forward and backward sighting in ordinary levelling.

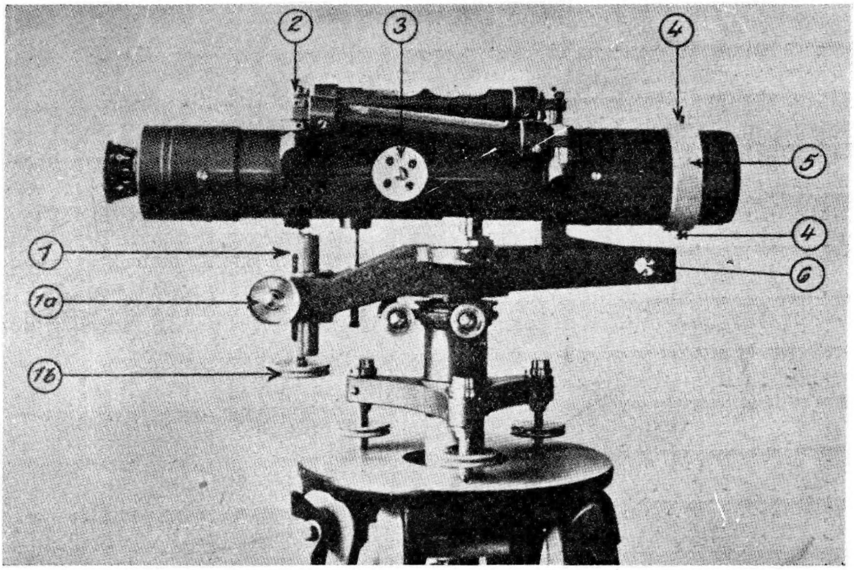


Fig. 1. — Levelling Instrument for hilly Country.
Sighting Axis horizontal.

Instrument de Nivellement pour Pays montagneux.
Axe de Visée horizontal.

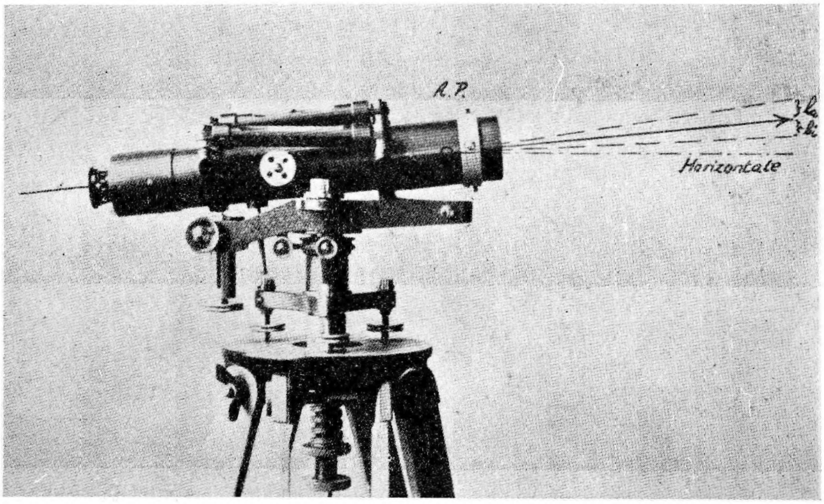


Fig. 3 a. — Levelling Instrument for hilly Country.
Centering Level " + "

Instrument de Nivellement pour Terrain montagneux.
Centrage de la Nivelle " + ".

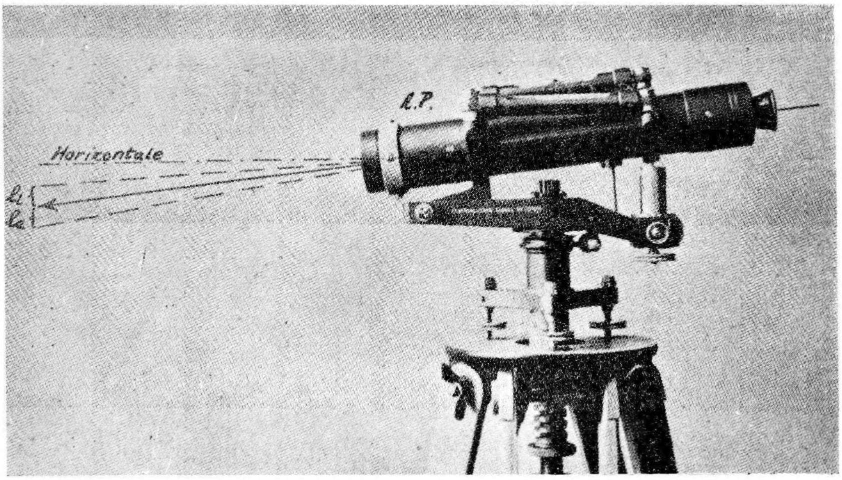


Fig. 3 b. — Levelling Instrument for hilly Country.
Centering Level " — ".

Instrument de Nivellement pour Terrain montagneux.
Centrage de la Nivelle " — ".

The quantities r' and v', depend on the constant angle α, and also on the distance, which is determined in minimum deviation position by means of a prism fitted in front on the objective. The prism covers half of the objective (see Fig. 5) and is rotatable through 180° about the axis of sight, so that, within

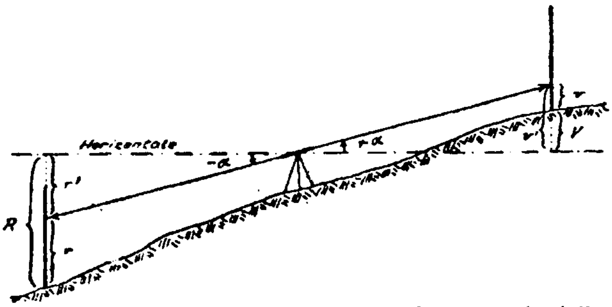


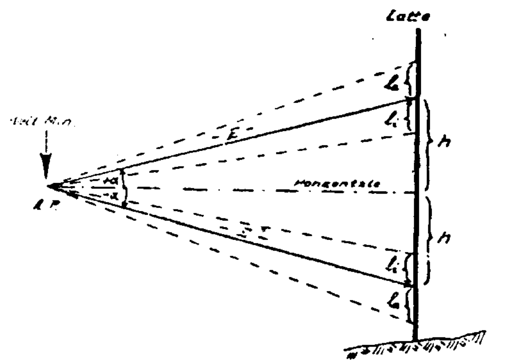
Fig. 2. — Difference of Height with Levelling Instrument for hilly Country

the field of view, the shifting in vertical direction of the image of the staff through the angle of deviation β can occur both upwards and downwards; this is necessary in order that for any value of the direct reading r or v, the locating of a section of the staff may be possible. Henceforth there will therefore be spoken of an outer staff section l_a and of an inner staff section l_i, according to whether the shifting of the image of the staff through the objective prism occurs towards, or away from, the horizontal (Figs. 3a & 3b). The differentiation as to whether, for the centering of one of the lateral levels, an inner or outer section of the staff is measured is done automatically. If, for instance, the level "+ α" on which the sign "+" is read (in which case the sign "+" has also become free on the tilting screw) is being centered, then, with the outer staff section the sign "+", and with the inner staff section the sign "—" on the upper attachment of the prism cell is visible. These reverse holds for centering level "—". Hence, when measuring an outer staff section there appear equal signs, and when measuring an inner staff section, different signs, hereafter designated as "crossed sign position" and labelled in the field books with a ×.

In order to distinguish between the two staff images (the direct one and that originating from the prism) in the telescope there is affixed, in front of the prism — and rotatable with it — a yellow-green plano-parallel glass plate. For the rest, the construction of the telescope is such that a sharp separation of the two images is obtained. Magnification of the telescope × 24.

To meet the special requirements involved, a 4-metre folding staff, whose graduation represents a combination of line and zone graduation, has been devised (see Fig. 5). This stave behaved very well indeed during the test measurements. On account of the double-image range finding procedure, the accurate adjustment of its circular level must be checked as often as possible.

The calculation of the additional quantity r' or v' (now generally termed h) from the staff section, and thus from the distance, is as follows (Fig. 4) :



$$E = l_a \frac{\cos(\alpha + \beta)}{\sin \beta} \tag{1}$$

$$E = l_i \frac{\cos(\alpha - \beta)}{\sin \beta} \tag{2}$$

$$E = \frac{h}{\sin \alpha} \tag{3}$$

Fig. 4. — Deduction of the mathematical Relationships.

(3) in (1) $$h = l_a \frac{\sin \alpha}{\sin \beta} \cos (\alpha + \beta) \qquad \text{(I)}$$

When l_i was measured, first l_a is calculated:

(1) = (2) $$l_a = l_i \frac{\cos (\alpha - \beta)}{\cos (\alpha + \beta)} \qquad \text{(II)}$$

and herefrom, according to (1), again h.

As regards angles α , and β , in the best interest of maximum simplification of the calculations of differences of height, the two following requirements have been laid down:

1. The quantity h must be an integral multiple of the outer staff section l_a ;
2. when the measurement of an inner staff section l_i only was possible, one should turn to the outer staff section by the addition of p %; or, mathematically expressed :

1. $$h = n\, l_a \qquad \frac{h}{l_a} = n$$

2. $$l_a = l_i + \frac{l_i p}{100} = \frac{100 + p}{100} l_i \qquad \frac{l_a}{l_i} = \frac{100 + p}{100}$$

These requirements introduced in equations (I) and (II) yield the equations of condition for α and β :

$$\left(\frac{h}{l_a} =\right) \frac{\sin \alpha}{\sin \beta} \cos (\alpha + \beta) = n$$

$$\left(\frac{l_a}{l_i} =\right) \frac{\cos (\alpha - \beta)}{\cos (\alpha + \beta)} = \frac{100 + p}{100}$$

from which the following values result :

a) for $n = 2$ and $p = 1\,°/_o$

$$\left(\frac{h}{l_a} = 2.000 \quad \frac{l_a}{l_i} = 1.010\right)$$

$\alpha = 5°44'21''$

$\beta = 2°50'02''$ (prism I)

b) for $n = 4$ and $p = 0.5\,°/_o$

$$\left(\frac{h}{l_a} = 4.000 \quad \frac{l_a}{l_i} = 1.005\right)$$

$\alpha = 5°44'21''$

$\beta = 1°25'17''$ (prism II)

As far as the present work is concerned, only the first pair of values is of interest, since prism II, which is more particularly designed for technical levelling, will only be tested later on. The further calculation of prism I for the condition of achromatism and its adaptation in minimum deviation position, will not be touched upon here.

2. *The Conditions of the Levelling Instrument for hilly Country and its Adjustment.*

As the prism cannot be produced to correspond exactly to the rough minimum deviation angle β , the question arises as to how an error in the edge angle of the prism acts on the procedure of levelling, i.e. to what extent the requirements (I) and (II) in para. I are influenced.

1. How does angle α vary for a given variation $d\beta$ in β in order that requirement (I) remain unchanged ?

By differentiation we have :

$$d\alpha = \frac{1}{2 \sin^2 \beta\, (\cot \beta \cot 2\alpha - 1)}\, d\beta$$

This gives for prism I : $d\alpha = 2.077\, d\beta$

and for prism II : $d\alpha = 4.115\, d\beta$

2. How does the discrepancy $d\,\beta$ from the rough value β influence the ratio

$$v = \frac{l_a}{l_i} = \frac{\cos(\alpha - \beta)}{\cos(\alpha + \beta)}\,?$$

The differentiation of the equation of condition (II) :

$$dv = \left(-\frac{m-1}{\rho}\cdot\frac{\sin(\alpha-\beta)}{\cos(\alpha+\beta)} + \frac{m+1}{\rho}\cdot\frac{\sin(\alpha+\beta)\cos(\alpha-\beta)}{\cos^2(\alpha+\beta)}\,d\beta\right)$$

$\left(\text{where } m = \frac{\alpha}{\beta}\right)$

shows that, for instance for $d\,\beta = 0.5'$, the error in the reduction of the inner staff section to the outer is $dv = 0.000\ 059$ for both prisms. This — again for the two prisms — gives, with sighting distances of approximately 40 m., an error in r', or v', of 0.24 mm. This error lies within the accuracy of observation and may therefore be neglected.

But, as the prism-edge angle $\gamma \overset{\prime\prime}{=} 2\,\beta$, $d\gamma$ is also $\overset{\prime\prime}{=} 2\,d\,\beta$, and therefore both equations of condition retain their validity, even with an error of 1' of the edge angle ; this accuracy can be attained by the optician when grinding the prism.

The mounting of the prism in its cell must be carried out in such a manner that the anallactic point, for a horizontal sighting, lies perpendicularly over the axis of tilt. Justification: With each forward or backward pointing with inclined line of sight, the difference of height of the point to be determined is measured relative to the anallactic point of the prism, since this point is the vertex of the angle of deviation β ; therefore, for a given setting up of the instrument, the height of the anallactic point must be constant whilst the telescope is tilted about the horizontal axis. This is however best performed — with a centering of the middle level as already stated — by placing the middle of the prism over the horizontal axis and mounting the prism in the position of minimum deviation, for then the anallactic point coincides with the middle of the prism. In this case the error in the height of the anallactic point engendered by the tilting of the telescope is unimportant (maximum 0.2 mm.). Such error remains, besides, below the accuracy of measurement when the anallactic point lies within 2 mm. from the vertical plane through the horizontal axis. The engineer when mounting the prism can also very well comply with this requirement.

Adjustment:

a) The line of sight must lie parallel to the axis of rotation of the prism. As this axis is an invariable mechanical axis, the line of sight alone can be set relatively to it, and not conversely. With the centering of the middle level bubble, the staff section in both positions "+" and "—" of the prism at any distance is determined: if a difference is apparent, then the line of sighting must be corrected by displacing the graticule until both staff sections are equal.

b) By levelling from mid-point and one end, the error in the parallelism of the sighting axis and middle level axis is determined, which error then may be eliminated alone by adjustment of the middle level, since the aiming axis has already been corrected.

c) The angle of inclination of the lateral axis against the line of sight must correspond to the *actual* angle of deviation β = (rough β) $-\ d\beta$. An exact adjustment as regards the determination of β occurs only during the manufacturing of the instrument; for field use, this method is not necessary. The field adjustment of the lateral levels is realized by levelling up with inclined line of sight from mid-point and one end independently for both lateral levels — exactly as when adjusting the middle level; the correction is effected hereby only on the levels and not on the aiming axis. For the correction of the lateral levels, two ground points with suitable differences of height should be selected. The procedure is not different from that of the adjustment of the middle level, for, even with a considerable error in the lateral level axes, the staff section 1 is obtained perfectly, and the aggregate error in the "direct" reading shows up and is eliminated.

3. *Practical Execution of the Measurement with the Levelling Instrument for hilly Country.*

After adjustment, the determination of a backward sighting R or of a forward sighting V is carried out as follows. Whenever possible, horizontal pointing should be used; otherwise "levelling up" is performed with inclined line of sight, and thereby that lateral level utilized with the centering of which the direct collimating ray strikes the levelling stave. The prism is then rotated to that position ("+" or "—") in which a section of the staff is obtained. If, as an exception, both staff sections l_i and l_a are measurable, the outer one l_a is to be preferred, since, in this case, the percentum calculation of the reduction is saved. When rotating the prism into its position of use, by means of a stop there is obtained that the double-image of the staff appears exactly as wide as the single staff image would be at the distance concerned. The readings are then made at the line of separation of the two staff images — i.e. in the true middle of the staff (see Fig. 5). After this preparatory work, the lateral level concerned is centered exactly and the direct reading *r* or *v* (white), as well as the reading of the prism image (in colour), are made at the horizontal thread; the difference is the staff section (Fig. 5). With this the actual measurement would be terminated, however without a check.

Due to the thickness of the horizontal thread, a certain amount of inaccuracy viciates the staff section thus determined; this inaccuracy can be done away with by making, in the vicinity of the thread, on a centimetre division-mark of the graduation of the staff, the corresponding prism image reading (colored); preference should be given this directly obtained staff section, which may occur as a result of a not exactly centered level bubble, as compared to that

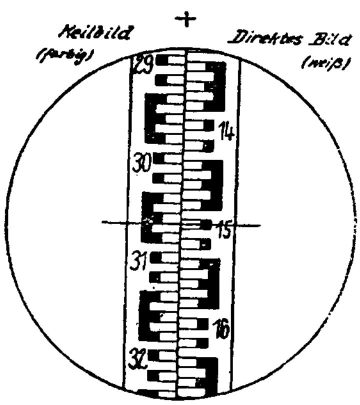


Fig. 5. — Field of view of the Telescope witht the Reading of an outer Staff Section by centered Level " + ".

previously determined from the readings at the horizontal thread. By the fact of making all of the four said readings (both stave images at the "thread" and in the "vicinity of the thread") a simultaneous perfect check is obtained. The formulary for the levelling instrument for hilly country (see Annexes pp. 54-57) has been drawn up according to these viewpoints.

The readings, white (column 2) and colored (column 3), are performed at the thread and in the vicinity of the thread ; with the agreement of the staff section (column 4) from both pairs of values, the measurement is terminated (maximum difference 1 mm.). For he computation of the levelling, only the direct thread reading (white *r* or white *v*) and the staff section 1 obtained from the readings in the "vicinity of the thread" are used. With the same sign position (level "+" and prism "+", or level "—" and prism "—") 1, as outer staff section ‘a with the correct sign by formation of the difference (reading white minus reading

4. *Result of the Test Measurements.*

colored), is determined; with unequal, i.e. crossed sign position, 1, as inner staff section l_i with correct sign from (reading colored minus reading white), is obtained. In this case the transformation to the outer staff section must be made by addition of 1 % of the staff section; for this reason

$$2l_a \text{ or } 2l_i + \frac{2l_i}{100} = 2l_a, \text{respectively,}$$

is formed in column 5.

This double outer staff section added to the direct thread reading *r* or *v* gives the horizontal backward sight R or forward sight V; further calculation, as with the ordinary method of levelling.

For the calculation of columns 5 and 8, a lateral sideways trial may be effected as a check.

By direct measurement of horizontal sightings with the levelling instrument for hilly country, only the direct thread reading is necessary, and, in the formulary R = *r* and V = *v*.

For the trying out of the instrument, an experimental stretch of ca. 1.7 km. in length, near Haimbach in the Eifel, with a difference of height of some 190-odd metres was reconnoitred and divided up into seven sections (tape lengths) by means of eight well-marked levelling points. The height of the position of origin (levelling point 1) was assumed to be 420.000 m. according to the topographic map to the scale of 1:25,000 available; the other levelling points were obtained with reference to this starting point by the ordinary method with horizontal sightings by properly adjusted instrument, but however with unequal sighting distances out and back. The results grouped in Table 1 offer a sufficient base for comparative measurements with the levelling instrument for hilly country, by means of which two levelling operations were carried out; their results, entered also in the table, reveal quite a good agreement between outward and return levelling. but,

Tabelle 1. Nivellementsergebnisse

Punkt Nr.	Entf. m	Gemess. Höhenunterschiede mit horiz. Zielungen 1. Niv.	2. Niv.	Diff. 1.—2. Niv. mm	Mittel I	Gemess. Höhenunterschiede mit Bergniv. 1. Niv.	2. Niv.	Diff. 1.—2. Niv. mm	Mittel II *Verb.*	Mittel I — verb. Mittel II mm
Niv.-P.1									*+16*	
	220	27,900	27,900	0	27,900	27,891	27,891	0	27,891	— 7
Niv.-P.2									*+16*	
	210	31,875	31,879	— 4	31,877	31,856	31,869	—13	31,862	— 1
Niv.-P.3									*+17*	
	230	27,919	27,923	— 4	27,921	27,904	27,895	+ 9	27,900	+ 4
Niv.-P.4									*+18*	
	240	27,554	27,549	+ 5	27,552	27,542	27,539	+ 3	27,540	— 6
Niv.-P.5									*+18*	
	240	35,457	35,466	— 9	35,461	35,443	35,434	+ 9	35,438	+ 5
Niv.-P.6									*+18*	
	240	15,855	15,856	— 1	15,855	15,838	15,841	— 3	15,840	— 3
Niv.-P.7									*+24*	
	320	20,744	20,745	— 1	20,745	20,721	20,721	0	20,721	0
Niv.-P.8										
									+127	
	1700	187,304	187,318	— 14	187,311	187,195	187,190	+ 5	187,192	— 8

Table I. — Results of Levellings.

on the other hand, in certain instances, also regular discrepancies as compared to levelling by the old method. The latter could be positively ascertained as due to the clumsiness of adjustment of the level in the test instrument, which was altered accordingly. After application of the graphically determined systematic error as a correction, the results agree quite well with the rough differences of height

A comparison of the practical measurements with horizontal sightings, on the one side, and with the levelling instrument for hilly country, on the other side, shows that the ordinary levelling method has managed the difference of height with 48 points of change, in the average, the new method, with 19 or 22, respectively. On the other hand, four readings (including test) were necessary with the levelling instrument for hilly country, against two only (including test) with horizontal sightings for a forward or a backward sight. The procedure in observing and plotting with the levelling instrument for hilly country, involves, after a short period of training, no special strain on the operator ; the position of the sign on the prism cell and on the level, or the tilting screw, are easily determined. From the experience gained in the trial measurements, the necessity of a few minor improvements to the experimental instrument was realized, after the carrying out of which the experiments will be continued. The instrument will undergo later on further development, especially in respect of protection of the levels and improvements of its external build.

The measurements carried out up to now with the first pattern of levelling instrument for hilly country have proved clearly that the accuracy of an ordinary levelling of average order, made with a well adjusted instrument, but carried out, however, regardless of equal sighting distances, may be expected, and that the saving of time, as compared with the "horizontal" levellings, in spite of the double-image reading, is considerable.