CONFORMAL PROJECTIONS OF THE ELLIPSOID.

by

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In order to establish a conformal projection of the terrestrial ellipsoid on the plane, we may employ the method indicated by Gauss, which consists in establishing first the conformal projection of the ellipsoid upon an intermediary sphere, and then the conformal projection of this sphere upon the plane.

If Z is the complex co-ordinate of the plane projection thus obtained, the scale (linear modulus) M_e of this projection will be equal to the scale M_s of the projection of the sphere on the plane, multiplied by a coefficient K, the expression for which, using the same notation as that in the Notes on Stereographic Projections (Hydrographic Review, vol. XVI, N° 1, page 33) is :

$$K = \frac{\alpha R \cos \varphi}{N \cos L},$$

and the development of which in the vicinity of the point of origin of latitude L_0, adopting for R the value $\sqrt{N_0 \rho_0}$ and for α^2 the value $1+e'^2 \cos^4 L_0$, will be, up to terms of the 4th order inclusive :

$$K = 1 - \frac{v^2}{3} e'^2 \frac{N_0}{\rho_0} \sin 2 L_0 \cos^3 L_0 + \frac{v^4}{6} e'^2 \frac{N_0^2}{\rho_0^2} \cos^4 L_0 (7 \sin^2 L_0 - 1).$$

On the other hand, the first two terms of the development of every conformal co-ordinate with respect to the powers of another conformal co-ordinate being the same, we may write :

$$\frac{Z}{N_0 \cos L_0} = \zeta - \frac{\zeta^2}{2} \sin L_0 ;$$

from which :

$$v = \frac{X}{N_0 \cos L_0} + \frac{X^2 - Y^2}{2 N_0^2 \cos^2 L_0} \sin L_0 ;$$

and consequently, by neglecting only the terms of the 5th order and above, we may write for every conformal projection of the ellipsoid, obtained by double projection :

$$(1) \quad K = 1 - \frac{e'^2 X^3}{3 N_0 R^2} \sin 2 L_0 + \frac{e'^2 X^2 Y^2}{N_0^2 R^2} \sin^2 L_0 - \frac{e'^2 X^4}{6 N_0^2 R^2} \left[1 + e'^2 (1 - 7 \sin^2 L_0)\right] \cos^2 L_0 .$$

I

GAUSS'S PROJECTIONS.

I. Jung has given a formula which allows the complex co-ordinate Z to be obtained for an entire series of conformal projections (called projections of separate variables) as a function of the complex co-ordinate z of the stereographic projection (1) having the same origin :

$$Z = \frac{2\,R e^{i\theta}}{\sqrt{n}} \text{ arc tang } \frac{\sqrt{n}\,z}{2\,R e^{i\theta}} \qquad (2)$$

This formula is established for a sphere of radius R ; n is a real parameter and θ the angle which the axis of X makes with the major axis of the indicatrix. We may agree to apply this to the plane projections of the ellipsoid by replacing z with a value which we have given to this complex co-ordinate (see formula (1), Hydrographic Review, vol. XVI, N° 1, page 34) and which has been established by double projection. We shall thus have a projection Z of the ellipsoid obtained by the Gauss method of double projection, to which the name of the corresponding projection of the sphere is generally given. Its scale possesses the property which we have just indicated.

In the formula (2) the quantity Z changes sign at the same time as z. The symmetry shows that the scale of the Jung projections of the sphere does not change when z changes sign and that, consequently, its development according to the powers of the co-ordinates, contains only unity and the terms of an even degree.

In order to obtain the scale of the projection of the ellipsoid, it is necessary to multiply this development by K. From this it follows that the term of the third order of the scale for every Jung projection of the ellipsoid, is :

$$-\frac{e'^2\,X^3}{3\,N_0\,R^2}\sin 2\,L_0$$

This is of a particular advantage; because, in view of the smallness of e'^2, we may consider this term as being of the same order as the following, namely the 4th order, and as a consequence, the projection may be employed at some distance from the origin, without the necessity, in practice, of taking into consideration the terms of the scale beyond those of the second order. If we develop as a series the second member of formula (2), the development will only contain the uneven powers of the variable z. All the terms of the development will still be of the uneven degree, if we replace z by any other projection of Jung. For the rest, we may stop the development at any desired term, and such a projection can always be rigorously termed a conformal projection. But its scale cannot be the same as though we had retained the entire series. If we suppress all the terms of the development of a higher degree than 2n + 1, there will be no change in the development of the modulus except beyond the terms of the order of 2n + 2 inclusive. If, for instance, we retain for Z simply the terms of the first and 3rd order, the modification of the scale will not be produced except from and after the 4th order inclusive,

(1) See *Hydrographic Review*, vol. X, N° 1, pages 83 to 87.

and the term of the 3rd order of the scale will still be that which we have just shown and which has e'^2 as a factor.

But this will not hold true if we replace z by the development which is given us by formula (3) in the Note cited above (Hydrographic Review, vol. XVI, N° 1, page 37), as a function of the complex co-ordinate of the Mercator projection. This last projection has its origin on the equator and not (in general) at the origin of the projection Z. The development of Z as a function of the powers of this co-ordinate, which we have called ζ, will contain the terms of all degrees in ζ. Consequently, it is necessary to retain the terms of this development at least up to the 4th order inclusive, if we desire that the term of the 3rd order of the scale shall remain unchanged. Therefore, we shall express Z by a series of four terms, in place of two ; but the tables of meridional parts eliminate the necessity of making the calculations for the co-ordinate z. If we retain but three terms, the term of the 3rd order of the scale will be different and will no longer have e'^2 as a factor.

We shall make an application of this method to the projection of Gauss, for which we have :

$$n = 1, \quad \theta = o$$

a) The equation (2) becomes :

$$Z = 2\,R \text{ arc tang} \frac{z}{2\,R} \cdot \tag{3}$$

To calculate the scale, we obtain :

$$\sqrt{\frac{d\,X^2 + d\,Y^2}{d\,x^2 + d\,y^2}} = \cos\frac{X + iY}{2\,R} \cos\frac{X - iY}{2\,R} \cdot$$

For the rest, the scale of the projection z of the ellipsoid is equal to :

$$K\left(1 + \frac{x^2 + y^2}{4\,R^2}\right);$$

but we deduce from the equation (3) :

$$\frac{x^2 + y^2}{4\,R^2} = \text{tang}\frac{X + iY}{2\,R}\,\text{tang}\frac{X - iY}{2\,R}\,.$$

We have therefore :

$$M_e = K\left(\cos\frac{X + iY}{2\,R}\cos\frac{X - iY}{2\,R} + \sin\frac{X + iY}{2\,R}\sin\frac{X - iY}{2\,R}\right) = K\cos\frac{iY}{R} = \frac{K}{2}\left(e^{\frac{Y}{R}} + e^{\frac{-Y}{R}}\right)$$

By taking for K the development up to the 4th order inclusive in formula (1), and developing similarly the parenthesis, we obtain :

$$M_e = K\left(1 + \frac{Y^2}{2\,R^2} + \frac{Y^4}{24\,R^4}\right) \cdot \tag{4}$$

b) If we develop the formula (3) in accordance with ascending powers of z, we shall have :

$$Z = z - \frac{z^3}{12\,R^2} + \frac{z^5}{80\,R^4} - \frac{z^7}{448\,R^6} + \cdots.$$

This formula appears advantageous because it contains only the uneven powers of z ; but it presupposes a rather long calculation of z. Let us substitue here for z a development which is given us by formula (3) in the above cited note (Hydrographic Review, vol. XVI, N° 1, page 37), and limit ourselves to the 5th order inclusive.

We then obtain :

$$(5)\quad \frac{Z}{N_0 \cos L_0} = \zeta - \frac{\zeta^2}{2} \sin L_0 + \frac{\zeta^3}{6}\left(\sin^2 L_0 - \frac{N_0}{\rho_0}\cos^2 L_0\right) + \frac{\zeta^4}{24}\left(5\,\frac{N_0}{\rho_0}\cos^2 L_0 - \sin^2 L_0\right) \sin L_0 + \frac{\zeta^5}{120}\left(\sin^4 L_0 - 18\,\frac{N_0}{\rho_0}\sin^2 L_0 \cos^2 L_0 + 5\,\frac{N_0^2}{\rho_0^2}\cos^4 L_0\right).$$

In this equation the coefficients of the successive powers of ζ depend only on the position of the origin and remain the same for the whole projection ; for the rest, the powers of ζ may be calculated very rapidly by means of the tables of meridional parts (Special Publication N° 21 of the International Hydrographic Bureau). For the rest, no matter what the number of terms retained in the development of the formula (5), the projection will always be rigorously conformal. If we retain five terms, the expression for the scale will be exactly that of formula (4). If we limit ourselves to four terms, the scale will still have for the first terms :

$$1 + \frac{Y^2}{2\,R^2} - \frac{e'^2\,X^3}{3\,N_0\,R^2} \sin 2\,L_0 .$$

By putting therefore :

$$v^2 + G^2 = d^2 , \qquad \frac{G}{v} = \text{tang}\,\delta ,$$

we shall be able to calculate the Gauss co-ordinates by the following formulae :

$$(6)\quad \begin{cases} \dfrac{X}{N_0 \cos L_0} = d \cos\delta - \dfrac{d^2}{2}\cos 2\delta \sin L_0 + \dfrac{d^3}{6}\cos 3\delta\left(\sin^2 L_0 - \dfrac{N_0}{\rho_0}\cos^2 L_0\right) \\ \qquad\qquad - \dfrac{d^4}{24}\cos 4\delta \sin L_0\left(\sin^2 L_0 - 5\,\dfrac{N_0}{\rho_0}\cos^2 L_0\right), \\ \dfrac{Y}{N_0 \cos L_0} = d \sin\delta - \dfrac{d^2}{2}\sin 2\delta \sin L_0 + \dfrac{d^3}{6}\sin 3\delta\left(\sin^2 L_0 - \dfrac{N_0}{\rho_0}\cos^2 L_0\right) \\ \qquad\qquad - \dfrac{d^4}{24}\sin 4\delta \sin L_0\left(\sin^2 L_0 - 5\,\dfrac{N_0}{\rho_0}\cos^2 L_0\right). \end{cases}$$

If we had retained the first three terms of formula (5), the scale would then have been :

$$M = 1 + \frac{Y^2}{2\,R^2} + \frac{XY^2}{2\,N_0^3}\,\text{tg}\,L_0\left(5\,\frac{N_0}{\rho_0} - \text{tg}^2 L_0\right) + \frac{X^3\,\text{tg}\,L_0}{6\,N_0^3}\left(\text{tg}^2 L_0 - \frac{N_0}{\rho_0} - 4\,\frac{N_0^2}{\rho_0^2}\right).$$

The terms of the 3rd order would not have been multiplied by e'^2

c) In paragraph a), we have projected the ellipsoid on to a sphere ; by this projection the element ds of the prime meridian has become Kds and the length of the arc of the meridian has become s_1 ; the projection of this sphere in the Gauss system has retained the prime meridian of the sphere in its true length. We have then :

$$X_0 = s_1$$

The value of s , or $\int_0^s K ds$ is, in accordance with the value of K given in the Hydrographic Review, vol. XVI, N° 1, page 43 :

$$s_1 = X_0 = s - \frac{e'^2 s^4 \sin 2 L_0}{12 R^2 N_0} - \frac{e'^2 s^5 \cos^2 L_0}{30 R^2 N_0^2} \left[1 + e'^2 (1 - 7 \sin^2 L_0)\right] + - - - .$$

The formula (5) furnishes the same result as the formula (3) to the nearest approximation resulting from the number of terms employed. The representation of the prime meridian, therefore, depends upon the latitude of the point of origin which is chosen for it. One might wish that it were otherwise, especially since, as is generally the case, we employ this projection to represent a long strip of terrain parallel to the meridian. By taking the origin in the middle of this strip, we shall be able to have values of X which are rather large and for which the term $\frac{e'^2 X^3 \sin 2 L_0}{3 N_0 R^2}$ of the scale can no longer be considered negligible. We shall show two methods for maintaining the prime meridian in the original size which it has on the ellipsoid ; that is, so that it shall be « automecoïc ».

In order that the projection shall be conformal, the complex co-ordinate Z must be a continuous function of ζ :

$$Z = f (v + i G)$$

When G is zero, we should have :

$$X_0 = s \ .$$

Therefore :

$$f (v) = s \ .$$

The Taylor development then gives us : (2)

$$Z = \zeta \left(\frac{ds}{dl}\right)_0 + \frac{\zeta^2}{2} \left(\frac{d^2 s}{dl^2}\right)_0 + \frac{\zeta^3}{6} \left(\frac{d^3 s}{dl^3}\right)_0 + - - - .$$

No matter what the number of terms retained in this series, the projection will be rigorously conformal, because Z will always be a function of ζ . The axis of X will represent exactly the meridian of the ellipsoid if we retain the entire series ; this condition will be approached more and more closely, the more the terms which are retained.

Let us calculate the successive derivatives of s with respect to the meridional part :

$$\frac{ds}{dl} = N \cos L \ , \ \frac{d^2 s}{dl^2} = - N \sin L \cos L, \ \frac{d^3 s}{dl^3} = N \cos L \left(\sin^2 L - \frac{N}{\rho} \cos^2 L\right),$$

$$\frac{d^4 s}{dl^4} = N \sin L \cos L \left(4 \frac{N^2}{\rho^2} \cos^2 L + \frac{N}{\rho} \cos^2 L - \sin^2 L\right),$$

$$\frac{d^5 s}{dl^5} = N \cos L \left[4 \frac{N^3}{\rho^3} \cos^2 L (1 - 7 \sin^2 L) + \frac{N^2}{\rho^2} \cos^2 L (1 + 7 \sin^2 L) - 2 \frac{N}{\rho} \sin^2 L \cos^2 L + \sin^4 L\right].$$

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(2) This procedure might have been employed in place of the quasi-stereographic projection (See Hyd. Rev., vol. XVI, N° 1, p. 42). If we desire that the representation of the prime meridian shall be : $X_0 = 2 R \operatorname{tg} \frac{s}{2 R}$ it would be necessary to take:

$$f (v) = 2 R \operatorname{tg} \frac{s}{2R}, \quad \left(\frac{df}{dl}\right)_0 = \frac{N_0}{\rho_0} \cos L_0 \ , \ \text{etc...}$$

But in this case, the method has not the same interest as for the Gauss projection.

We have therefore:

$$
(7)\left\{\begin{aligned}
\frac{Z}{N_0 \cos L_0} = \zeta - \frac{\zeta^2}{2} \sin L_0 + \frac{\zeta^3}{6}\left(\sin^2 L_0 - \frac{N_0}{\rho_0}\cos^2 L_0\right) + \frac{\zeta^4}{24}\sin L_0 \\
\left(4\frac{N_0^{\,2}}{\rho_0^{\,2}}\cos^2 L_0 + \frac{N_0}{\rho_0}\cos^2 L_0 - \sin^2 L_0\right) \\
+\frac{\zeta^5}{120}\left[4\frac{N_0^{\,3}}{\rho_0^{\,3}}\cos^2 L_0\left(1 - 7\sin^2 L_0\right) + \frac{N_0^{\,2}}{\rho_0^{\,2}}\cos^2 L_0\left(1 + 7\sin^2 L_0\right)\right. \\
\left. - 2\frac{N_0}{\rho_0}\sin^2 L_0 \cos^2 L_0 + \sin^4 L_0\right] + \cdots
\end{aligned}\right.
$$

This expression for Z does not differ from that in formula (5) except from the terms of the 4th order onward.

The scale, limited to the 3rd order, will be therefore :

$$
M = 1 + \frac{Y^2}{2R^2} - \frac{e'^2 XY^2 \sin 2L_0}{R^2 \rho_0} .
$$

We see that within this limit, it is equal to unity if Y is zero.

The expression (7) for Z, limited to 4 terms, is almost as easy to calculate as the expression (5) and appears to be more suitable when the strip to be represented is very long. Moreover, it is not entirely independent of the latitude of origin selected.

d) In order to obtain a representation entirely independent of the latitude of origin, we shall develop the function :

$$
Z = f(v + iG)
$$

according to the Taylor theorem, with respect to iG :

$$
Z = f(v) + iG\frac{df}{dv} - \frac{G^2}{2}\frac{d^2 f}{dv^2} - \frac{iG^3}{6}\frac{d^3 f}{dv^3} + \frac{G^4}{24}\frac{d^4 f}{dv^4} + \frac{iG^5}{120}\frac{d^5 f}{dv^5} + \cdots,
$$

in which $f(v)$ is equal to s. Therefore, noting that $GN \cos L$ is the length of the arc of the parallel which we shall call u, we shall have :

$$
(8)\left\{\begin{aligned}
X = s + \frac{u^2}{2N}\operatorname{tg} L + \frac{u^4}{24N^3}\operatorname{tg} L\left(4\frac{N^2}{\rho^2} + \frac{N}{\rho} - \operatorname{tg}^2 L\right) \\
+ \frac{u^6}{720 N^5}\operatorname{tg} L\,(61 - 58\operatorname{tg}^2 L + \operatorname{tg}^4 L) + \cdots, \qquad (3) \\
Y = u + \frac{u^3}{6N^2}\left(\frac{N}{\rho} - \operatorname{tg}^2 L\right) + \frac{u^5}{120N^4}\left[4\frac{N^3}{\rho^3}(1 - 6\operatorname{tg}^2 L) + \frac{N^2}{\rho^2}(1 + 8\operatorname{tg}^2 L)\right. \\
\left. - 2\frac{N}{\rho}\operatorname{tg}^2 L + \operatorname{tg}^4 L\right] + \cdots .
\end{aligned}\right.
$$

The projection thus defined does not depend any longer upon the latitude of origin, no matter at which term the progression is stopped, and for it the definition of the prime meridian will always be : $X_0 = s$. But only the complete series will yield a projection which is rigorously conformal ; limited, it is no longer a function of ζ . It would therefore be necessary to take a rather large number of terms in order to

(3) We have replaced in the factor of the term in u^6, $\frac{N}{\rho}$ by unity.

be certain that the conformality of the projection might be assured within the limits of accuracy desired. This object will be attained if the terms neglected are without appreciable value. The « Landesaufnahme of Prussia » has adopted the number of terms of the above formula.

Over the entire prime meridian, the projection is rigorously conformal and its scale equal to unity. If one grants that the conformality is sufficiently realised, we may then calculate the scale for the variation du. We shall have :

$$M = 1 + \frac{u^2}{2 R^2} + \frac{u^4}{6 N^4}\left[tg^4 L - 3 \frac{N}{\rho} tg^2 L + \frac{N^2}{\rho^2} \frac{5 + 56 \, tg^2 L}{4} + 5 \frac{N^3}{\rho^3} (1 - 6 \, tg^2 L)\right] + \cdots$$

This expression does not contain terms of the 3rd order.

e) Commandant P. Tardi, at the Sixth General Assembly of the Association of Geodesy, held at Edinburgh in September 1936, proposed that all triangulations of the countries comprised between latitudes — 36° and + 36°, and particularly of the continent of Africa, should be calculated on the Gauss system, by spindles of 6° of longitude. (See Proceedings of the Association of Geodesy, Volume 14).

The origin should be, therefore, situated at the equator, which will result in bringing about considerable simplification to the greater part of the formulae.

In the method of double projection, the expression for the linear distortion produced by the projection of the ellipsoid on the sphere (see formula (1)) becomes reduced to the 4th order:

$$K = 1 - \frac{e'^2 X^4}{6 a^4} (1 + e'^2)^2 ;$$

which, for X = 4000 km., will give :

$$K = 1 - \frac{1}{5655} \quad (4)$$

Formula (5) then becomes :

$$\frac{Z}{a} = \zeta - \frac{\zeta^3}{6} (1 + e'^2) + \frac{\zeta^5}{24} (1 + e'^2)^2 + \cdots$$

It no longer contains the even terms and is very easy to calculate.

We might also employ the formula (7) which becomes :

$$\frac{Z}{a} = \zeta - \frac{\zeta^3}{6} (1 + e'^2) + \frac{\zeta^5}{24} (1 + e'^2)^2 \left(1 + \frac{4}{5} e'^2\right) + \cdots .$$

With these two formulae, the scale contains only terms of the 2nd order, of the 4th and above.

The same will be true of the scale if we establish this projection by the method employed by M. Roussilhe for the quasi-stereographic projection (see Hydrographic Review, vol. XVI, N° 1, page 42), by defining

(4) and not 1/1,800 as the author states.

the representation of the prime meridian by $X_o = s$, and writing that the projection is conformal. We have then the formula :

$$(9)\left\{\begin{aligned} X = {} & s + \frac{u^2 s}{2a^2}(1+e'^2) + \frac{u^2 s^3}{6a^4}(1+e'^2)^2(1-2e'^2) + \frac{u^4 s}{72a^4}(1+e'^2)^2(11+8e'^2) \\ & + \frac{u^2 s^5}{15a^6}\left(1+e'^2\right)^3\left(1-e'^2+\frac{11}{2}e'^4\right) + \frac{u^4 s^3}{18a^6}(1+e'^2)^3\left(11+\frac{15}{2}e'^2-2e'^4\right) \\ & + \cdots, \\ Y = {} & u + \frac{u^3}{6a^2}(1+e'^2) - \frac{u^3 s^2}{18a^4}(1+e'^2)^2(1+4e'^2) \\ & - \frac{u^3 s^4}{18a^6}(1+e'^2)^3(11+12e'^2-2e'^4) + \frac{u^5}{360a^4}(1+e'^2)^2(11+8e'^2) \\ & + \frac{u^5 s^2}{2160a^6}(1+e'^2)^3(52+55e'^2-4e'^4) + \cdots, \end{aligned}\right.$$

with :

$$M = 1 + \frac{u^2}{2a^2}(1+e'^2) + \frac{u^4}{72a^4}(1+e'^2)^2(11+8e'^2) + \frac{2}{9}\frac{u^2 s^2}{a^4}(1+e'^2)^2(2-e'^2) + \cdots;$$

modulus which is equal to unity throughout the entire length of the prime meridian. The term of the 4th degree which depends upon s never exceeds 1/3150.

These rather complicated formulae have the advantage of having constant coefficients which are all calculated once and for all. Commandant Tardi has preferred to employ formulae (8) which give the same accuracy with a few less terms, but in which the coefficients are functions of the latitude. Profiting by the fact that the quantities vary little with this, he has been able to construct very condensed tables which greatly facilitate the calculation of the co-ordinates. But formula (8) and formula (9) give, within the same limits of approximation, the same result, because the fact of fixing the representation of the prime meridian completely defines the conformal projection.

f) *Historical:* The so-called Gauss projection, or the transverse Mercator, or the orthomorphic cylindrical projection of Lambert, was discovered by J. H. Lambert (Beyträge zum Gebrauche der Mathematik und deren Anwendung — Berlin, 1772). C. F. Gauss mentioned it in 1822 in a Note which he presented to the Academy of Sciences at Copenhagen and which was awarded the prize by this Academy. This note was published by Schumacher in 1825. Gauss treated the question again in 1843 in his « Untersuchungen über Gegenstände der höheren Geodäsie ».

General Oscar Schreiber again took up the study in his work « Theorie der Projektionsmethode der hannoverschen Landesvermessung - 1866 » and completed it in 1878.

Professor Dr. L. Krüger brought a new contribution to the study of this projection in the publication of the Geodetic Institute of Potsdam : « Konforme Abbildung des Erdellipsoids in der Ebene (1912) ». He had abandoned the double projection of Schreiber in order to adopt the « automecoïc » representation of the prime meridian, that is, in which all the parts retain their exact length on the projection.

II

JUNG PROJECTIONS.

The development of the formula (2) up to the third order gives us :

$$Z = z - \frac{n e^{-2i\theta}}{12 R^2} z^3 .$$

If, in this formula, we take n = 1 and θ = o, it will give the first terms of the development of the Gauss coordinate :

$$z_g = z - \frac{1}{12 R^2} z^3 .$$

From which, to the same approximation :

$$z = zg + \frac{1}{12 R^2} z_g^3 .$$

And, consequently, for Z the following expression, as a function of the Gauss co-ordinate, may be calculated for the ellipsoid.

$$Z = z_g + \frac{1}{12 R^2} \left(1 - n e^{-2i\theta}\right) z_g^3 .$$

This is the formula which Col. Laborde employed for the triangulation of Madagascar (5). Tissot had already proposed (6) an analogous formula in which the variables were the distances of the parallel from the origin, reckoned along the prime meridian, and the distance of the meridian, from the origin reckoned on the parallel of origin. Tissot performed a great service by showing that for this type of projections the linear distortions were only of the second order, and from this he deduced the projections called projections of minimum deformation.

M. Courtier, Ingénieur Hydrographe Général (7), has made a study of the Tissot developments by making use of other variables ; the co-ordinates of Cassini-Soldner, also called orthogonal or geodetic (8).

But the formulae of Tissot and Courtier do not provide projections which are rigorously conformal, the angular distortions in them being of the third order, while the formulae of Col. Laborde, and all those which may be deduced from formula (2) are rigorously conformal regardless of the number of terms used in the development.

Let us replace z_g in this last formula by the value (5), obtained for this co-ordinate as a function of the Mercator co-ordinate. We shall obtain, to the same approximation :

$$(10) \quad \frac{Z}{N_0 \cos L_0} = \zeta - \frac{\zeta^2}{2} \sin L_0 + \frac{\zeta^3}{12} \left(2 \sin^2 L_0 - \frac{N_0}{\rho_0} \cos^2 L_0 - n e^{-2i\theta} \frac{N_0}{\rho_0} \cos^2 L_0 \right)$$
$$+ \frac{\zeta^4}{24} \sin L_0 \left(2 \frac{N_0}{\rho_0} \cos^2 L_0 - \sin^2 L_0 + 3 n e^{-2i\theta} \frac{N_0}{\rho_0} \cos^2 L_0 \right) + \cdots .$$

(5) *Traité des projections des Cartes géographiques*, by MM. L. Driencourt and J. Laborde. — Hermann & C°, Paris 1932, 4th part, pages 234 & subs.

(6) *Mémoire sur la représentation des surfaces et des projections des cartes géographiques*, Paris, 1881.

(7) *Annales Hydrographiques*, Paris 1912, pages 40 and subs.

(8) *Hydrographic Review*, vol. VII, N° 1, pages 13 and subs.

This expression is somewhat complicated, but it is not less rigorously conformal ; and it presents the advantage of avoiding the necessity for passing through the double projection or through the calculation of the Gauss co-ordinate ; it suffices to make use of a reasonably accurate table of meridional parts for the ellipsoid.

The scale therefore becomes, when not exceeding the third order :

$$M = 1 + \frac{X^2}{4 N_o \rho_o}(1 - n \cos 2\theta) + \frac{Y^2}{4 N_o \rho_o}(1 + n \cos 2\theta) - \frac{XY}{2 N_o \rho_o} n \sin 2\theta - \frac{e'^2 X^3 \sin 2 L_o}{3 N_o^2 \rho_o} + - - - .$$

The expressions for X and Y are easier to calculate and comprise only 6 terms, if we put :

$$v^2 + G^2 = d^2 \ , \qquad \frac{G}{v} = \mathrm{tg}\ \delta \ .$$

We then have $\zeta = d\, e^{i\delta}$; and the formula becomes :

$$\frac{Z}{N_o \cos L_o} = d\, e^{i\delta} - \frac{d^2}{2} \sin L_o\, e^{2i\delta} + \frac{d^3}{12} e^{3i\delta}\left(2 \sin^2 L_o - \frac{N_o}{\rho_o}\cos^2 L_o\right) + \frac{d^4}{24} e^{4i\delta} \sin L_o \left(2\frac{N_o}{\rho_o}\cos^2 L_o - \sin^2 L_o\right) - \frac{nd^3}{12}\frac{N_o}{\rho_o}\cos^2 L_o\, e^{i(3\delta - 2\theta)} + \frac{nd^4}{8}\frac{N_o}{\rho_o} \sin L_o \cos^2 L_o\, e^{i(4\delta - 2\theta)} .$$

We then deduce :

$$\frac{X}{N_o \cos L_o} = d \cos \delta - \frac{d^2}{2} \sin L_o \cos 2\delta + \frac{d^3}{12} \cos 3\delta \left(2 \sin^2 L_o - \frac{N_o}{\rho_o}\cos^2 L_o\right) + \frac{d^4}{24}\cos 4\delta \sin L_o \left(2\frac{N_o}{\rho_o}\cos^2 L_o - \sin^2 L_o\right) - \frac{nd^3}{12}\frac{N_o}{\rho_o}\cos^2 L_o \cos(3\delta - 2\theta) + \frac{nd^4}{8}\frac{N_o}{\rho_o}\sin L_o \cos^2 L_o \cos(4\delta - 2\theta) ,$$

$$\frac{Y}{N_o \cos L_o} = d \sin \delta - \frac{d^2}{2} \sin L_o \sin 2\delta + \frac{d^3}{12} \sin 3\delta \left(2 \sin^2 L_o - \frac{N_o}{\rho_o}\cos^2 L_o\right) + \frac{d^4}{24}\sin 4\delta \sin L_o \left(2\frac{N_o}{\rho_o}\cos^2 L_o - \sin^2 L_o\right) - \frac{nd^3}{12}\frac{N_o}{\rho_o}\cos^2 L_o \sin(3\delta - 2\theta) + \frac{nd^4}{8}\frac{N_o}{\rho_o}\sin L_o \cos^2 L_o \sin(4\delta - 2\theta) .$$

The terms of the 2nd order of M show us that the isometre curve of this projection is an ellipse of which the major axis, proportional to $\frac{1}{\sqrt{1-n}}$, makes an angles θ with the axis of X, and of which the minor axis is proportional to $\frac{1}{\sqrt{1+n}}$. If *a* and *b* are the axes of this ellipse, we have :

$$n = \frac{a^2 - b^2}{a^2 + b^2} ,$$

By choosing for the representation of the given terrain, the ellipse which encloses the totality of the terrain and in which the diameters bissecting the axis have the minimum length, we shall obtain a projection of minimum distortion according to the ideas of Tissot. By giving in

$$X = N_0 \cos L_0 \, d \sin 1' \left[\cos\delta - \frac{\sin L_0}{2} d \sin 1' \cos 2\delta + A d^2 \sin^2 1' \cos 3\delta - B d^2 \sin^2 1' \cos(3\delta - 2\theta) + C d^3 \sin^3 1' \cos 4\delta + D d^3 \sin^3 1' \cos(4\delta - 2\theta) \right]$$

$$Y = N_0 \cos L_0 \, d \sin 1' \left[\sin\delta - \frac{\sin L_0}{2} d \sin 1' \sin 2\delta + A d^2 \sin^2 1' \sin 3\delta - B d^2 \sin^2 1' \sin(3\delta - 2\theta) + C d^3 \sin^3 1' \sin 4\delta + D d^3 \sin^3 1' \sin(4\delta - 2\theta) \right]$$

$$A = \frac{1}{12}\left(2 \sin^2 L_0 - \frac{N_0}{\rho_0} \cos^2 L_0\right), \quad B = \frac{1}{12} \frac{N_0}{\rho_0} \cos^2 L_0, \quad C = \frac{\sin L_0}{24}\left(2 \frac{N_0}{\rho_0} \cos^2 L_0 - \sin^2 L_0\right), \quad D = \frac{1}{8} \frac{N_0}{\rho_0} \cos^2 L_0 \sin L_0$$

$$L_0 = 41° - 40'$$

$lg \cos L_0$	9,87333518
$lg N_0$	6,80535706
$lg N_0 \cos L_0$	6,67869224

$lg \cos^2 L_0$	9,74667036
$lg N_0$	6,80535706
$colg \rho_0$	3,19628016
$lg \frac{N_0}{\rho_0} \cos^2 L_0$	9,74830758
$lg\ 12$	1,07918125
$lg B$	8,66912633

$lg \sin L_0$	9,82268831
	9,74830758
$colg\ 8$	9,09691001
$lg D$	8,66790590

$lg \sin^2 L_0$	9,64537662
$\sin^2 L_0$	0,44195354
$2 \sin^2 L_0$	0,88390708
$\frac{N_0}{\rho_0} \cos^2 L_0$	0,56015418
$12 A$	0,32375290
$lg\ 12 A$	9,51021370
$lg\ 12$	1,07918125
$lg A$	8,43103245

	0,44195354
$2 \frac{N_0}{\rho_0} \cos^2 L_0$	1,12030836
$24 C$	0,67835482
$lg\ 24 C$	9,83145691
$lg \sin L_0$	9,82268831
$colg\ 24$	8,61978876
$lg C$	8,27393398

$L = 43°-50'-29''68$	$\ell = 2916,59242$	$\delta = -59°-23'-29''432$	$\theta = 133°-30'$
$G = 4°-59'-21''73$	$\ell_0 = 2739,49024$	$2\delta = -118°-46'-58''865$	$2\theta = 267°-00'$
$lg G$ 2,47619692	$v = 177,10218$	$3\delta = -178°-10'-28''297$	$3\delta - 2\theta = -85°-10'-28''297$
		$4\delta = -237°-33'-57''730$	$4\delta - 2\theta = -144°-33'-57''730$

$lg\, d \sin 1'$	9,00508810		9,00508810
$lg \cos 2\delta$	9,68259081	$lg \sin 2\delta$	9,94272687
$lg \sin L_0$	9,82268831		9,82268831
$colg\ 2$	9,69897000		9,69897000
lg 2e term	8,20933722		8,46947348

$lg\, v$	2,24822391
$lg\, tg\, \delta$	0,22797301

$lg \cos \delta$	9,70686193	$lg \sin \delta$	9,93483494
$lg\, v$	2,24822391	$lg G$	2,47619692
$lg\, d$	2,54136198		2,54136198

	2,54136198
$lg \sin 1'$	6,46372612
$lg\, d \sin 1'$	9,00508810
	6,67869224
$lg\, N_0 \cos L_0\, d \sin 1'$	5,68378034

$lg \cos 3\delta$	9,99977954	$lg \sin 3\delta$	8,50317932	$lg \cos(3\delta - 2\theta)$	8,92490373	9,99845793
$lg A$	8,43103245		8,43103245	$lg B$	8,66912633	8,66912633
$lg\, d^2 \sin^2 1'$	8,01017620		8,01017620		8,01017620	8,01017620
lg 3e term	6,44098819		4,94438797	lg 4e term	5,60420626	6,67775646

$lg \cos 4\delta$	9,72942982		9,92634767	$lg \cos(4\delta - 2\theta)$	9,91104263	9,76325144
$lg C$	8,27393398		8,27393398	$lg D$	8,66790590	8,66790590
$lg\, d^3 \sin^3 1'$	7,01526430		7,01526430	$lg\, d^3 \sin^3 1'$	7,01526430	7,01526430
lg 5e term	5,01862810		5,21554595	lg 6e term	5,59421283	5,44642164

1er term	+ 0,509168968	− 0,860666574
2e term	+ 0,016193370	+ 0,029476321
3e term	− 0,000276050	− 0,000008798
4e term	− 0,000040198	+ 0,000476168
5e term	− 0,000010438	+ 0,000016427
6e term	− 0,000039284	− 0,000027953
Σ	+ 0,524996368	− 0,830734409
$lg \Sigma$	9,72015630	9,91946220
$lg\, N_0 \cos L_0\, d \sin 1'$	5,68378034	5,68378034
$lg X$	5,40393664	5,60324254
	+ 253.475,881	− 401.090,556
	− 12,674	− 20,055
	X = 253.463,207	Y = 401.070,501

formula (10), the value 0 to n, we re-discover the stereographic projection for which the isometer is a circle ; by giving to n the value 1 and to θ the value zero, we shall obtain the formula (5) of the Gauss projection, for which the isometer curve is formed by two straight lines parallel to the axis of X.

In the case of hydrographic surveys, the region which is to be represented as perfectly as possible, is usually a strip parallel to the coast. If, on the other hand, the coast is approximately rectilinear, the isometers should be two straight lines nearly parallel to the coast. We give therefore to n a value 1, and we take for θ the angle of inclination of the direction of the coastline to the prime meridian.

If the coastline has a direction sensibly North-South, we shall adopt the projection of Gauss : $n = 1$, $\Theta = 0$. If it lies East-West, we shall use $n = 1$ and $\theta = \frac{\pi}{2}$, and the co-ordinate will be then :

$$X = d \cos \delta - \frac{d^2}{2} \sin L_0 \cos 2\delta + \frac{d^3}{6} \sin^2 L_0 \cos 3\delta - \frac{d^4}{24} \sin L_0 \cos 4\delta \, (1 + e'^2 \cos^4 L_0) + --,$$

$$Y = d \sin \delta - \frac{d^2}{2} \sin L_0 \sin 2\delta + \frac{d^3}{6} \sin^4 L_0 \sin 3\delta - \frac{d^4}{24} \sin L_0 \sin 4\delta \, (1 + e'^2 \cos^4 L_0) + --.$$

We give below an example of the calculation, by applying the formula (10) at the same point at which we have already calculated the co-ordinates by passing through the double projection and the Gauss co-ordinates. (See Hydrographic Review, vol. VII, N° 1, pages 26 and subs.). The results do not differ by more than a few decimeters. This difference is due to the terms of the order higher than the 4th, which we have not employed in the formula (10). The suppression is further without detrimental effect because the projection furnished by formula (10) is always rigorously conformal, no matter what the number of terms retained. The two projections employed for this new calculation and for the old are slightly different : but both are rigorously conformal and the scale is the same up to the third order inclusive.

We have diminished the values found for the co-ordinates by 1/20,000 of their value, in order to take into account the factor called K_0 in the example cited, and which has the purpose of reducing to about half the linear distortion at the calculated point.

Note. — We may rediscover the formula (10) by another way ; the complex co-ordinate Z of any conformal projection of the ellipsoid around the point of origin may be expressed, a priori, in an absolutely general manner, by a sum of four terms arranged according to the ascending powers of ζ :

$$\frac{Z}{N_0 \cos L_0} = \zeta - \frac{\zeta^2}{2} \sin L_0 + \frac{A \zeta^3}{12} + \frac{B \zeta^4 \sin L_0}{24}$$

The first two terms are common to all conformal projections : A and B are the functions of the latitude L_0 which may contain imaginary quantities.

Let :

$$A = \alpha + \gamma i \quad , \qquad B = \beta + \delta i .$$

The calculation of the scale for such a projection, as a function of the co-ordinates gives us :

$$(11) \quad M = 1 + \frac{X^2}{4 N_0^2 \cos^2 L_0}\left(2 \frac{N_0}{\rho_0} \cos^2 L_0 - 2 \sin^2 L_0 + \alpha\right) - \frac{\gamma X Y}{2 N_0^2 \cos^2 L_0}$$
$$+ \frac{Y^2}{4 N_0^2 \cos^2 L_0} (2 \sin^2 L_0 - \alpha)$$
$$+ \frac{X^3 \sin L_0}{6 N_0^3 \cos^3 L_0}\left(5 \frac{N_0}{\rho_0} \cos^2 L_0 - 4 \frac{N_0^2}{\rho_0^2} \cos^2 L_0 - 5 \sin^2 L_0 + 3\alpha + \beta\right)$$
$$+ \frac{X Y^2 \sin L_0}{2 N_0^3 \cos^3 L_0}\left(5 \sin^2 L_0 - \frac{N_0}{\rho_0} \cos^2 L_0 - 3\alpha - \beta\right)$$
$$- \frac{X^2 Y \sin L_0}{2 N_0^3 \cos^3 L_0}(3\gamma + \delta) + \frac{Y^3 \sin L_0}{6 N_0^3 \cos^3 L_0}(3\gamma + \delta) + - - - .$$

The terms of the 2nd degree furnish the equation of the conic indicatrix (9), which characterizes the projection. To reduce the value of M, it is essential to reduce that of the terms of the third degree. We arrive at this by means of the conditions :

$$3\alpha + \beta = 5 \sin^2 L_0 - \frac{N_0}{\rho_0} \cos^2 L_0 + p\left(\frac{N_0}{\rho_0} - 1\right),$$
$$3\gamma + \delta = 0 ,$$

in which p is a real arbitrary coefficient, and consequently :

$$3 A + B = 5 \sin^2 L_0 - \frac{N_0}{\rho_0} \cos^2 L_0 + p\left(\frac{N_0}{\rho_0} - 1\right).$$

The expression of the terms of the third degree of the scale becomes :

$$-\frac{e'^2 X^3 \sin 2 L_0}{3 N_0^2 \rho_0} + \frac{e'^2 X \sin 2 L_0}{3 N_0^2 \rho_0} \times \frac{p \rho_0}{4 N_0 \cos^2 L_0} (X^2 - 3 Y^2) ;$$

and the general expression of the projection will be :

$$\frac{Z}{N_0 \cos L_0} = \zeta - \frac{\zeta^2}{2} \sin L_0 + \frac{A}{12} \zeta^3 \left(1 - \frac{3\zeta}{2} \sin L_0\right)$$
$$+ \frac{\zeta^4}{24} \sin L_0 \left[5 \sin^2 L_0 - \frac{N_0}{\rho_0} \cos^2 L_0 + p\left(\frac{N_0}{\rho_0} - 1\right)\right].$$

If we adopt for p the value zero, we shall rediscover the formulae provided by the method of the double projection, and only a term with $e'^2 X^3$ subsists as term of the third degree of the scale. If we adopt for p the value $4 \frac{N_0}{\rho_0} \cos^2 L_0$, it is the term in $e'^2 X Y^2$ which exists alone, as was found in the case of the quasi-stereographic projection and for the projection of Gauss when one requires that the prime meridian should be « automecoïc » (c). I does not seem to be of interest to give to p another value than this ; the latter is interesting when X may become appreciably greater than Y.

The coefficient A may be considered as characterizing really the projection. It does not contain imaginary parts unless the axis of the conic

(9) Also called the isometer curve or conic of equal linear distortion.

indicatrix is not in the direction of the meridian. For the stereographic projection, we shall have :

$$A = 2 \sin^2 L_0 - \frac{N_0}{\rho_0} \cos^2 L_0 .$$

For the Gauss projection :

$$A = 2 \sin^2 L_0 - 2 \frac{N_0}{\rho_0} \cos^2 L_0 .$$

For the Jung projections :

$$A = 2 \sin^2 L_0 - \frac{N_0}{\rho_0} \cos^2 L_0 \left(1 + ne^{-2i\theta}\right) .$$

III

CONFORMAL CONIC PROJECTIONS ON THE TERRESTRIAL ELLIPSOID.

a) We have examined in the Hydrographic Review, vol. XIV, N° 1, pages 39 and following, the conformal conic projections of the sphere comprised in the formula :

$$(12) \qquad Z = a\,(z + 2 \operatorname{cotg} C)^p ,$$

in which z is the complex co-ordinate of a stereographic projection of the centre i ; and Z that of a conic projection of origin j ; j i x being the direction of the axis of x for the two projection, and O being the point of the sphere which corresponds to a minimum deformation (See fig. 1).

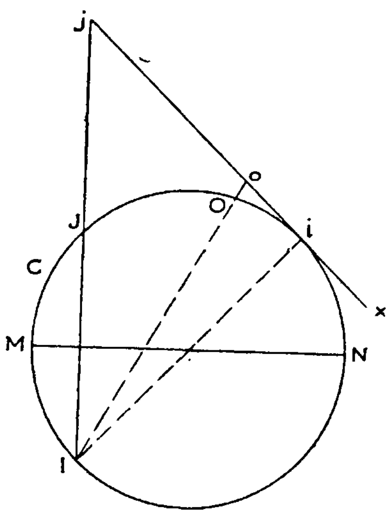

Fig. 1

The arc MJ being equal to C and the arc NO to μ ; if, for the rest, at the point corresponding to O, the scale is equal to unity, the length corresponding to j o, which we can call r_0 , may be expressed for the sphere of radius R by :

$$r_0 = R \frac{\cos\mu + \cos C}{\sin\mu} , \quad \text{et} \quad p = \frac{\sin\mu}{\sin C} .$$

The quantities μ and C, which have a geometrical signification on the sphere, will no longer retain these when we pass to the ellipsoid, and consequently it would be better to replace them by the two parameters r_0 and p . From the two above equations, we deduce :

$$\operatorname{tg} \frac{\mu + C}{2} = \frac{p+1}{p} \frac{R}{r_0} , \quad \operatorname{tg} \frac{\mu - C}{2} = \frac{p-1}{p} \frac{R}{r_0} , \quad \operatorname{tg} C = \frac{2\,p\,R\,r_0}{p^2 r_0^2 + (p^2 - 1) R^2} ,$$

$$\operatorname{tg} \mu = \frac{2\,p^2\,R\,r_0}{p^2 r_0^2 - (p^2 - 1) R^2} .$$

For the rest, in place of considering the stereographic projection of the sphere with the centre i, let us call z_1 that which will have O for centre. z and z_1 are related by the expression :

$$z = \frac{z_1 - 2\,R\,tg\,\frac{\mu - c}{2}}{1 + \frac{z_1}{2\,R}\,tg\,\frac{\mu - c}{2}}.$$

Substituting in formula (12) the above value, and replacing μ and C by their values in p and r_0, it follows :

$$Z = r_0 \left(1 + \frac{z_1}{p r_0 + \frac{p-1}{2} z_1} \right)^p.$$

If we take, in place of the axis ox_1, the axis ox_2, tangent on the meridian OP and directed towards the pole, this meridian making an angle θ with the great circle M J O N, we will have : (See fig. 2).

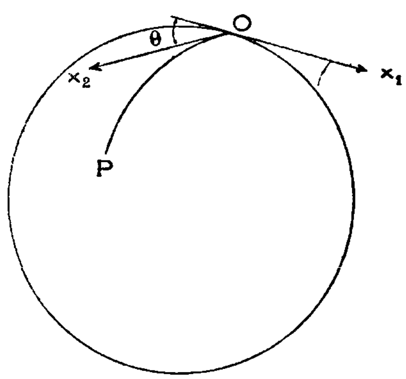

Fig. 2

$$z_1 = - z_2\, e^{-\theta i}.$$

and consequently the equation (12) becomes :

$$(13) \qquad Z = r_0 \left(1 + \frac{z_2}{\frac{p-1}{2} z_2 - p r_0 e^{i\theta}} \right)^p.$$

This expression is valid for the sphere ; we will have a rigorously conformal projection of the ellipsoid, and we may then call this the conic projection of the ellipsoid, if we substitue in equation (13), for the value z_2 the expression (1) which we have given in Hydrographic Review of May 1939, page 30, for the stereographic projection of the ellipsoid.

In order to calculate the scale, we deduce from equation (13) :

$$z_2 = 2\,p r_0\, e^{i\theta} \frac{\left(\frac{Z}{r_0}\right)^{\frac{1}{p}} - 1}{(p-1)\left(\frac{Z}{r_0}\right)^{\frac{1}{p}} - (p+1)}$$

from which :

$$\frac{dz_2}{dZ} = -4\, e^{i\theta} \frac{\left(\frac{Z}{r_0}\right)^{\frac{1-p}{p}}}{\left[(p-1)\left(\frac{Z}{r_0}\right)^{\frac{1}{p}} - (p+1)\right]^2}$$

By multiplying this expression by the conjugate expression and taking the square root of this product, we will have the ratio $\frac{m_2}{M}$ of the scales of the projections z_2 and Z (10):

$$\frac{m_2}{M} = \frac{4\left(\frac{\rho}{r_0}\right)^{\frac{1-p}{p}}}{(1-p)^2\left(\frac{\rho}{r_0}\right)^{\frac{2}{p}} + 2\,(1-p^2)\left(\frac{\rho}{r_0}\right)^{\frac{1}{p}}\cos\frac{\omega}{p} + (1+p)^2}.$$

For the rest :

$$m_2 = K\left(1 + \frac{x_2^2 + y_2^2}{4\,R^2}\right)$$

$$= K\,\frac{\left(\frac{\rho}{r_0}\right)^{\frac{2}{p}}\left[(1-p)^2 + \frac{p^2 r_0^2}{R^2}\right] + 2\left(\frac{\rho}{r_0}\right)^{\frac{1}{p}}\left[1 - p^2 - \frac{p^2 r_0^2}{R^2}\right]\cos\frac{\omega}{p} + (1+p)^2 + \frac{p^2 r_0^2}{R^2}}{(1-p)^2\left(\frac{\rho}{r_0}\right)^{\frac{2}{p}} + 2\,(1-p^2)\left(\frac{\rho}{r_0}\right)^{\frac{1}{p}}\cos\frac{\omega}{p} + (1+p)^2}.$$

Dividing these two equations, the one by the other, we then obtain .

(14)
$$M = K\,\frac{\left(\frac{\rho}{r_0}\right)^{\frac{2}{p}}\left[(1-p)^2 + \frac{p^2 r_0^2}{R^2}\right] + 2\left(\frac{\rho}{r_0}\right)^{\frac{1}{p}}\left[1 - p^2 - \frac{p^2 r_0^2}{R^2}\right]\cos\frac{\omega}{p} + (1+p)^2 + \frac{p^2 r_0^2}{R^2}}{4\left(\frac{\rho}{r_0}\right)^{\frac{1-p}{p}}}.$$

b) The conical projection is particularly interesting when the scale does not depend on ω, but simply on ρ ; that is to say if the isometers are circles. This condition, which may be realised for the sphere (it suffices to give to C the value $\frac{\pi}{2}$, see Hydrographic Review, vol. XIV, N° 1, page 41) cannot be obtained in all rigorousness for the ellipsoid, except when the origin is at the pole, for the same reason which has been given for the stereographic projection (vol. XVI, N° 1, page 27). But if, in the above expression, we eliminate the term in $\cos\frac{\omega}{p}$, there will remain only K which depends upon ω, and in terms having e'^2 as factor. This leads us to consider as specially interesting the conical projections for which p and r_0 are related by the equation :

(15)
$$R^2\,(1-p^2) = p^2 r_0^2.$$

The expression for M then becomes :

$$M = \frac{K}{2}\left(\frac{\rho}{r_0}\right)^{\frac{p-1}{p}}\left[(1-p)\left(\frac{\rho}{r_0}\right)^{\frac{2}{p}} + 1 + p\right].$$

(10) In the following formulae, ρ and ω, designate the polar co-ordinates which have been employed in Hyd. Rev., vol. XIV, N° 1, page 40

By limiting ourselves to the first two terms of the equation (1), we may write :

$$K = 1 + e'^2 \frac{r_0^3 \sin 2 L_0}{3 N_0 R^2} \left[\frac{\rho}{r_0} \cos(\theta + \omega) - \cos\theta\right]^3 .$$

c) It is only in the value of K that the angle θ occurs in equation (14). If this angle is zero, we have the great advantage of having the meridian as axis of x.

If the shape of the terrain, which should then be symmetrical with respect to the meridian, compels us to adopt the value r^0 for the radius of curvature of the isometers, the equation (15) will supply us with the value of p .

But if this value of r^0 is not imposed, a considerable simplification may be obtained by choosing r_0 in such a manner that the origin of the co-ordinates Z shall be at the pole, for the meridians will then be strictly concurrent straight lines and the parallels will be concentric circles.

At the origin of coordinates, we have, according to equation (13) :

$$z_2 = \frac{2\, p r_0}{1 + p}$$

We have seen in studying stereographic projections, (formula 1), that at the pole :

$$z_2 = \frac{2 N_0 \cos L_0}{\alpha + \sin L_0} \qquad (\alpha = 1 + e'^2 \cos^4 L_0)$$

We have, therefore, between p and r_0 , the relation :

$$\frac{p\, r_0}{1 + p} = \frac{N_0 \cos L_0}{\alpha + \sin L_0} .$$

By adding to this, the relation (15), we shall have for p and r_0, the values :

$$p = \frac{\sin L_0}{\alpha}, \qquad r_0 = N_0 \operatorname{cotg} L_0 .$$

The equation (13) then becomes, by substituting in it these values of p and r_0 , and z_2 by its value as a function of ζ :

$$Z = N_0 \operatorname{cotg} L_0\, e^{-\zeta \sin L_0} ,$$

$$\rho = N_0 \operatorname{cotg} L_0\, e^{-v \sin L_0} , \qquad \omega = - G \sin L_0$$

$$Lg\, \rho = Lg\, N_0 \operatorname{cotg} L_0 - v \sin L_0 .$$

With :

$$M = \frac{K}{2} \frac{\alpha - \sin L_0 + (\alpha + \sin L_0)\, e^{2\alpha v}}{(\alpha + \sin L_0)\, v\, \alpha\, e}$$

$$= \frac{K}{2\alpha}\left[(\alpha - \sin L_0)\left(\frac{\rho}{N_0} \operatorname{tg} L_0\right)^{1 + \frac{\alpha}{\sin L_0}} + (\alpha + \sin L_0)\left(\frac{\rho}{N_0} \operatorname{tg} L_0\right)^{1 - \frac{\alpha}{\sin L_0}}\right],$$

and :

$$\frac{K}{2} = 1 - \frac{e'^2}{3} \frac{N_0}{\rho_0} v^3 \sin 2 L_0 \cos^3 L_0 + \frac{e'^2}{6} \frac{N_0^2}{\rho_0^2} v^4 \cos^4 L_0 (7 \sin^2 L_0 - 1) + \cdots .$$

In this case, K no longer depends upon G, that is ω .

d) It is perhaps not without interest to compare the conical projection of the ellipsoid with the Jung projections which have their origin at the point O. Let Z_1 be a projection of Jung of which the axis x_1 is directed along the meridian. It is characterised by the parameter n , related to the axes *a* and *b* of the conic indicatrix, by the relation :

$$n = \frac{a^2 - b^2}{a^2 + b^2},$$

and by the angle θ which the meridian makes with the major axis of this conic.

If z_2 is the stereographic co-ordinate of the same origin :

$$z_2 = \frac{2 R e^{i\theta}}{\sqrt{n}} \operatorname{tg} \frac{Z_1 \sqrt{n}}{2 R e^{i\theta}}.$$

In order to simplify, let us put :

$$\lambda = \frac{\sqrt{n}}{2R e^{i\theta}}, \text{ d'où : } z_2 = \frac{1}{\lambda} \operatorname{tg} \lambda Z_1 .$$

Let us replace in equation (13), the coordinate z_2 by this value :

$$\frac{Z}{r_0} = \left(1 + \frac{\sin \lambda Z_1}{\frac{p-1}{2} \sin \lambda Z_1 - \lambda p r_0 e^{i\theta} \cos \lambda Z_1}\right)^p .$$

We then have the expression of the co-ordinate Z of the conic projection as a function of any Jung projection whatever. Let us develop this expression in accordance with the ascending powers of Z_1 :

$$(16) \qquad (r_0 - Z) e^{\theta i} = Z_1 + \frac{Z_1^3}{12 r_0^2 e^{2\theta i}} \left(\frac{1 - p^2}{p^2} + n \frac{r_0^2}{R^2}\right) + \frac{Z_1^4}{24} \frac{1 - p^2}{p^2 r_0^3 e^{3\theta i}}$$

$$+ \frac{Z_1^5}{240} \frac{3(p^4 - 1) + 5 np^2 (1 - p^2) \frac{r_0^2}{R^2} + 2 n^2 p^4 \frac{r_0^4}{R^4}}{p^4 r_0^4 e^{4\theta i}} + \dots .$$

This formula shows that the generalised conic projection does not differ except in the third order from a Jung projection, having the same centre (the point where the scale is a minimum) and of which the major axis of the indicatrix is directed towards the origin of the conic projection. This difference will be lowered to the 4th order, if we have, between p and r_0 the relation :

$$(17) \qquad R^2 (p^2 - 1) = n p^2 r_0^2 .$$

This expresses the fact that the two projections have indicatrix ellipses of the same form. Only in this case may we compare them.

We have seen (Hydrographic Review, vol XIV, N° 1, page 47) that if μ is (11) greater than C, the ratio $\frac{a^2-b^2}{a^2+b^2}$ between the axes of the conic indicatrix of the generalised conic projection, is equal to :

$$\frac{\cos C - \cos \mu}{\cos C + \cos \mu} = \operatorname{tg} \frac{\mu - C}{2} \operatorname{tg} \frac{\mu + C}{2} = \frac{p^2 - 1}{p^2} \frac{R^2}{r_0^2}.$$

This relation is, in effect, equal to n if the above condition is realised. Formula (16) then becomes :

$$(r_0 - Z) e^{\theta i} = Z_1 - \frac{n Z_1^4}{24 R^2 r_0 e^{3\theta i}} + \frac{n Z_1^5}{40 R^4 e^{4\theta i}} \left(\frac{R^2}{r_0^2} - n\right) + \cdots.$$

In order to have a development in ζ, it will suffice to replace Z_1 by its value given in formula (10). We then obtain :

(18)
$$\frac{(r_0 - Z) e^{\theta i}}{N_0 \cos L_0} = \zeta - \frac{\zeta^2}{2} \sin L_0 + \frac{A \zeta^3}{12}$$

$$+ \frac{N_0^4}{24} \sin L_0 \left[2 \frac{N_0}{\rho_0} \cos^2 L_0 - \sin^2 L_0 + n \frac{N_0}{\rho_0} \cos^2 L_0 \, e^{-3\theta i} \left(3 e^{\theta i} - \frac{N_0}{r_0} \operatorname{cotg} L_0\right)\right] + \cdots.$$

The term of the 4th order of equation (18) differs from that of equation (10) and no longer satisfies the condition which we have found necessary for the term of the 3rd order of the scale having e'^2 as a factor. The generalised conic projection is therefore less advantageous, from this point of view, than the corresponding Jung projection. But this will not remain true, however, if the scale can be expressed as a function of ρ only (except terms which have e'^2 as a factor) ; that is to say if the isometer is a double circle having its centre at the origin of the Z.

The projection of Jung with which the comparison will be made, will then be a projection in which n will be equal to —1 and of which the part of the 2nd order of the isometer will represent two parallel straight lines. Condition (17) then becomes identical with condition (15).

The quantities cited above : α, β, γ, δ, will then have for values :

$$\alpha = 2\left(\sin^2 L_0 - \frac{N_0}{\rho_0} \cos^2 L_0 \sin^2 \theta\right), \qquad \gamma = -\frac{N_0}{\rho_0} \cos^2 L_0 \sin 2\theta,$$

$$\beta = 2 \frac{N_0}{\rho_0} \cos^2 L_0 - \sin^2 L_0 - 3 \frac{N_0}{\rho_0} \cos^2 L_0 \cos 2\theta + \frac{N_0^2}{r_0 \rho_0} \cos^2 L_0 \operatorname{cotg} L_0 \cos 3\theta,$$

$$\delta = \frac{N_0}{\rho_0} \cos^2 L_0 \left(3 \sin 2\theta - \frac{N_0}{r_0} \operatorname{cotg} L_0 \sin 3\theta\right),$$

$$3\alpha + \beta = 5 \sin^2 L_0 - \frac{N_0}{\rho_0} \cos^2 L_0 + \frac{N_0^2}{r_0 \rho_0} \cos^2 L_0 \operatorname{cotg} L_0 \cos 3\theta,$$

$$3\gamma + \delta = -\frac{N_0^2}{r_0 \rho_0} \cos^2 L_0 \operatorname{cotg} L_0 \sin 3\theta.$$

The scale will therefore be, in accordance with formula (11), and replacing X by $r_0 \cos \theta - \rho \cos(\theta + \omega)$ and Y by $r_0 \sin \theta - \rho \sin(\theta + \omega)$:

$$M = 1 + \frac{(r_0 - \rho \cos \omega)^2}{2 R^2} + \frac{r_0 - \rho \cos \omega}{6 R^2 r_0} \left[(r_0 - \rho \cos \omega)^2 - 3 \rho^2 \sin^2 \omega\right] + \cdots.$$

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(11) If μ is smaller than C, the conic projection may be compared to a Jung projection in which the minor axis of the conic indicatrix makes an angle θ with the prime meridian. This is the same as changing the sign of n.

This may be written by neglecting, as in the above expression, the terms of the 4th order, and by considering $r_0 - \rho$ and ω as infinitely small of the same order :

$$M = 1 + \frac{(r_0 - \rho)^2}{2 R^2} + \frac{(r_0 - \rho)^3}{6 R^2 r_0},$$

quantity which is independent of ω and of θ .

It appears interesting to employ formula (18) for the calculation of an oblique conic projection, in the case where the coast to be represented has the form of an arc of circle, for which one takes for the radius its value of r_0 . Further, it will be necessary to retain the first four terms.