TIDE MODELS

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The representation of tidal motion by means of small-scale models is of increasing importance, so that it is desirable for Hydrographic Services to understand the limits of usefulness of such models. Hitherto they have been developed by civil engineers who have tended to be more concerned with the engineering problems than with the effects on tides and who have approached the problems with an inadequate knowledge of tidal theory and problems.

Figure 1 gives an illustration of a model used to investigate a proposed barrage across the River Severn. The mechanism used to produce a motion of the water in the model, and which is presumed to be the correct generator of tidal motion in the estuary, is illustrated in figure 2. It consists of a plunger which is moved up and down in the water through variable ranges from springs to neaps. The plunger is shaped according to the shape of the tide curve at the seaward end of the model. Certain relations between the horizontal scale, the vertical scale, and the time scale, are derived from the equations of tidal motion, and thus in order to understand the underlying theory of models it is necessary to consider these equations. Though these may be unfamiliar to hydrographic officers, it is not necessary to understand all the symbols such as those for the differential operators, and sufficient will be said to make clear what the terms mean. The equations will be stated for a channel of uniform breadth but this limitation does not affect the deductions to be made from the equations.

We adopt the following notation :—

x	Distance along a channel ;
t	The time ;
h	The mean level of the surface above the bed of the channel, taken over a cross-section for all values of the time, at a distance x along the channel :
b	The mean breadth of the cross-section at x ;
ζ	The mean elevation of the surface above the level h, taken at time t along the cross-section at x ;
u	The mean velocity in the cross-section at time t, measured in the direction of increasing x ;
g	The gravitational acceleration ;
k	A constant which has the value of about 0.002 when all variables are measured in centimetres and seconds ;
Q	The total flow of water per second across the section :
d	The length of a short portion of the channel ;
u'	The absolute value of u, that is, ignoring the sign.

The tidal equations are :—

$$\frac{\delta \zeta}{\delta t} + \frac{\delta}{\delta x}\left\{ h + \zeta \right\} = 0. \qquad \text{..........(1)}$$

$$\underset{(a)}{\frac{\delta u}{\delta t}} = - g \underset{(b)}{\frac{\delta \zeta}{\delta x}} - u \underset{(c)}{\frac{\delta u}{\delta x}} - \underset{(d)}{\frac{k u' u}{h}} \qquad \text{..........(2)}$$

The first of these equations simply states that within a small portion of the channel, the rate at which the elevation is increasing depends upon the rate at which water accumulates by horizontal flow. The second equation states that (a) the rate at which the velocity of the water is increasing depends upon (b) the loss of potential energy due to a fall of the surface, (c) the rate of change of the kinetic energy along the channel, and (d) the loss of energy due to the friction caused by the water moving over rough surfaces. In very deep water, where the velocity is small, the terms (c) and (d) can be ignored.

If we wish to consider the motion on a different scale, we can apply the factors X, T, and Z to horizontal lengths, the time, and vertical lengths, respectively. Since the velocity is a change in distance per unit of time its appropriate factor is X/T.

In equation (1) the three terms have the factors

$$\frac{Z}{T} \qquad \frac{Z}{X}\cdot\frac{X}{T} \qquad \frac{Z}{X}\cdot\frac{X}{T}$$

And these are obviously all equal to Z/T. The separate terms of the equation have thus been multiplied by the same factor, and therefore the equation is valid for all scales.

In equation (2) the four terms have the factors

$$\begin{array}{cccc} (a) & (b) & (c) & (d) \\ \frac{X}{T^2} & \frac{Z}{X} & \frac{X}{T}\cdot\frac{X}{T}\cdot\frac{1}{X} & \frac{X}{T}\cdot\frac{X}{T}\cdot\frac{1}{Z} \end{array} \quad \ldots\ldots\ldots\ldots(3)$$

and these are equal to

$$\frac{X}{T^2} \qquad \frac{Z}{X} \qquad \frac{X}{T^2} \qquad \frac{X}{T^2}\,\frac{X}{Z} \quad \ldots\ldots\ldots\ldots(4)$$

Since the equation is only valid if we have a constant factor for all terms then we must have, from the terms (a) and (b) the relation

$$ZT^2 = X^2 \qquad \ldots\ldots\ldots\ldots(5)$$

The fourth term demands the relation X = Z, and if this were feasible then the water model would satisfy the tidal equations exactly, but any model of a large estuary demands a small horizontal scale, and an equal scale for vertical lengths would render the tidal motion almost imperceptible. The practice has been to exaggerate the vertical scale and to ignore the effect that this has upon the frictional losses, so far as the satisfaction of the tidal equations section by section is concerned. The exaggeration varies according to the size of the estuary but for the Severn model it was such that the vertical scale was 42.5 times the horizontal scale. This means that the frictional term in the equation was reduced to the fraction 1/42.5 of its value in nature.

A very simple calculation from equation (2) shews that the frictional term is very important. It has been pointed out that in deep water the main terms are (a) and (b) and we can readily compare the first of these with the frictional term. If we are dealing with a semi-diurnal stream, then its increment of phase can be denoted by n radians per second, and this has the value of 1.4×10^{-4} for the principal lunar constituent. If the amplitude of stream is denoted by A then the amplitude of the first term (a) is nA, the amplitude of the frictional term is $0.002\ A^2/h$ and the ratio of the frictional term to the main term (a) is then 14 A/h. In the Bristol Channel west of Avonmouth the value of A is about 150, in cms per sec., and the value of h is about 2000, in cms. This means that the frictional term is just as important as any other term in the equation.

An investigation by Dr. S. F. Grace, of the University of Liverpool, on " Friction in the Tidal Currents of the Bristol Channel ", gives ample confirmation of this conclusion. For a typical cross-section we can deduce from his results values for the terms of the equation for each of two phases, as follows :—

Term	(a)	(b)	(c)	(d)
First phase	— 0.019	0.000	....	— 0.019
Second phase	— 0.003	— 0.040	....	0.037

It is therefore beyond all question that the neglect of the frictional term under such conditions is quite unjustifiable. The ratio 14A/h is a useful criterion of the importance of the term.

Engineers who have experimented with models have claimed that the models, even with such conditions as exist in the Bristol Channel, do give a close representation of the tides, even though it is conceded that the tidal equations are not satisfied section by section. While this claim is not fully accepted by tidal experts there is some reason for thinking that the tidal model gives better results than might at first sight be expected from the above investigation. There is a need for more exhaustive examination of the factors involved, and more experiments are needed to determine precisely how and where the failure to account for the frictional term is to some extent counteracted. One possible explanation, in particular, needs consideration, and that is the effect of the exaggeration of vertical scale in producing losses of energy by shock; that is, by the direct impact of water upon banks which in nature are gently sloping but which in the model are so steep as to be approaching the vertical. It is well known that tidal streams are reflected by steep banks. The result is a loss of energy, and it is possible that such energy losses may make up for the deficient energy loss due to friction. If the compensation of the deficient losses in a particular section were made up within that section then the compensation would be perfect, but what is most likely is that we get only an over-all result. For true similitude it is essential to have the losses of energy taking place in the right sections. If, for instance, the main losses of energy by shock occur above the point where a barrage is to be erected then the indications of the model after the erection of such a barrage will be misleading, as the losses of energy below the barrage will not be correctly given.

Another important consideration regarding tidal models is that engineers generally assume that a plunger will give a rise of level which will automatically take care of tidal streams. This is a very naive assumption. All tidal authorities recognise that it is vital to have the right relations of tide and stream, both in amplitude and in phase if right results are to be obtained. Hydrographic officers ought to be on their guard and ascertain if any provision has been made for this condition to be satisfied. I am not aware of any model which has any but the most elementary representation of tides. This caution has most weight when the model has to do with an area of open water on either side of the deep-water approach to a river through sand-banks for a considerable distance. A model which erects walls on either side of the main channel and simply generates tides over a short portion of the end wall is simply ignoring the tidal streams which must exist throughout the whole area, transversely also to the side walls, in nature. See figure 3.

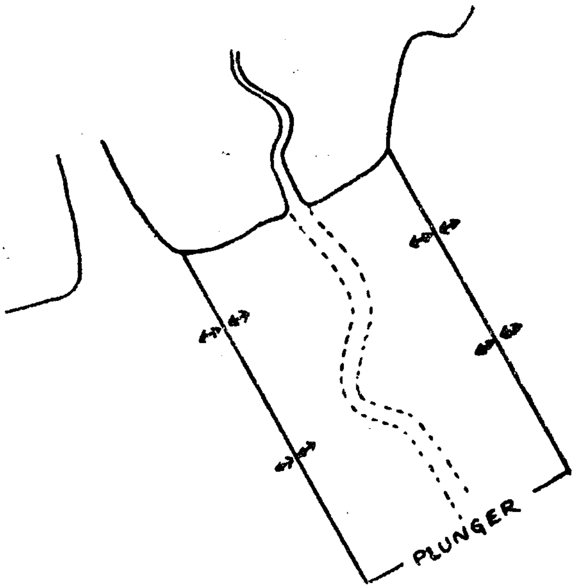

Fig. 3
Model which ignores the tidal streams transverse to the walls.

Obviously, in such a case as this it is vital that stream and tide observations should be made over the whole area before the model is commenced, for without these the indications of the model may be highly unreliable. The observations should be reduced by trained officers and not by the engineers. The mechanism for generating the water movement needs to be elaborated, and a vertical plunger should be supplemented by a horizontal piston so that the tidal streams will be correctly related to the elevation in both amplitude and phase. The bounding walls should be made double so that water can be forced in at the sides to ensure that the side streams are correct in amplitude and phase. They may have to be totally enclosed compartments from which water is forced by a piston geared to the main mechanism. The side walls can be pierced with holes whose sizes and positions will depend upon the observed rates of stream. See figure 4.

The question now arises as to whether the models can be modified, or if any technique can be adopted, by which the earlier mentioned difficulties may be evaded. Hitherto, far too many things have been attempted in one operation, for the model has been expected to deal simultaneously with the changes in the tides and in the bed of the channel. The proper course seems to be to proceed by a series of experiments.

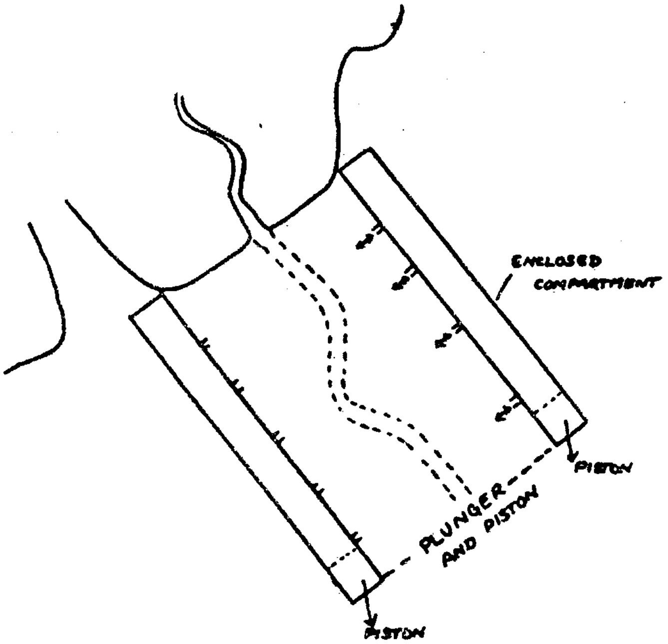

Fig. 4

Suggested modification of model to give transverse streams.

The first essential is to make certain that the tidal equations are being satisfied. There should be the minimum distortion of scale, and artificial roughening should be resorted to in order to increase the friction. Banks which are nearly vertical should be smoothed off so as to reduce the un-natural shock losses. Every effort should be made to satisfy the equations for each section even if separate models have to be made for two halves of an estuary, one below a proposed barrage and one above. The effects of barrages or other alterations in the estuary may then be examined as regards their effects on tides and tidal streams, supposing that there is no change in the bed of the channel by silting or erosion. Laboratory experiments should be made to determine the limits of amplitude of tidal streams within which will be no resultant silting or erosion, and the effects of tidal streams outside these limits should be estimated. The bed should then be made deeper or shallower in the places where it is estimated that changes will take place and a fresh experiment be made to ascertain the new values of amplitudes of tidal streams. This process should be continued until a stable state is reached.

When the model has given satisfactory values for the tidal streams, the problem of erosion and silting should be examined by larger-scale models of specially important areas. Such models should have minimum distortion and the streams on both sides should be regulated by means of plungers and pistons so as to give the streams correctly. In my judgment it is necessary to consider silt problems quite separately from the tidal problems.

Greater attention needs to be paid also to the extreme conditions associated with meteorological disturbances. There is no doubt that changes in the bed of a channel are not simply due to the normal to-and-fro movements of the tide. Cyclical changes must have a cyclical origin and if the only cycle is that of the average springs to neaps cycle then no long-period changes can be expected. Exceptional conditions cause local changes and often start new cycles. This means that the model must be caused to operate with a right frequency of amplitude of tide, and not merely with average values. To sum up, I believe it to be essential to proceed by a series of carefully controlled experiments and not to expect a model to yield all the desired information in one operational process.

We now consider a totally different kind of model which gives an electrical representation of tides and streams in a channel. The apparatus was designed by Dr. Joh. Van Veen, Chief of the Research Bureau for Tidal Rivers, Rijkswaterstaat, Holland, and I am indebted to him for such details as are here given. Dr. Van Veen's own exposition is given in terms and notations familiar to the hydraulic engineer, but I shall attempt a fresh presentation of the principles in terms of the tidal equations already used, even at the risk of not doing full justice to the method. For our present purposes the simplest exposition ensues if we assume that we can neglect terms of the equations which involve the product ζu and the square u^2, and if we assume that we can take the flow of water across a section as :—

$$Q = bhu$$

so neglecting one of the second-order terms mentioned. The equations become

$$-\frac{\delta Q}{\delta x} = b\frac{\delta \zeta}{\delta t} \qquad \text{..........(8)}$$

$$-bgh\frac{\delta \zeta}{\delta x} = \frac{\delta Q}{d t} + \frac{Ku'Q}{h} \qquad \text{..........(9)}$$

The variable δx denotes a short length of the channel, which we shall call d, so that we have

$$-\delta Q = b\alpha\frac{\delta \zeta}{\delta t} \qquad \text{.........(10)}$$

$$-\delta \zeta = \frac{d}{bgh}\frac{\delta Q}{\delta t} + \frac{d\,Ku'}{bgh^2}Q \qquad \text{.........(11)}$$

Let

$L = bd$ ....(12)

$K = d/bgh$ ....(13)

$R = bgh^2/dku'$ ....(14)

and the equations become

$$-\delta Q = L\frac{\delta \zeta}{\delta t} \qquad \text{.........(15)}$$

$$-\delta \zeta = K\frac{\delta Q}{\delta t} + \frac{Q}{R} \qquad \text{.........(16)}$$

Now the forms of these equations suggest well-known electrical equations, the first for the relation between the potential ($-\delta Q$) across an inductance L through which flows an electrical current (ζ), and the second for the current ($\delta\zeta$) flowing through a resistance R in parallel with a condenser K, across which is a potential Q. The appropriate circuit is shewn in figure 5. It is supposed that an electrical current flows along P_1 P_2 and some of it leaks to "earth", the line of zero resistance and of zero potential, P'_1 P'_2. Through the points P_1 P_2 there are currents (ζ_1) and (ζ_2) and the leak to earth is $\zeta_1 - \zeta_2 = -\delta\zeta$.

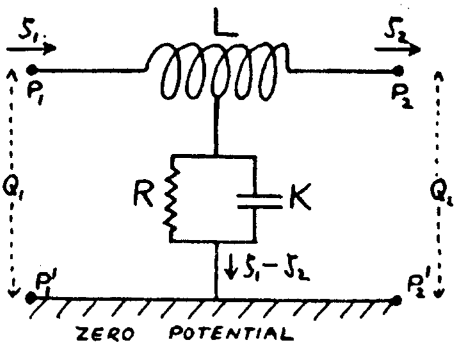

Fig. 5
Electrical Unit.

The mean current through the inductance is ζ, the mean of ζ_1 and ζ_2, and the potentials of the points P_1 and P_2 relative to earth are Q_1 and Q_2. This is one form of the circuit which satisfies the equations (15) and (16), so that if P_1 and P_2 correspond to points in the river, and if L, R and K are computed according to the values of the mean width, mean

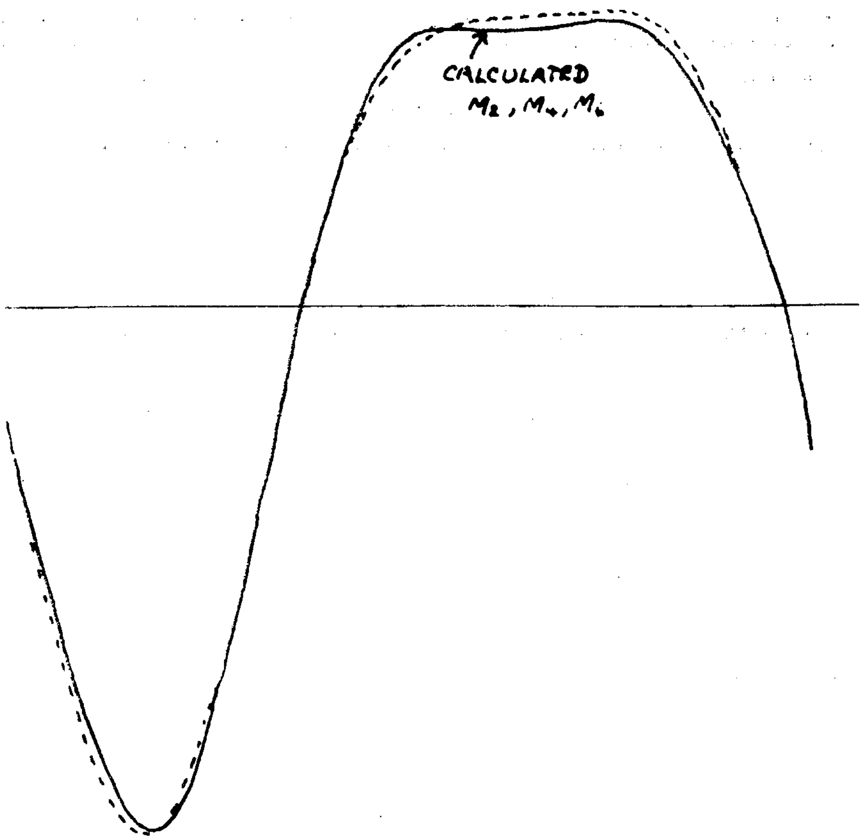

Fig. 7
Streams in river Lek, showing accuracy of electrical method.

length, mean depth, and mean absolute velocity in the section, then we have satisfied the tidal equations (subject to the validity of the approximations made).

The whole river can be dealt with by stringing together a number of such units, and at the seaward end an alternator can be arranged to give an alternating current which expresses the tide at the mouth, while at the other end is a battery to express the steady flow of the

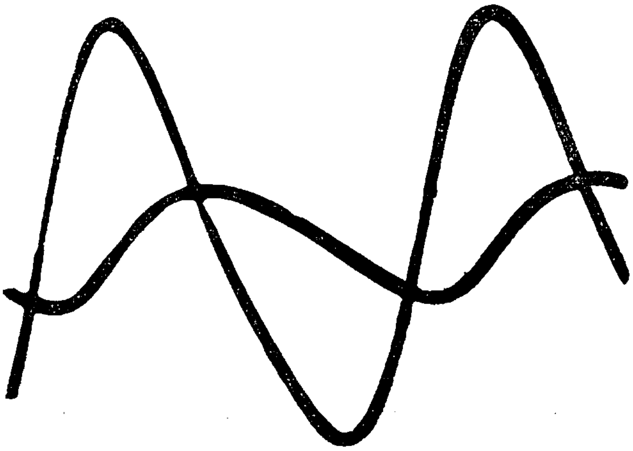

Fig. 8a
Vertical tide in the river Lek initiated by electricity.

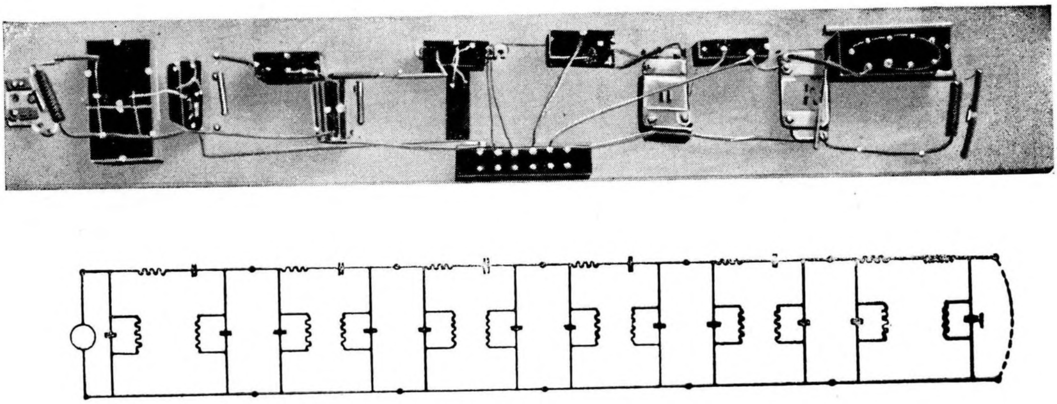

Fig. 6
River Lek.

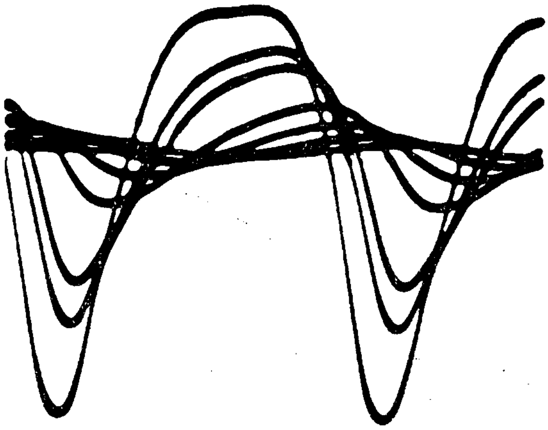

Fig. 8b

Streams in the river Lek initiated by electrical currents.

river. A simple example is given in figure 6 for the river Lek. It will be noticed that Dr. Van Veen uses two sub-units of resistance and condenser on either side of the inductance, but the principle of action is the same as that outlined above. At any point P the introduction of an ammeter will give the mean amplitude of current, corresponding to the elevation of tide, and a voltmeter from P to earth will give a potential corresponding to the tidal flow, from which the mean amplitude of stream may be derived by dividing by bh. The accuracy of the results is illustrated in figure 7.

If it is desired to study the tide and stream moment by moment then an oscilloscope can be used, and two illustrations are given in figure 8.

The simplicity of the apparatus used is very striking, and it is quite evident that the tide and stream can be observed anywhere with great ease and speed. If an alteration in a channel is proposed the values of L, K and R can be readily changed also. The same methods of estimating the effects of silting or of erosion can be adopted as those already outlined above, but the ease with which the elements can be changed make the process very rapid.

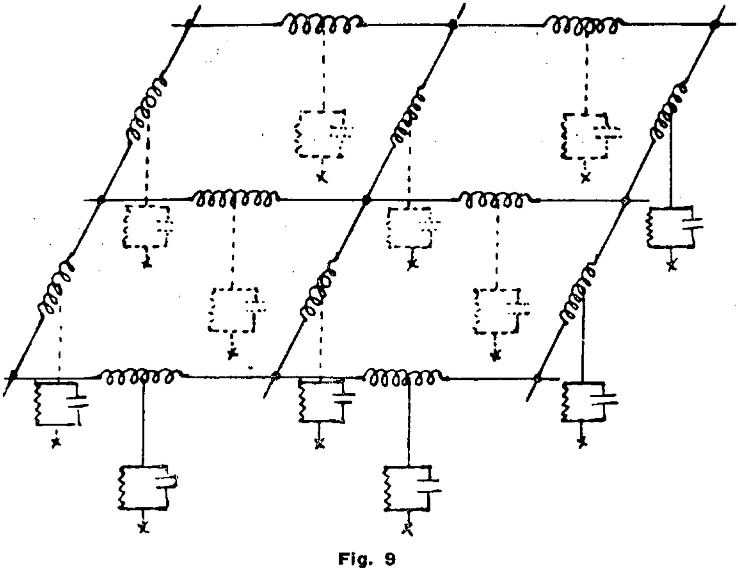

Fig. 9

Network covering an area.—The inductances are in one plane and the resistances and condensers drop to points (X) on a lower plane which is at zero potential.

The method is capable of extension to all kinds of networks as shewn in figure 9. It will be noticed that in this diagram Dr. Van Veen replaces the resistances by rectifying units which have the advantage of varying their resistance according to the current flowing through

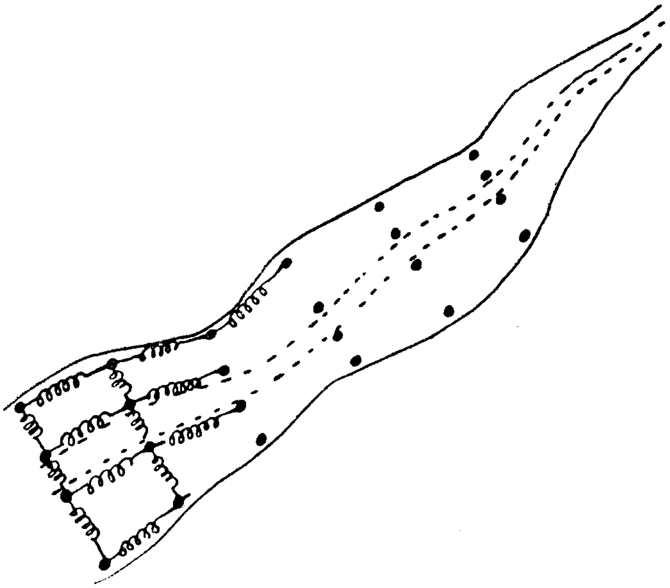

Fig. 10
Electrical network for an estuary.

them. This is a big step forward as it is now no longer necessary to estimate u' which can be deleted from (14).

One very great advantage of this method is that the frictional term retains its full value. The terms which are neglected in the electrical method are adequately expressed in the water-model so that the two methods can be regarded as supplementary to one another. The terms neglected in the electrical method, however, are usually small, and they affect principally the quarter-diurnal tides. The method tends to give inaccuracies if (ζ/h) is not small. The term (c) in equation (2) is neglected, and the criterion is that u^2 must be small compared with $(g\,\zeta)$. Approximately, if the amplitude of stream is one knot, then the elevation must have an amplitude greater than one foot, for the neglected term to be less than 10 per cent of the main terms. Dr. Van Veen believes that after the electrical model has given its indications and all adjustments have been made the final solution should be checked by the exact equations by direct calculation, with which opinion I strongly concur. I would suggest that the use of an electrical model as a check upon a water model and of calculation to check both, is the ideal combination.

It is not known whether the apparatus has been developed for use with a wide estuary, but the application can readily be made, as is seen from figure 9. A network of inductance in one plane is connected to earth in a lower plane by means of sub-units of resistance and condenser in parallel from the centre point of each inductance. The application to a converging estuary with a tidal channel of some depth in the middle, and shelving banks, is indicated in figure 10.

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