RADIO AIDS AND GEODESY

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Yesterday I gave a summary of the new position fixing developed by radio engineering during the war.

The relative accuracy afforded by these position fixing systems will, of course, depend upon the positions of the radio stations used being known exactly, as the ships' position is determined in relation to these known positions. As in the case of all other relative fixing methods it is necessary to know the base line, i.e. its bearing and length.

We observed that in all these position fixing systems very long base lines were used in comparison with those used in ordinary trigonometrical fixes, in which the base line is at most a first-order side and always shorter than the range of vision.

It may therefore be of interest to investigate in detail the accuracy with which these long base lines are known, that is to say, the accuracy of the actual geodetic basis of our charts. We must therefore first examine what geodesy is, its development and its aims and means.

Geodesy is the science of determining the shape and size of the Earth. This science is studied by the geodetic institutes of various countries which pass on their results to other scientific institutions or to the practical workers, partly in the form of data (coordinates, levels, etc.), and partly in the form of graphical reproductions (maps, graphs, etc.).

In practice the work of the geodetic institutes may briefly be said to be that of "levelling" and "coordinating" the country in question. In this connection we must distinguish between different kinds of coordinates, partly plane rectangular chart coordinates, partly geographic coordinates, which, as we shall see, comprise many different classes.

In geodesy we work with several different globes, which we shall now briefly explain :—

1. **The physical surface of the Earth** is the surface of the actual Earth. It is on this surface that we stand, walk, navigate, set up our instruments and make our observations.

This Earth is very much "alive". Houses are being built and disappear, roads and railroads are being established, forests grow up and are being felled, excavations and reclamations are being made, the sea removes and deposits soil, large ice masses are deposited and melt, and the attraction of Sun, Moon and planets pulls at it and changes its shape.

Owing to its irregular shape it is quite unsuitable to form the basis of surveys, and it will therefore be necessary to introduce an imaginary (idealized) surface of so regular a shape as to make it possible to develop a geometry on it.

At every point of space the actual Earth gives rise to an acceleration due to gravity, represented by the vector function g (x, y, z); the direction of gravity at any point is determined by the tangent to the "gravity line" through that point and is indicated by the plumb line. Gravity lines defined in this way all meet at the centre of the Earth.

All surveys of the Earth are based on the actual gravity lines inasmuch as all our instruments are set up and adjusted by means of spirit levels, as they indicate the sole direction given by nature with sufficient accuracy at any point.

Surfaces which are at all points at right angles to the gravity lines are termed level or potential surfaces. All points on the same surface have the same potential and no potential work is expended in moving a mass around on the surface; the mass will at all places on the surface maintain its potential energy. The surface is indicated by the free surface of a quiescent liquid.

On the other hand, the length of the actual gravity acceleration vector g will vary when we move around on a potential surface; this fact is due partly to the uneven distribution of the masses in the crust of the Earth, which must be assumed to extend to a depth of nearly

100 kilometers, partly to the fact that the gravity acceleration is composed *both* of the attraction of all the mass particles of the Earth *and* of the centripetal force due to the rotation of the Earth. The numerical value of the gravity acceleration is in fact greater at the poles than its mean value at the equator.

An infinite number of levels may be laid, namely one through every point of any plumb line right from the centre of the Earth to a point infinitely remote. All these actual levels are closed surfaces which cannot touch or intersect each other, and which at no place can have peaks, edges or breaks, on account of the laws of nature applying to potential functions; they are all continuous and differentiable. The actual gravity acceleration is a function of force of these potential surfaces, which means that the total differential of g will be zero everywhere in the same potential plane.

Among the infinite number of levels it will be natural to choose that which coincides with the mean water level in the open sea as plane of projection. The surface thus defined is called :—

2. **The mathematical Earth surface** or the **geoid** and the aim of higher geodesy is to determine the shape and size of that surface.

If the water in the seas was not affected by winds, currents, tides, differences in salinity and temperature, etc., the surface of the sea would form part of the geoid, but as the sea does not fulfil these ideal requirements, the geoid and the immediate surface of the sea need not coincide. On the other hand, to a high degree of approximation the geoid can be considered as coinciding with the mean water level of the sea at places where no special one-way currents prevail.

The geoid, like the Earth, will be a " living " globe. It will not only be subject to periodical changes due to the attraction of Sun, Moon and planets, but also to other changes due to great earthquakes, melting of the ice masses at the poles, etc. It is therefore very loosely defined by the words : " coinciding with the mean water level in the open sea ", which it is very difficult to determine and which must presumably be a function of time.

The heterogeneous distribution of masses in the crust of the Earth will — as is known from physics — give rise to irregularities in the shape of the geoid. An increase in the density of the Earth around a point will attract the gravity lines and thereby raise the geoid in the vicinity of the point, and conversely a decrease in density around a point will cause a lowering of the geoid in the vicinity of the point.

Thus, not only *visible* irregularities in the distribution of masses (such as mountains, valleys, etc.), but also invisible (subterranean) irregularities will cause undulations in the shape of the geoid.

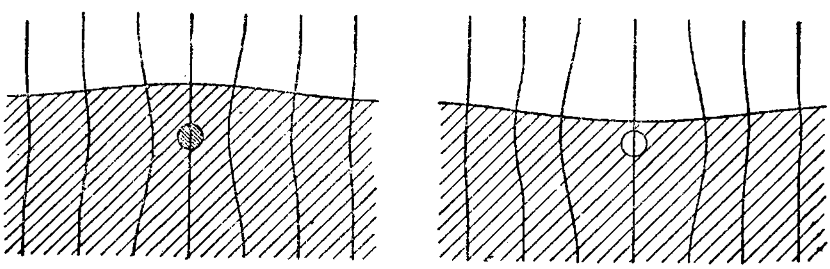


Fig. 1

We must in other words visualize the geoid as a large globe endowed with many "bumps" or irregularities, and it will "breathe" very faintly. These " bumps " are partly of very large (continental) extent, partly of small (local) extent. The building of houses would, of course, theoretically cause an attraction of the lines of force and thus a raising of the geoid, but it will be so infinitesimal as to be without practical geodetic importance.

It is impossible without exaggeration to illustrate a cross-section through the geoid, as the " bumps " even on a large scale drawing will be quite small, as the concavity of the geoid must at all times turn inwards; for the gravity vector is to be the normal vector of the geoid, and it is directed inwards at all points of the Earth. Thus, the geoid will at no place show mountains and valleys like the physical earth, but only an increase or decrease of the ever-concave curvature towards the centre of the Earth. The greater the curvature, the smaller the length of the actual gravity acceleration vector (the mean radius of curvature).

A drawing must therefore always be grossly exaggerated and misleading, which is only natural as angles of a few seconds of arc are so small that they cannot be shown.

It must further be realized that meridians and parallels of latitude on the physical or mathematical earth must not be visualized as an intersection between these globes and meridian planes and latitude planes respectively, there are thus only " geometrical locations ".

We may, for example, define the equator on the physical Earth surface as the locus of the points the plumb lines of which are at right angles to the axis of the Earth and define the parallel of latitude at latitude φ as the locus of the points the plumb lines of which form an angle of $\frac{\pi}{2} - \varphi$ with the axis of the Earth.

Correspondingly, the meridian through Greenwich on the physical Earth surface may be defined as the locus of the points the plumb lines of which are parallel to a plane through the axis of the Earth parallel to the plumb line through Greenwich and define the meridian of longitude λ as the locus of the points the plumb lines of which are parallel to a plane through the axis of the Earth and forming the angle λ with another plane through the axis of the Earth and parallel to the plumb line through Greenwich. (In this connection the axis of the Earth is understood to mean the fixed axis through the Earth connecting the geodetical poles (mean poles) of the Earth).

On account of its irregularities the geoid is not a suitable basis for a survey; no geometry can be developed on this globe and it cannot be depicted simply by a couple of parameters on a plan (chart). It will therefore be necessary further to idealize the Earth in order to obtain a globe suitable as a basis for surveying.

We might accomplish this idealization by letting the Earth be alone in space, by letting it maintain its rate of rotation and then letting all solids change into fluids while maintaining their densities.

The idealized Earth thus defined is called :—

3. **The Spheroid.**— From the theory of elementary potentials we know that if this idealized globe were stationary so as to be subject only to the mutual attraction of the mass particles, it would assume the shape of a sphere.

It has further been shown by Professor Helmert :—

1° That the rotating globe, thus idealized and subject only to the mutual attraction of the mass particles and the centrifugal force, will be a body of rotation.

2° That if the rotating globe thus idealized is homogeneous, it will assume the shape of an ovaloid of revolution, the meridian section of which will be given by the equation $r \doteq a\,(1 - \alpha \cos \varphi)$ in polar coordinates, a being the radius at the equator and α the flattening. Such ovaloid of rotation will, if it has the constants applying to the Earth (velocity of rotation, mean density, etc.) deviate only slightly from an ellipsoid of rotation.

3° That the rotating globe thus idealized may very well assume the shape of an ellipsoid of rotation if the density of the Earth is increasing inwards towards the centre, and from the results of geo-physics there is reason to assume that it does so increase.

4° That the mass of the air envelope is so insignificant in proportion to the mass of the Earth that in view of the accuracy of observation attained up to the present it may be entirely disregarded.

Such an idealized globe (the Spheroid, i.e. a figure like a sphere) will only be very faintly " alive "; only if the Earth receives masses from space or gives off masses to space or changes its velocity of rotation will it change its size and shape.

The spheroid will also be a suitable basis for surveying. By the idealization " by letting the Earth melt ", the " bumps " of the geoid (the mathematical Earth surface) will disappear, so that the curvature will vary so regularly and according to law from point to point as to make it possible to develop a geometry on the surface and depict it simply on a plane. The actual plumb lines will " straighten up " and the equator, meridians and parallels of latitude will be " straightened out " so as to constitute not only " geometrical loci ", but also intersections between the equator plane, meridian planes, latitude planes and the spheroid, respectively. Latitude and longitude will be so-called curved spheroidal coordinates on the surface.

It may be popularly expressed as a paradox if we say " that on the actual Earth the plumb line is not vertical and the water level not horizontal ", as by vertical and horizontal we understand the direction of the gravity line and a plane at right angles thereto on the idealized globe.

The spheroid is not only *a surface* of an idealized globe, but at the same time *a potential plane* (one of an infinite number) in the idealized gravity system (vector field γ (x, y, z), which it creates in space. Each of the idealized gravity lines will (in contradistinction of the actual ones) have its course entirely within the same meridian plane and will there have only a very slight curvature, which will have its maximum at latitude $\varphi = 45°$ (about 0".016 per 100 m.) and its minimum at latitude $\varphi = 0°$ and $\varphi = 90°$.

The horizontal component of the vector difference between the actual gravity acceleration vector g (x, y, z) and the idealized gravity acceleration vector γ (x, y, z) (the angle between the actual and the idealized zenith) at any point (x, y, z) in space may be called the absolute plumb line deflection at that point and it may be given by its value θ in seconds of arc and its azimuth ε, or by its components in direction N-S and E-W ξ and η, respectively.

The determining elements or dimensions of this idealized globe are not known, nor is it known whether it assumes the shape of an ellipsoid of rotation or some other body of rotation, nor how the gravity varies on its surface; one of the aims of geodesy is to determine these facts.

In order to make a survey at all it is necessary to have a basis, we must establish the shape and dimensions (determining elements) of a :—

4. **Reference ellipsoid** or **computation globe.**—Such a computation globe, defined by figures, is completely "dead".

To recapitulate, geodesy is operating with the following four different globes :—

1. The physical Earth surface (the actual Earth).	Common gravity system g (x, y, z).
2. Geoid or mathematical Earth (potential surface to the actual Earth).	Common gravity system g (x, y, z).
3. Spheroid (idealized Earth, at the same time idealized potential surface).	Common gravity system γ (x, y, z).
4. Reference ellipsoid (computation globe, now also computation potential surface).	

and they are more or less "alive" in the order given.

On the basis of the reference ellipsoid we define the chart projections, partly in order to provide a plane graphical reproduction, partly in order to have a chart plan in which to compute, plane geometry being considerably simpler than the geometry of an ellipsoid of rotation, which requires development in series with more or less terms depending upon the distance and the accuracy desired.

Development of Geodesy.

In order to obtain information as to the shape and size of the spheroid several countries have previously undertaken degree measurements, i.e. they have by a combination of terrestrial (trigonometrical) and astronomical observations determined the ellipsoid of rotation which on the basis of the observations made was *in the best conformity* with the curvature of the geoid *within the country or territory in question.* Thus, each country determined its own reference ellipsoid or computation globe on which it bases its maps and charts.

Thus, each country used its own globe and orientated it to make it fit best to the geoid within their territory; this means that the reference ellipsoid was not orientated exactly with its axis and equator plane coinciding with those of the actual Earth, but only parallel with them, and it was therefore impossible to obtain conformity between the geographical (curved ellipsoid) coordinates or geographical grid of the different countries.

The Danish longitudes were thus perfectly correctly defined from the "Round Tower" in Copenhagen, inasmuch as so remote a reference station as Greenwich was not, of course, included in the Danish trigonometrical system.

Each country made its maps and charts on the basis of its own private globe, and it was impossible to obtain a total picture of the Earth with any reasonable degree of accuracy.

Later on gravity measurements began to be developed and to assume importance in the determination of the shape of the Earth "as a whole", in other words, in the determination of the flattening of the Earth, and still later they found practical appplication in geological investigations of the sub-soil.

This epoch in the development of geodesy (we may call it the degree-measurement epoch) *has now been completed.* Like almost all other forms of science, geodesy also is international, and at a congress of the " Union Géodésique et Géophysique Internationale " at Madrid in 1924 it was agreed in all international geodetic work to use the same reference ellipsoid, the so-called international ellipsoid which is defined as an ellipsoid of rotation with the semi-major axis (the radius at the equator) a = 6 378 388 metres and the flattening $\alpha = 1:297{,}0$.

This is thus at present *the closest possible approximation to the spheroid,* being determined on the basis of a very large number of degree measurements and gravity measurements distributed all over the Earth.

While thus the reference ellipsoid of the Danish degree measurement made on the basis of the observations of the Danish degree measurement affords the closest possible approximation to the *geoid below Denmark, the* international ellipsoid affords the closest possible approximation to the *spheroid.* But from this follows that it should also be orientated differently, namely absolutely, i.e. with its axis and equator plane *coinciding* with the axis of the Earth and the equator plane of the Earth (i.e. a plane at right angles to the axis of the Earth through its centre).

Subsequently, the " International Hydrographic Bureau " sponsored the International ellipsoid, and it has been adopted for scientific and practical work in almost all countries with but few exceptions (e.g. Germany). It might be more correct to call it " *the international reference spheroid* " in order to indicate the principal difference from the many different reference ellipsoids previously used.

The adoption of the international spheroid is of great value, inasmuch as the international institutions established for the purpose (the " Union Géodésique et Géophysique Internationale " and the " International Hydrographic Bureau ") have computed very accurate tables for this ellipsoid (tables of radii of curvature of meridian and normal sections, ρ and N, of mean curvature, K, equivalent length on the parallel of one minute of arc, functions W and V, length of meridian arc from the equator, meridional parts, etc.).

By means of these tables numerical solutions of geometrical computations on the surface are facilitated, and it is possible to plot the geographical grid in the two projections most commonly used, namely the normal orthomorphic cylinder projection (the Mercator projection) and the transverse orthomorphic cylinder projection (transverse Mercator projection or the meridian strip projection) with an accuracy of a centimetre, and to transfer any given point on the idealized globe to the chart, and conversely, with an accuracy of a centimetre.

Further, it has been possible to produce a formula for the numerical value of the idealized gravity acceleration vector $|\gamma_0|$ for the surface of this globe. This international gravity formula :—

$$\gamma_0 = 978{,}049\ (1 - .005\,2884\ \sin^2 \varphi - .000\,0059\ \sin^2 2\,\varphi)\ \text{gal}$$

(in which φ means the idealized latitude) has made it possible to use the international reference spheroid also as potential surface for computational purposes, and thus made it possible *to include gravity measurements and potential theory among the aids of geodesy,* thereby providing increased absolute accuracy, as will be explained in detail later.

Accuracy requirements to the coordinates.

The various institutions making detailed surveys (in this country the district surveyors, the cadastral survey institution and the Hydrographic Office), which by means of arithmetical (in contrast to graphical) methods of surveying calculate in the plane of the chart or map on the natural scale 1 : 1, desire to have the rectangular, plane coordinate defined with an accuracy in terms of centimetres. It must be realized that this is nothing but a relative accuracy, accuracy of computation or reading of the list of coordinates, for the absolute accuracy can, of course, never be greater than that with which the position of a single point on the globe may be determined, as, naturally, it must be more difficult to determine the positions of thousands of points than to determine that of a single point.

In actual fact, these institutions are really not users of coordinates, but producers thereof, as they need only the *difference in coordinates* to define the length and direction of the base lines by means of which they produce coordinates of new points, and thus continue building on the basis provided by the Geodetic Institute. If the differences in coordinates are given with an accuracy of a centimetre, then the direction of a base line of about 2 kilometres will be defined with only an accuracy of about 1 second of arc. In Germany they had therefore adopted an accuracy of computation in terms of millimetres, but the German charts cannot for all that be said to have a higher absolute accuracy than those of other countries.

For these institutions the chart or map (the rectangular, plane coordinate) is an aim, not a means, and it will therefore be necessary to investigate what requirements the actual users of charts and maps want to have fulfilled.

For most users the map is only a plane picture showing everything visible on the surface of the Earth, by means of which it is easy to observe at a glance the relative positions of various conspicuous points, so that one may use the picture for finding one's way and for measuring distances.

Only to the scientist and the navigator the chart is something more, namely a means (a scientific instrument) enabling them to determine their position on the idealized globe (the international reference spheroid).

A scientist desiring to determine his position on the globe will buy a map on the largest possible scale of the locality in question (in this country a survey map on a scale of 1 : 20 000), determine his position in the map and then by means of a pair of compasses and the meridians and parallels of latitude given in the map measure his latitude and longitude on the globe.

In this manner the Geodetic Institute has itself used the map for the reduction of its gravity measurements and for the subsequent graphical reproduction of the gravity anomaly in Denmark.

If we reckon that the least measurement that may be made by the compasses is 0,2 millimetre, an accuracy on the globe of 4 metres will be required, or as a second of arc of a méridian is about 30-31 metres and of a parallel about 16-18 metres in Denmark, the second decimal of a second of arc may be uncertain in the geographical coordinates; but this is an absolute accuracy requirement.

There is thus a great disparity between the two different requirements of the surveying institutions and the users of the map or chart. One is a relative accuracy (accuracy of computation) in terms of centimetres, i.e. a " small-scale " accuracy in the projection, the other is an absolute accuracy, i.e. a " large-scale " accuracy on the globe.

The projection used is of quite subordinate importance to the scientist, if only it is provided with a geographical grid, i.e. if only the picture is orientated on the globe. A map (picture) without a geographical grid to indicate the longitude and latitude is of no scientific value.

As the navigator in navigating his ship is always using the compass, he wants a special projection (Mercator's projection) in which lines of constant bearing are reproduced as straight lines. The navigator uses the chart in the same manner as the scientist for determining his position on the idealized globe, but he further uses it for laying down or computing (by means of a table of meridional parts) the courses between different points; not because it is the angular direction in this special projection, but because on account of the conformity of the projection it is at the same time the azimuth of the compass line (loxodrome) on the idealized globe.

A chart differs from a map in that essentially it depicts matters *not visible to the eye*, meridians, parallels of latitude, magnetic meridians, isogonic lines, and banks and rocks hidden beneath the surface of the sea. " A chart, therefore, must be considered not as a pictorial expression of natural objects, but as analogous to a scientific instrument that has been correctly designed and calibrated ". (Admiralty Manual of Hydrographic Surveying.)

As a thermometer made by a skilled instrument maker may be able to show whether the relative temperature difference between different points is large or small, thus a map (plane picture) covering a small area and made by a careful surveyor will be able to indicate the relative bearings between the various distinctive points and whether their relative distance is great or small.

But just as the thermometer would be useless as a scientific instrument if it were not provided with a finely graduated and correctly mounted scale, thus the plane picture would be useless as a chart if it were not provided with a finely graduated and correctly placed geographical grid.

« No chart, except possibly a plan of a very small area, can be called complete unless it is « graduated » in order that the geographical position of any point may be readily seen. An orderly arrangement of meridians and parallels, correctly disposed in relation to the physical features represented on the chart, is a natural and essential feature of any map projection » (Admiralty Manual of Hydrographic Surveying).

As our largest scale chart (Svendborg Sound) is on 1 : 10.000 and the Danish Hydrographic Office in its survey computes in the Mercator plane on the natural scale 1 : 1, the Hydrographic Office, in order to satisfy the demands of shipping, desires an absolute accuracy of about 2 metres and a computational accuracy in terms of centimetres.

Finally, as regards the cadastral survey institution, a projection is required the maximum distortion of which is less than 1 : 20 000. Careful tape measurements in the field involve an inaccuracy of this magnitude, so that linear measurements in the field may, the demands to the projection being fulfilled, be transferred to the plane direct.

As calculations in the plane are always on a scale of 1 : 1, an accuracy of calculation of 1 centimetre is desired, but as surveyors and the cadastral survey institution never use the absolute astronomical observational methods, there are no requirements whatever for an absolute accuracy. The position of the point on the globe is of no interest, only its position in the fictitious projections is of interest. The relative rectangular coordinates (the picture on a scale 1 : 1) actually constitute a scheme of calculations for taxation purposes and for the settlement of boundary disputes.

The different terrestrial (relative) methods of surveying.

In surveying different methods are used according to the size of the area to be surveyed.

If we make a survey of a room, we confine ourselves to measuring along the walls with a rule.

If a small street is to be surveyed, the orthogonal method will be used, i.e., a straight line will be run down the street, and the line and perpendiculars to the gutter, house corners, etc. will be measured by tape measure.

In surveying a large park, a closed polygonal measurement will be used in connection with tachymetry or the orthogonal method, and if the area is still larger, a basis for the survey will be established by a detailed triangulation with appropriate base line measurement.

All these terrestrial, relative methods are well known from land surveying and they have been developed to a high level of perfection. It is a characteristic of them all that they involve a greater or smaller accumulation of errors, but they are indispensable when it is a question of making a plane picture with relative accuracy, as they quickly, simply and plainly connect one point with another.

As will be known, first order triangulation with the use of the largest possible triangles involves less accumulation of errors than detail triangulation with several smaller triangles, and the detail triangulation is therefore made to conform to this superior method.

Polygonal measurement involves a large accumulation of error, as an error in the angle first measured will turn the entire subsequent polygonal line, and a surveyor will therefore never use this method without letting it begin and end at the same point or at two points determined by the superior method, i.e. triangulation.

Only methods constantly based on an absolute direction, like Boussole surveying, which is based on the magnetic meridians, or the geometrical levelling based on the actual plumb lines will have a favourable law of accumulation of error. Boussole surveying is therefore more favourable " on a large scale " than polygonal measurements, which are only profitable " on a small scale ".

Each method has its own distinct field in which it may be favourably employed. No surveyor would ever think of using triangulation in surveying a room or a small street, as siting errors would make this method unfavourable and only involve increased observation and computation work without improving the accuracy.

Our charts are at present based on the first order triangulation and we can therefore confine ourselves to examine this net-work when we want to investigate the accuracy " on a large scale ", and it is evident that if absolute accuracy is wanted great stress must be placed on the determination of this net-work on which the entire chart is based.

The Danish triangulation net-work of the first order is determined *solely* by terrestrial, relative observation; 5 baseline measurements and horizontal, relative direction measurements at all base end points and first order points.

A net-work as large as the Danish first order triangulation was not subjected to one common adjustment, as this would give so many hundreds of conditional equations that the computations required to solve them would present a major task. For practical reasons it was therefore subdivided into several chains of triangles, each with its base line, each of which was adjusted and then combined into the net-work.

The law of the accumulation of errors in such a chain of triangles is not easy to determine as it depends on the form of the individual triangles.

By using the base line as a foundation it is possible to compute the error ellipses for the vertices of the triangles and thereby to give a graphical representation of the mean error of position in the different directions.

This is a major task which has been performed only in respect of the old first order triangulation in the conical-orthomorphic projection and the method is not a popular one, as it proved that when only quite few triangles had been computed the point error exceeded 1 metre, and gradually as the chain extended a positional mean error of up to about 30 metres resulted. Just as an error in the first angle measured in a polygonal line turns the entire subsequent part of the line and causes a considerable accumulation of errors, so an error in a direction measured in the first triangle of a chain will affect the entire subsequent part of the chain and be able to twist and deform it.

And the positional mean error may even greatly exceed that shown in the error ellipse, as all the observations, as will be shown, *may* involve one-sided errors.

" When observations subject to one-sided errors are adjusted in accordance with the usual rules, the result will be that the system at hand is determined in a homogeneous manner and in close conformity with the observation, but the question as to whether the best approximation to truth is attained depends upon pure accident and cannot as a rule be presupposed. At any rate, the mean errors in observations and determining elements and functions thereof appearing from the adjustment will be quite misleading and cannot at all serve as a criterion of the accuracy of the results obtained by the adjustment. If, for instance, an examination of the errors gives rise to the presumption that the observations are subject to one-sided errors, all possible means must be attempted to have them removed from the observation results " (N.P. Johansen : *Udjævning af Observationsstørrelser* (The Adjustment of Observations).

As far as our charts are concerned the presumption of such one-sided errors will arise when the location of a definite point is compared in the Danish, German and Swedish geographical grid, and a difference of several seconds of arc, i.e. a difference of up to some hundred metres will be found.

Let us fiirst consider the base line measurements which, as will be known, are made on the physical surface of the Earth. They are normally subject to a mean error of about 1 millimetre per kilometre, ascertained by checking the measurements and measuring with different tapes.

A long base line of 10 kilometres will thus be inaccurate by centimetres at one end if we consider the other end to be fixed. This is the so-called " internal mean error ".

So far this base line has only been reduced to mean sea level, i.e. to an approximation to the geoid (the imaginary continuation of the sea beneath the continents) by means of the geometrical levelling, which (the actual gravity lines being used) will give the level above the mathematical surface of the Earth, and *not* above our calculation globe.

Fig. 2

As in one part of the country the geoid may be several metres above and in another several metres below the spheroid, this fact may give rise to one-sided errors in the individual base lines.

This will particularly be the case if a reference sphere is used instead of a reference ellipsoid and the base lines are not first (as done in the orthomorphic double projections previously used in Germany) transferred from the reference ellipsoid to the sphere before they are again transferred to the plane of the chart.

The mean error of measurement of the relative, horizontal direction observations in the case of fine first order measurements will (after horizon adjustment) range between $\pm$ 0".2 and $\pm$ 0".5.

If, for argument's sake, we imagine Denmark covered by one large triangle having its angles at the Scaw, on the island of Rømø and on the island of Møen and we make horizontal direction observations between these three points, we will be unable to observe any difference, whether these three points are lying on the international reference spheroid or on a reference sphere with radius of curvature equal to the mean radius of curvature at the centre of gravity of the triangle, the angle between the normal sections and the geodesics being smaller than the normal inaccuracy of measurement.

This demonstrates that the relative, trigonometrical method of surveying is so uncertain or imperfect that over an area as large as Denmark it is unable to distinguish between a reference ellipsoid and a reference sphere, but *it does not* prove that we may just as well use a reference sphere instead of the reference spheroid.

For, *firstly*, we have not paid any regard to the second part of the relative, trigonometrical method, namely the base line measurements, in which the direct application of a reference sphere instead of a reference ellipsoid, as stated, will give rise to the introduction of considerable one-sided errors.

Secondly, we cannot make observations over such long distances and we can only observe a number of considerably smaller triangles, giving rise to accumulation of errors. All the relative, horizontal direction observations are, of course, made on the physical Earth surface with instruments orientated in accordance with the actual plumb lines which are not identical with the idealized plumb lines of the reference spheroid or (still more) with those of the reference sphere.

" In the observation of the horizontal angles an inclination of the vertical axis will cause errors in the angles observed, which errors are a definite function of the inclination of the axis and the location on the circle of the projection of the inclination of the axis on the observed directions. "

" The treatment of the systematic errors consists therein that through special investigations the causes of the errors are determined, the errors are calculated and applied as corrections to the results of observation, which will then appear freed from these errors......" (N.P. Johansen: *Udjævning of Observationsstørrelser*).

If the plumb line deflection is given by the components ξ and η the formula for the correction will be : $(v) = (\xi \sin \alpha - \eta \cos \alpha) \tan h$, in which α is the azimuth of the direction in question.

Let us consider an example: For a normal first order side (about 70 kilometres = about 40 nautical miles) η is numerically equal to the angle of depression, which will be of the order of (1) (about 20' = 1/3°). (In this connection we assume that we sight along the straight line (the chord) between the points, and that both points are on the reference surface. The fact that refraction will raise the distant point closer to the horizon and that it is not on the reference surface cannot possibly increase the accuracy). If we then assume a plumb line deflection of the order of (2) (about 9"), the correction may be an angle of the order of (3) (about 0".06), in other words an angle smaller than the measuring accuracy, but by no means insignificant in geodetics.

From the above it will appear that the relative, trigonometrical method is so inaccurate or imperfect that within a range of about 70 kilometres it cannot distinguish between the reference ellipsoid and the reference sphere. It is therefore understandable that mean error in relative, horizontal direction observations *is not*, as we take it in this country, *independent of the distance*, but as also proved in Germany, it increases with the distance due to the factor tan h. If we consider that the first order sides are of a length of only about 15 kilometres, the direction correction of an orthomorphic, transverse cylinder projection will be smaller than the measuring accuracy as long as we remain in the neighbourhood of the initial meridian. Within this distance the relative, trigonometrical method cannot distinguish the Earth from a plane, but nevertheless it will not be advisable to replace the reference ellipsoid or the reference sphere with a reference plane, as the accumulation of errors would then, of course, become even greater when the observations are extended to comprise larger areas.

Thirdly, we have paid no regard to the other aids to geodesy, the astronomical observations and the gravity measurements which, as will be shown, are capable of distinguishing between geoid and reference ellipsoid and between reference ellipsoid and reference sphere.

In our first order net-work of triangles no regard has been paid to these systematic errors, for which no corrections have been applied and therefore give what are called one-sided errors.

" One-sided errors are very dangerous for the observations, for not only may they encumber the observational results with errors which at times may be very considerable, but also they are difficult to ascertain and are therefore often overlooked.

In the planning of observational work the danger of introducing one-sided errors must therefore always be carefully considered and measures taken to avoid or eliminate them or to ascertain them definitely. " (N.P. Johansen : *Udjævning af Observationsstørrelser.*)

By means of trigonometry only a picture can be produced. In order to produce a chart it is necessary to make astronomical observations of latitude, longitude and azimuth at at least one point of the surface of the Earth in order to have the location of the picture established.

It will therefore be necessary to examine also the astronomical observations in detail.

Astronomical observations.

By an astronomical latitude observation the angle between the actual plumb line and the equatorial plane of the celestial sphere is ascertained (by means of the declinations of the stars).

If the observations are continued through a considerable time (about 1 year or more), it will be found that this " observed latitude " is not a constant but that it is a function of time. As will be known, this is due to the polar movement, inasmuch as the world axis (axis of rotation) and the axis of the Earth (the line connecting the permanent " mean poles " or " geodetic poles " of the Earth) and thus the equatorial planes of the celestial sphere and the Earth (fundamental planes) do not coincide, but have one point only, the centre of the Earth, in common. Popularly speaking, the Earth is not fixed on the world axis, but " rocks " slightly (about 0".05 — 0".03) about the centre of the Earth.

The cause of the polar movement is, as will be known, that the Earth is not alone in space, but is influenced by other celestial bodies. As the Earth is neither entirely fluid nor entirely rigid, but is an elastic body, the polar movement cannot be calculated, but it may be and is being observed by internationally arranged observations, so that all latitude observations may be reduced to the permanent mean poles of the Earth and the permanent Earth equatorial plane appertaining thereto.

The observed latitude thus referred to the equator plane of the Earth has hitherto in this country been termed the " astronomical latitude ". In my opinion this term is misleading, as this latitude is no longer referred to the celestial sphere (it is only an observed latitude which should always be accompanied by a statement of time), and I will therefore in the following use the term " the physical Earth latitude " in order to designate the globe to which it belongs. As long as the plumb line (Zenith) remains unchanged (in relation to the Earth), the physical Earth latitude will also remain constant.

If we connect all points on the physical Earth surface having the physical Earth latitude zero by a line, we will have a curve showing the geometrical locus of the points the actual plumb line at which is parallel to the equatorial plane of the Earth, but *not* the intersection between the physical Earth surface and the equator plane of the Earth. Similarly the parallels of latitude on the physical Earth surface will be only geometrical loci.

If we make the mental experiment of taking the latitude observations on the geoid and correcting them for the polar movement, we will obtain a " geoidic latitude ". The geoidic equator (the geometrical locus of the points on the geoid the plumb lines at which are parallel to the equatorial plane of the Earth or at right angles to the axis of the Earth) will be a geodesic on the geoid, inasmuch as its osculating plane at any point of the curve will contain the normal of the surface. On the other hand, the geoidic parallels will only be " geometrical loci ".

If we consider the geoid idealized into a spheroid, the " spheroidic equator " will at the same time be the geometrical locus of the points on the spheroid, the idealized plumb lines at which are at right angles to the Earth axis (which is also the axis of the spheroid) and the intersection between the equatorial plane of the Earth (being at the same time the equatorial plane of the spheroid) and the spheroid; further, it will at the same time be a geodesic on the spheroid. Likewise, the spheroidic parallels will at the same time be "geometrical loci" and intersections between the parallel planes and the spheroid.

If we consider the geoidic equator and the geoidic parallels "depicted", "transferred" or "projected" to the spheroid along the plumb lines (it is of no importance in this connection whether they are the idealized or actual plumb lines, as the surfaces of the globes are so relatively close to one another and the actual and the idealized plumb lines are at angles of only a few seconds of arc with one another), then they will not coincide with the corresponding curves on the spheroid, in the same way as a geodesic between two points on the spheroid or a great circle on a sphere reproduced on a chart plane will not coincide with the straight line (the geodesic in the plane) between the corresponding points. At all the points at which the N-S component ξ of the absolute plumb line deflection is zero, the corresponding curves will intersect, and only at the few points in which both the N-S component ξ and the E-W component η are zero and the actual and the idealized plumb lines thus coincide, will the corresponding curves touch one another and thus coincide over a short length.

The most careful astronomical latitude observations made with portable instruments for geodetical purposes are subject to a mean observation error of $\pm$ 0".02-0".05 or about 1 metre on the globe. This is again an "internal mean error" ascertained on the basis of the discrepancies in a series of observations between the various pairs of stars and on the basis of the probable errors of the declinations of such stars entered in the star catalogue, and on the basis of the coordinates given for the polar movement. If the declinations are subject to considerable one-sided errors, the inaccuracy of the observations will, of course, be greater than that calculated. This shows that astronomy and geodesy are sciences so nearly related that progress within the field of astronomy will be of benefit to geodesy, and although I am not an astronomer, I am of the opinion that the opposite will also be the case, and that the astronomers in their charting of the celestial sphere will be interested in ascertaining how much the actual plumb line and thus their instruments are inclined to the idealized plumb line.

When using a physical Earth latitude (so-called "astronomical latitude") direct as an "idealized latitude" (hitherto also called "geodetic latitude" or "geographical latitude") for the purpose of orientating a point on the reference ellipsoid and thus in the chart plane in direction N-S, we introduce an error of the same magnitude as the angle between a plane at right angles to the meridian plane containing the actual plumb line at the point in question and another plane at right angles to the meridian plane containing the normal of the spheroid at the corresponding point of the spheroid. This angle will also be equal to the N-S component ξ of the actual plumb line deflection at the point + the curvature of the idealized gravity line from the physical surface of the Earth to the spheroid. The idealized gravity lines lie entirely within the meridian planes and have there only a very slight curvature which may be given by the formulae :—

$$\frac{\Delta h}{5820} \sin 2\varphi \text{ (given by Helmert) or}$$

$$1070'' \frac{H}{a} \sin 2\varphi \text{ (given by Gauss).}$$

It will be seen that the curvature of the idealized gravity lines will have its maximum at $\varphi = 45°$ when it amounts to only about 0".016 per 100 metres and its minimum at $\varphi = 0°$ and $\varphi = 90°$. In some countries these formulae have hitherto been used for the purpose of reducing the latitude observations to mean sea level (the geoid), but in Denmark no reduction has been applied, the correction for the small Danish heights being less than the inaccuracy of observation.

As far as I know, no one has so far made any correction for the absolute plumb line deflection (I have been unable to find any reference to such correction in the literature) and this correction which is introduced with its full value may amount to some seconds of an arc, in other words, considerably more than the mean error of observation; indeed, it will frequently be much greater than the correction for the polar movement.

By thus transferring a latitude directly from one globe to another (physical Earth or geoid to reference ellipsoid (reference spheroid, computation globe)) an error is made which may amount to some seconds of arc or a couple of hundred metres. It is assumed that the plumb line "is vertical" in direction N-S, i.e. that it has no inclination in that direction relative to the idealized plumb line.

Considerations somewhat similar to those applying to astronomical latitude observations apply also to astronomical longitude observations.

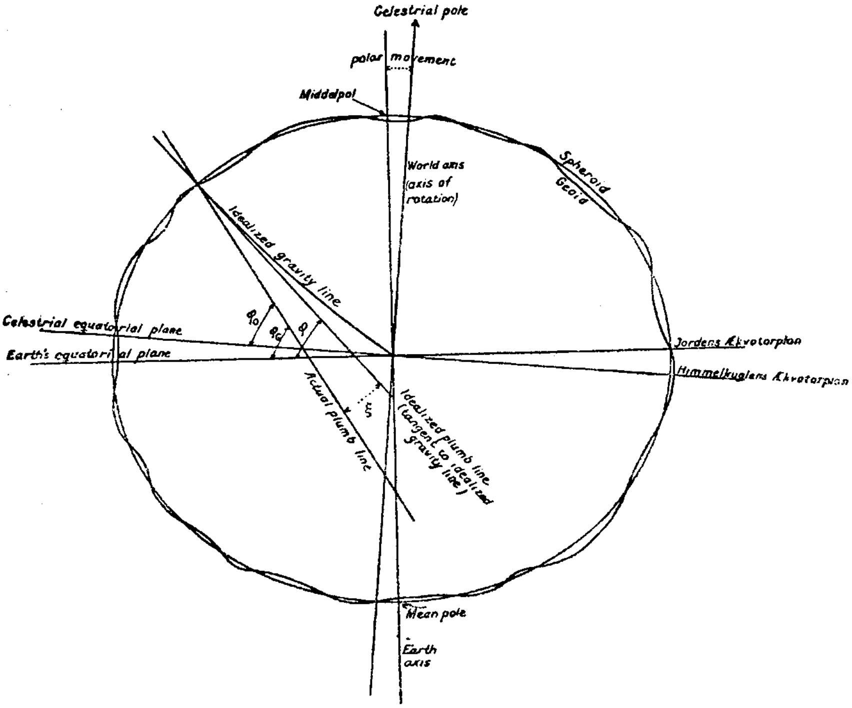


Fig grossly exaggerated. World-axis and actual plumb line need not lie within the plan of the paper.
φ_0 = observed (astronomical) latitude φ = geoidic latitude φ = idealized (geographical) latitude.

Fig. 3

By an Astronomical Longitude observation we ascertain (by means of the right ascensions of the stars and on the presumption that the Earth rotates uniformly) the angle (expressed in time) between a plane at right angles to the equatorial plane of the celestial sphere containing the actual plumb line of the reference station and another plane at right angles to the equatorial plane of the celestial sphere containing the actual plumb line of the point of observation. Otherwise expressed, we ascertain the arc on the infinitely remote equator of the celestial sphere cut off between one plane determined by the world axis and the *direction of* the actual plumb line of the reference station and another plane determined by the world axis and the *direction* of the actual plumb line of the point of observation, as it must be realized that the actual plumb lines at these two points by an extension need not necessarily intersect either the world axis or the axis of the Earth, being " inclined " in relation to the idealized plumb lines.

If the observations are continued through a considerable space of time (about 1 year or more), it will be found that this " observed longitude " is not a constant, but a function of time, and this is again due to the polar movement giving rise to the fact that world axis—Earth axis and thus the celestial equator plane and the Earth equator plane (fundamental planes) do not coincide.

By means of the internationally observed coordinates of the polar movement it is possible to reduce the observed longitude from the celestial equator plane to the Earth equator plane fixed in the Earth.

The " observed longitude " thus referred to the equatorial plane of the Earth has hitherto in this country been termed the " astronomical longitude ". In my opinion this term is also misleading, it being no longer attached to the celestial sphere (but only an observed longitude which should always be accompanied by a statement of time), wherefore I will call it the " physical Earth longitude " in order to designate the globe to which it is attached. As long as the actual plumb line (and thus the zenith) at reference station and point of observation remains unchanged in relation to the Earth, also the "physical Earth longitude" will remain a constant.

As a reference station for longitude the Royal Observatory at Greenwich which has a longitude zero is being used internationally in science (astronomy, geodesy, geophysics, etc.), and in navigation. If we consider all points on the physical surface of the Earth with zero physical Earth longitude connected by a line, we will have a curve indicating the geometrical locus of points on the physical Earth surface the actual plumb lines at which are parallel to a plane through the Earth axis parallel to the actual plumb line through Greenwich Observatory, but *not* the intersection between this plane and the physical Earth surface. We may call this curve the zero-meridian of the physical Earth surface and the plane "the Greenwich meridian plane" in contradistinction to the "meridian plane *through Greenwich*", which (like the zero-meridian of the physical Earth surface) does pass through the Greenwich Observatory, but need not contain the plumb line of that point and only will do so if the E-W component η of the absolute plumb line deflection at that point is O; the two planes, the "Greenwich meridian plane" and the "meridian plane *through Greenwich*" will then coincide. All points on a physical Earth meridian have a common "meridian plane", whereas many "meridian planes through any of them" will pass through the various points of a physical Earth meridian.

If for argument's sake we make the longitude observations on the geoid and correct them for the polar movement, we will obtain a "geoidic longitude". All the "geoidic meridians" (geometrical loci) will be geodesics on the geoid, their osculating planes containing the normal of the surface at any point of the curves. All the "geoidic meridians" and "geoidic parallels" will intersect at right angles and thus form an orthogonal grid on the geoid.

If we consider the geoid idealized to the spheroid, all these "geoidic meridians" (geometrical loci) which are at the same time geodesics on the geoid will straighten themselves out and become "spheroidic meridians" which at the same time are geometrical loci, geodesics on the spheroid and lines of intersection between the spheroid and the meridian planes.

If we consider the geoidic meridians depicted or projected on the spheroid along the plumb lines, they will not coincide with the corresponding spheroidic meridians, but will intersect them at all points at which the E-W componant η of the absolute plumb line deflection is zero. Only at the few points at which both the N-S. component ξ and the E-W component η are simultaneously zero will the two corresponding curves touch one another and thus coincide over a short length.

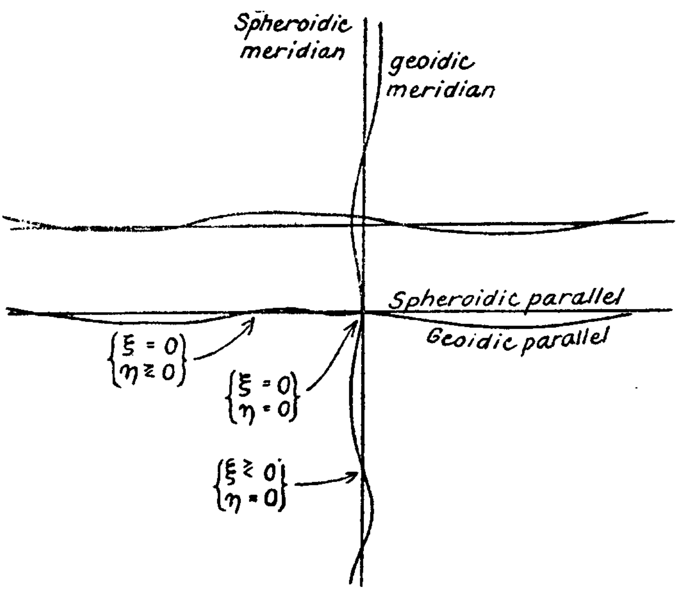


FIG. 4

The careful longitude observations made for geodetic purposes by means of portable instruments, radio time signals and chronographs are subject to an observational mean error of the order of ± 0.003 — 0.005 seconds of time, which on the globe at our latitude will give a linear inaccuracy similar to that of the latitude observations of about 1 metre. This is again an "internal mean error" determined on the basis of the discrepancies of a series of observations of many stars and on the basis of the probable errors in the right ascensions of such stars as given in the star catalogue, and on the basis of the coordinates given for the polar movement. The value of this "internal mean error" depends upon the value of the probable errors in the right ascensions.

As the linear length on the globe of a minute of longitude decreases by cos φ, the error committed by making direct use of a "physical Earth longitude" (so-called astronomical longitude) as "idealized longitude" (hitherto also called "geodetical longitude" or "geographical longitude") for the purpose of orientating a point on the reference ellipsoid and thus on the plane of the chart in direction E-W without regard to the fact that the plumb line (zenith) is thereby changed, will amount to η sec. φ. As η may amount to some seconds of arc, the error may become considerably greater than the inaccuracy of observation and may give rise to a linear error of a couple of hundred metres. It is "assumed" that the plumb line is vertical in direction E-W, i.e., that it has no inclination in that direction in relation to the idealized plumb line.

I shall next pass to an examination in detail of the astronomical azimuth observations, the determination of an absolute direction or the direction of the normal section through the point of observation and another point.

By an astronomical azimuth observation the angle is determined by means of the calculated azimuth of a star between a plane at right angles to the equatorial plane of the celestial sphere, containing the actual plumb line of the point of observation, and another plane containing the actual plumb line of the point of observation and the direction of the other point in question.

In other words the angle is measured in the actual horizontal plane of the point of observation between the direction to the pole of the celestial sphere and the direction in question. The azimuth of the star is calculated on the basis of the declination of the star (taken from the star catalogue), the local hour angle of the star (which in connection with a time signal is determined by means of the right ascension of the star and the geographical (idealized) longitude of the point of observation) and the geographical (idealized) latitude of the point of observation. An error in the geographical coordinates of the point of observation will therefore affect the calculation of the azimuth of the star and thereby the value of the observation. In order to minimize this influence (as far as longitude is concerned), a pole star at its maximum easterly or westerly elongation should preferably be used for the observation.

Such an astronomical azimuth will doubtless be a function of time, as the pole of the celestial sphere is not stationary in relation to the mean pole of the Earth. Hitherto no correction has been made in this country for the polar movement (whether it has been done abroad I do not know), presumably because the correction is much smaller than the uncertainty of observation. For while in this country a relative direction of a first order side is determined with 24 repetitions, an absolute direction (azimuth) is determined with only 4-6 repetitions, presumably because the observational work is much greater.

Actually, however, it is not, of course, an astronomical azimuth, the angle in the actual horizontal plane between the direction to the celestial pole and the direction in question which is required, but an idealized azimuth, i.e. the angle in the idealized horizontal plane between the stationary mean pole of the Earth (at the same time the pole of the spheroid) and the direction in question. When a geophysicist from the "Meteorological Institute" studying Earth magnetism and the course of the magnetic meridians (the variation) on the globe, requests the "Geodetic Institute" to provide him with an azimuth at the magnetic observatory in the Rude Forest, then it is the absolute direction on the idealized globe used for international scientific work he desires, in order that he may compare the results of scientific observations in Denmark with those of other countries.

When the physical Earth is idealized to the spheroid, the actual plumb line (the zenith) and thus also the actual horizontal plane change position and the angle in that plane will thereby be changed. The correction of the angle required in such an idealization process (adjustment of the "inclined" plumb line) is given by the formula η tan φ.

As the E-W component η of the absolute plumb line deflection may amount to some seconds of arc, the error committed by the direct use of an astronomical azimuth as an idealized azimuth in orientating a direction on the reference ellipsoid and thus in the plane of the chart may amount to some seconds of arc in our latitude, and at high latitudes to still more, an error which is considerably in excess of the mean error of observation.

It is "assumed" that the actual plumb line at the point of observation is vertical in direction E-W, i.e. has no inclination in that direction relative to the idealized plumb line. Besides, an error (due to both components, ξ and η of the plumb line deflection) in the geographical (idealized) coordinates of the point of observation will affect the value of the calculated azimuth of the star and thus the results of the observation.

As the linear observational mean error in the case of fine astronomical latitude and longitude observations amounts to about 1 metre, we see that this astronomical method of surveying when extended beyond a few metres is capable of distinguishing the surface of the Earth from a plane in which all plumb lines are parallel. As the difference in absolute plumb line deflections (the relative plumb line deflection) between two neighbouring first order points may amount to several seconds of arc, we see that the survey method at these distances is capable of distinguishing plainly between the geoid and the reference ellipsoid both as regards coordinates and directions.

This fact has previously been used for the purpose of obtaining knowledge of the shape of the geoid (the mathematical earth surface) in relation to the reference ellipsoid under limited areas by the so-called "astronomical levelling" e.g. by the investigation of the shape of the geoid in the Harz. The method is, however, not particularly suitable for this purpose as it involves an immense amount of astronomical observation, and it has therefore been used rarely and not at all in Denmark.

Thus, a plumb line deflection (and in this connection it is of no importance whether it is relative or absolute) may be "felt" in the astronomical observations, as it may be relatively great in proportion to the observation mean error and may give rise to :—

A plumb line deflection in latitude : ξ ;

A plumb line deflection in longitude : η sec φ and

A plumb line deflection in azimuth : η tan φ .

A point on the physical Earth surface is (at a given time, or as long as the plumb line remains unchanged) uniquely determined by its physical Earth latitude and its "physical Earth longitude" (from any reference station), as all potential surfaces have a concave curvature towards the centre of the Earth, and as the physical Earth surface (apart from perfectly vertical rock faces, subterranean grottoes and caves and the like) is not in several "storeys" and as the "geometrical loci" (position lines) intersect at right angles.

It is impossible by calculation to move a point determined on the physical Earth surface by physical Earth latitude and physical Earth longitude a given distance in a given direction, as the physical Earth surface cannot be stated in terms of mathematics and as the absolute plumb line deflection expressed as a function of place on that surface is unknown.

Nor is it possible by calculation to move a point given on the geoid by geoidic latitude and geoidic longitude a given distance in a given direction, it being not as yet possible (and it is doubtful whether it ever will be) to express the form of the mathematical Earth surface and thereby a geodesic on that surface in mathematical terms.

It will be possible to move by calculation only a point on the idealized Earth (the reference ellipsoid or the reference spheroid) given by geographical (idealized) latitude and longitude a given distance in a given direction. And this given direction is *not* a faultlessly observed astronomical azimuth to the other point (the point arrived at) corrected for polar movement, absolute plumb line deflection, curvature of the idealized gravity line and the height of the other point over the ellipsoid to the idealized azimuth, which gives *the direction of the normal section* from the point of departure, but *the direction of the geodesic* from the former point to the latter. *This direction cannot be observed* and the geodesic is therefore of significance only as a mathematical aid.

If, for argument's sake, we observe all directions in a large first order triangle on a spheroid (idealized globe) without any error, then the triangle will not be closed, the sights being taken along mutual normal sections.

If we imagine that the points of the triangle, with the observed directions corrected by a faultlessly calculated direction correction, were transferred orthomorphically to a chart plane, the triangle would not be closed there either. Only the adjustment which applies a "correction" to the faultlessly measured and transferred directions will close it. Only when the spheroid (the idealized globe) assumes the simple form of a sphere, the normal sections and geodesics of which coincide (great circles) the triangle will be closed before the adjustment is made.

But as far as the Earth spheroid is concerned the eccentricity is so small we may *locally* and *for each* first order triangle or *for each* first order point of observation use a spherical calculation, the angle between the mutual normal sections and the geodesic being much smaller than the observational mean error even in the case of the finest direction observations (both absolute and relative).

But the radius of curvature in the "computation globes" used or the arbitrary constants in the formulae for direction and distance correction (if we desire to calculate in a chart plane) must vary with our movements in the first order net of triangles, so that they will correspond to the mean radius of curvature in the centre of gravity of the triangle, or of the point of observation on the spheroid, in order to obviate an accumulation of one-sided errors.

In this country astronomical observations have not so far been used for the purpose of giving the picture (the relative coordinates) a greater absolute accuracy, but solely for the purpose of orientating the picture on the globe, by determining at one point the physical Earth latitude, physical Earth longitude and astronomical azimuth and then using these values direct as idealized values. For such purpose the astronomical observations *alone* will be unsuitable.

By thus transferring a physical Earth latitude from one globe (the physical Earth surface) direct to another (the reference spheroid), an error will be committed equal to the plumb line deflection in latitude ξ less the curvature of the idealized gravity line from the Earth surface to the spheroid, and, as regards longitude, an error equal to the plumb line deflection in longitude η sec φ, and as regards azimuth an error equal to the plumb line deflection in azimuth η tan φ ; in other words, errors which linearly as regards the coordinates may amount to some hundreds of metres and as regards direction (azimuth) to several seconds of arc.

Whereas the relative direction of any first-order side in the chart plane of our system is determined by 0".001, the most fundamental of all directions, e.g. the direction of the Earth axis, and the direction of the geographical coordinate system (grid), in which other countries are placed in relation to Denmark, and whence navigators plot or calculate their courses, are determined only with an absolute accuracy of several seconds of arc.

It is "assumed" that the plumb line at this point of departure is "vertical", i.e. has no inclination in relation to the idealized plumb line on the idealized globe.

It will never be possible to determine a position on the idealized globe absolutely orientated with greater accuracy than that with which it is possible to determine the components of the absolute plumb line deflection.

Every navigator will know an analogous case, even if it is here only an example in the plane (the horizontal plane) and not in space.

A compass placed on board a ship will *by itself* be of little value as an indicator of the absolute direction. The compass does not attain any value as a direction indicator until the variation, which (being common to all ships) like the polar movement (common to all countries) is being checked by international observations; and deviations, which each ship must determine by observations on the different courses (just as each country must determine the plumb line deflection by observations in the neighbourhood around the various points of observation) are known.

Even the most skilful helmsman cannot keep his course with a greater accuracy than that with which the deviation is known; if there is an (unknown) deviation of, say 5°, the course will be 5° in error.

If we then successively calculate the geographic coordinates of all first order points from this point, then it is "assumed" that the picture (the relative coordinates) in the chart plane is free from error, but, of course, it can only be assumed to be so provided that all the relative, terrestrial observations are free from one-sided error, and this they can be assumed to be only if the geoid underlying the area partakes of the form of the reference surface (the computation globe).

In the placing of the ellipsoid of the Danish degree measurement (the conical orthomorphic system) in this country, it has been "assumed" : Firstly, that the plumb line in the Round Tower in Copenhagen was vertical (i.e. that this point was subject to no absolute plumb line deflection, that the actual gravity line and the idealized gravity line corresponding to the ellipsoid of the Danish degree measurement, coincided), and, secondly, that the geoid partakes of the form of the ellipsoid of the Danish degree measurement.

When the system was established in 1841, it was impossible to reckon on the basis of other assumptions. This ellipsoid was determined by the Danish degree measurement to be that form of ellipsoid which gave the best conformity between the relative terrestrial and the astronomical observations in Denmark, and it was in those days impossible to determine the absolute plumb line deflection.

Perhaps the geographical latitude of the "Round Tower" 55°40'53" 0000, has not been determined by observation at all, but has been "chosen" or "fixed" as a suitable value like the geographical longitude 0°00'00" 0000. At that time, more than 100 years ago, when the system was established, the declinations of the stars were rather indefinitely determined, the inaccuracy of astronomical observations was considerably greater than to-day, and the polar movement was not known to exist.

As far as I know, no astronomical longitude observations whatever were used in the establishment of the conical orthomorphic system; any such observations, at a time when telegraphy was in its infancy and the introduction of radio time signals still a very long way ahead, so that journeys with chronometers had to be resorted to, were at a very primitive stage compared with those of our days.

It is possible that in the extremities of the grid astronomical azimuth observations have been used at discretion in order to counteract an accumulation of errors of relative direction observation, but as shown in the foregoing these astronomical azimuth observations may be subject to one-sided errors (the plumb line deflection in azimuth $\eta \tan \varphi$) of several seconds of arc.

As a modern chart, our conical orthomorphic system must therefore now be considered obsolete and defective, but it was a very fine and beautiful piece of work at the time when it was created, a time when they had no close knowledge of the shape of the Earth "on the whole", nor of the polar movement, when astronomy and observation-and instrument-technique were far behind their present-day state, and when gravity measurements had not been developed at all. The system has served the country well and faithfully for several generations, and all honour and credit are due to the Danish degree measurements and the topographical department of the Danish general staff for the great pioneer work which was eminent for that age.

A chart must, however, always to a certain extent be a "snapshot", (1) partly on account of the cultural development on the actual surface depicted; (2) partly on account of movements in the actual surface of the "living" globe (in German termed "sekuläre Bodenbewegungen"), for even an actual plumb line is not something permanent and unchangeable, and (3) partly on account of the scientific development in the fields of geodesy, physics, astronomy and instrument making.

The greater part of all scientific research is necessary for practical life and will redound to its benefit, and thus the most recent results of geodesy are necessary in order that science and practical life (shipping) may be provided with the best possible charts, which must be based on a scientific foundation.

Hitherto it has been impossible to produce a relatively accurate chart of the Sound, as the Danish and Swedish coasts geodetically have been lying each on its own globe (reference ellipsoid) and it has therefore been impossible at all points to obtain conformity between the geographical coordinates belonging to these globes. Only at a single point conformity may be obtained by a "shifting" of Sweden.

Just as we in Denmark have "assumed" that the plumb line through the "Round Tower" was vertical and that the geoid partook of the shape of the Danish degree measurement reference ellipsoid, so they have in Sweden "assumed" that the plumb line at Stockholm was vertical and that the geoid partook of the shape of the reference ellipsoid used in Sweden.

It is therefore no wonder that the discrepancies between the geographical grids of the two countries are so great; the reference ellipsoids are not uniform and not orientated absolutely with their axes and equatorial planes coinciding with those of the Earth; *the geographical grids* (the geographical coordinates) *are consequently* not absolute, but *relative,* i.e. attached to the picture and *related to the point of observation used by the country in question as a geodetic datum for its calculations.*

It has always been the practice to let the geographical coordinates (the grid) be functions of the different relative (local) coordinate systems, whether they are rectangular, spherical coordinates (the so-called Cassini-Soldner coordinates), rectangular spheroidic coordinates (the so-called Gauss-Krüger coordinates) or rectangular, plane coordinates.

If we imagine that another point of departure had been chosen for the calculations and astronomical observations had been made at that point, entirely different geographical coordinates (an entirely different geographical grid) would be obtained with the same relative terrestrial observations, not so much on account of the observational inaccuracy of the astronomical observations, which is perceptible only in the second decimal of the second of arc (about 1 metre), but on account of the relative plumb line deflection (i.e. the difference between the absolute plumb line deflections) between the two points of departure, which may amount to several seconds of arc (up to some hundred metres).

While Denmark and Sweden thus have considered the method of relative terrestrial surveying (triangulation) the most accurate practical method, and (apart from the point of departure) one-sidedly have let the geographical coordinates (the geographical grid) be functions of the relative coordinates (the picture), they have in other countries (e.g. Germany and Finland) been aware of the fact that this relative terrestrial method is subject to accumulation of errors, have tried to counteract this and thus to create a more accurate picture by means of absolute astronomical observations in which no accumulation of errors will take place. They have " coordinated " or " cooperated " the relative terrestrial and the absolute astronomical observations by a so-called Laplace—or geodetic astronomical—grid adjustment.

This method of adjustment was first introduced by Professor Helmert and subsequently developed by others.

In this method of adjustment the plumb line deflection is the angle between the *normal of the reference ellipsoid* and the actual plumb line, i.e. something relative, dependent upon the elements determining the reference ellipsoid used and its location or orientation.

In the Danish conical orthomorphic system this relative plumb line deflection may be found at any point from astronomical latitude and longitude observations made at that point and corrected for the polar movement, by collating the results with the " geodetic " latitude and " geodetic " longitude, i.e. the values for the point in question taken from the chart plane (by means of the geographical grid) or calculated; thereby the relative plumb line deflection in latitude ξ and the relative plumb line deflection in longitude η sec φ will be obtained, whence the components ξ and η of the plumb line deflection or the total plumb line deflection $\theta = \sqrt{\xi^2 + \eta^2}$ and its azimuth ε (having $\xi = \theta \cos \varepsilon$ and $\eta = \theta \sin \varepsilon$) may be calculated.

If we consider an astronomical latitude observation made at our geodetic datum, the " Round Tower " in Copenhagen, and corrected for polar movement and height above " sea level ", the peculiar fact may emerge that even the " Round Tower " has a relative plumb line deflection in latitude ξ, whether this may be due : (1) to the fact that the geodetic latitude of the " Round Tower ", 55°40'53",0000 has not at all been determined by observation, but has been fixed or adopted as a suitable value, or (2) to the fact that in those days the latitude could not be determined with the same accuracy as nowadays, or finally (3) to the fact that the plumb line through the " Round Tower " in the course of time has changed in direction N-S in relation to the Earth.

The longitude of the " Round Tower " with that tower as a reference station will, of course, continue to be 0°00'00".00. But if we imagine an astronomical longitude observation with Greenwich as a reference station made at the " Round Tower " and corrected for the polar movement, then it may similarly be possible to ascertain a relative plumb line deflection in longitude from the geodetic longitude of the " Round Tower " on the basis of Greenwich given in more modern charts, whether this is due to the fact that the plumb line at the Observatory of Copenhagen, at the " Round Tower " or at Greenwich has changed in direction E-W in relation to the Earth, since the longitude of the Observatory was determined, or to the fact that in those days the astronomical longitude could not be determined with the same accuracy as nowadays.

The basis for the orientation of the reference ellipsoid of the Danish degree measurement (the conical orthomorphic system) must therefore probably now be said to have been lost.

By the Laplace adjustment we find the components ξ and η of the relative plumb line deflection, and thus the plumb line deflection in latitude ξ, the plumb line deflection in longitude η sec φ and the plumb line deflection in azimuth η tan φ (i.e. the " corrections " to be applied to the astronomically observed values of latitude, longitude and azimuth corrected for polar movement in order to convert them into " geodetic " values (i.e. located on the reference ellipsoid), at all the so-called " Laplace Points ", at which astronomical latitude, longitude and azimuth observations have been made, by an adjustment of these observations as well as of the relative terrestrial first order observations before any chart (system) whatever has been drawn or has become available.

By a degree measurement, the elements are determined of that ellipsoid of rotation which will give the best possible conformity between some given astronomical and relative terrestrial observations within a certain limited area. The Laplace adjustment proceeds in the opposite direction and finds that orientation of a *given* reference ellipsoid (on the basis of a point of departure or point of reference considered to be without error), which will give the best possible conformity between the astronomical and the relative terrestrial observations within the area and the correction to be applied to these observations in order to give such conformity.

This orientation or location of the reference ellipsoid is *not* absolute (i.e. having the axis and equatorial plane coinciding with those of the Earth), and the geodetic, geographical coordinates (latitude and longitude) will therefore not be absolute either, but will depend upon : (1) the elements determining the reference ellipsoid, and (2) the point of departure (geodetic datum) chosen for the calculations.

If we choose another point of departure for the calculations we will (on a reference ellipsoid with the same determining elements and the same astronomical and terrestrial observations) obtain an entirely different system of plumb line deflections and thus quite different geodetic-geographical coordinates. In this principle of adjustment it is " assumed " that the relative plumb line deflection at all Laplace points in the area assume the same character as an accidental error ($\sum \theta^2$ = minimum i.e. $\sum \xi = 0$, $\sum \eta = 0$), and that the plumb line at the point of departure is " vertical ".

In the European plumb line deflection system created by Germany (which system comprises also Denmark and in which the astronomical observations and the base line measurements are given the weight ∞), it has been " assumed " that the elements are those of the Bessel reference ellipsoid, that the plumb line at Greenwich is vertical (i.e. in respect of Greenwich astronomical latitude = geodetic latitude, astronomical longitude 0° = geodetic longitude 0°, and relative plumb line deflection 0), and that the relative plumb line deflection at all Laplace points distributed all over Europe has the same character as an accidental error. Thus, this plumb line deflection system can never be extended to areas which cannot be trigonometrically (visually) connected with the European continent (in other words, islands so far out in the ocean that they cannot be seen from the continent).

Even if the Greenwich Observatory is being used internationally as a reference point in determining longitude and time, it is improbable that the plumb line for that reason will be " vertical " just at that point, and it can by no means be taken for granted that the plumb line deflections at the relatively few, casually selected Laplace points assume the character of an accidental error.

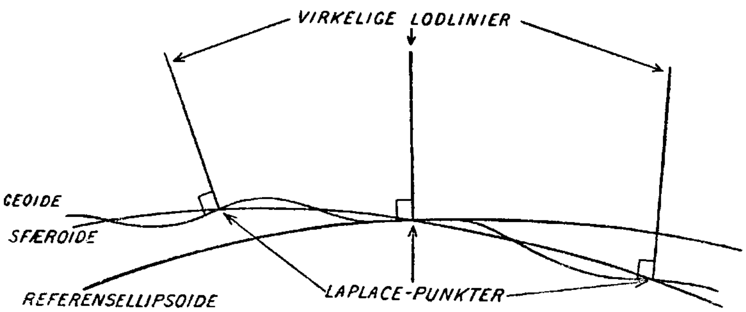


Fig. 5

If. for example, the geoid is regularly undulating in any direction (say E-W) (i.e. has a continually varying radius of curvature, in this case radius of normal curvature) and all the Laplace points about that direction (parallel of latitude) happen to lie on the same side of the undulations, this will give rise to a one-sided orientation of the reference ellipsoid in relation to the spheroid. Or if the geoid under the territory in question has a continental rise (and rises are found to such an extent that it has been considered that the best *geometrical* approximation to the geoid might be obtained by a 3-axial ellipsoid), this fact may also cause a one-sided orientation of the reference ellipsoid in relation to the spheroid. The axis and equatorial plane (fundamental plane) of the reference ellipsoid will *not* coincide with the axis and the equatorial plane, respectively, of the Earth.

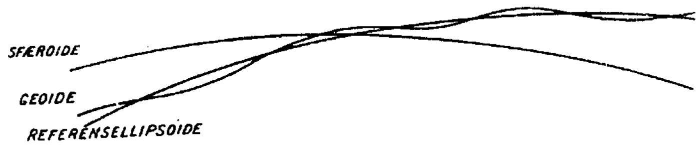


Fig. 6

The adjustment principle is closely related to the degree measurement epoch and cannot, therefore, in my opinion be used in a country which has (internationally) adopted the international reference spheroid; this, of course, is *not* to be orientated so as to give the best possible conformity between the terrestrial and astronomical observations at relatively few points within the small part of the Earth covered by the country. If so the internationally adopted common computation globe (the reference spheroid, at present the physically best possible approximation " on the whole " to the mathematical Earth surface, applicable to computations) would be split up into many small pieces, and every country would orientate its little piece of " shell " in such a way as to give the closest possible conformity with the terrestrial and astronomical observations within its area and without connection with those of surrounding countries. But this spheroid must be *as far as possible orientated absolutely*, i.e. so that the axis and equatorial plane of the reference spheroid will coincide with those of the Earth.

Thereby the plumb line deflection will become an absolute quantity, the angle between the idealized plumb line belonging to the reference spheroid and the actual plumb line, or the horizontal component of the vector difference between the normal vectors (gravity vectors) of the idealized and actual potential surfaces.

As both the reference spheroid and the geoid are potential surfaces with equatorial plane (fundamental plane) and axis in common, and as the difference between two potential functions is a potential function, then in this case it is the integral of the plumb line deflection θ, taken over all surface elements on the globe, that is to be zero.

The question is then whether it is possible by observation to determine at the various points the absolute plumb line deflection so that it will be possible to reduce or refer the direction observations there made (both astronomical and terrestrial) to the reference spheroid before an adjustment is made either here or in a chart plane defined on the basis of the reference spheroid in order to obtain conformity between all the observations made.

As the actual and the idealized gravity are functions of force for the actual and the idealized potential surfaces, respectively, then it would be prima facie probable that gravity observations might solve this question, and it will therefore be of interest to examine gravity observations in detail.

Gravity Observations

A mathematical pendulum is defined as a material point suspended by a thread without weight (of length l) which, solely subject to the acceleration of gravity (g) makes oscillations (of amplitude v) on a horizontal axis.

The period of oscillation T of such a mathematical pendulum is given by the formula :

$$T = \pi \sqrt{\frac{l}{g}} \cdot \left(1 + 1/4 \sin^2 \frac{v}{2} + \frac{9}{64} \sin^4 \frac{v}{2}\right).$$

If we let the amplitude converge towards zero the period of oscillation will also converge towards a definite limit, T_0, determined by :—

$$T_0 = \pi \sqrt{\frac{l}{g}}$$

The observation of the period of oscillation of a pendulum must always be made at small finite, not infinitely small amplitudes, but by means of the former equation it is possible to reduce or refer the observations to infinitesimal amplitudes, so that for the calculation of the acceleration of gravity it is only necessary to use the second equation from which we obtain :—

$$g = \pi^2 \frac{l}{T_0{}^2}$$

From this the dimension of the acceleration of gravity may be seen directly. If we take the cm. as a unit of length and the second as a unit of time, the dimension of the acceleration of gravity will be given in cm/sec². This physical unit. cm/sec², is also called the gal (as a tribute to Galilei) with the smaller unit the milligal (1 gal = 1000 mgal.).

Unfortunately, it is impossible to construct such an idealized mathematical pendulum; only physical pendulums can be constructed. The measurement of the absolute gravity at any point is therefore one of the most difficult observations in physics, and it is made by means of several sets of reversible quartz pendulums in thermostats almost exhausted for air. The observation therefore takes a very long time and comprises almost all of the finest measurements known to physics (temperature, time, air pressure, volume, mass, and linear measurements and determination of the co-oscillation of the point of suspension, etc.).

Therefore, as will be mentioned later, only a few absolute gravity measurements have been made all over the Earth, i.e. measurement of the numerical value of the vector of gravity (the length of the normal vector to the actual level through the point in question) expressed in the physical unit gal.

The relative gravity measurements have assumed much greater practical importance. For this purpose physical pendulums of a far simpler construction were used at first and are still being used extensively.

If the value (g_1) of gravity is given at a point in space and the period of oscillation of a pendulum T_1 has been determined at that point, then it will be possible by moving to another point in space and there determine the period of oscillation T_2 of the same pendulum, to calculate the value of the gravity at that point as

$$\frac{T_1^2}{T_2^2} = \frac{g_2}{g_1}, \text{ or } g_2 = g_1 \frac{T_1^2}{T_2^2}$$

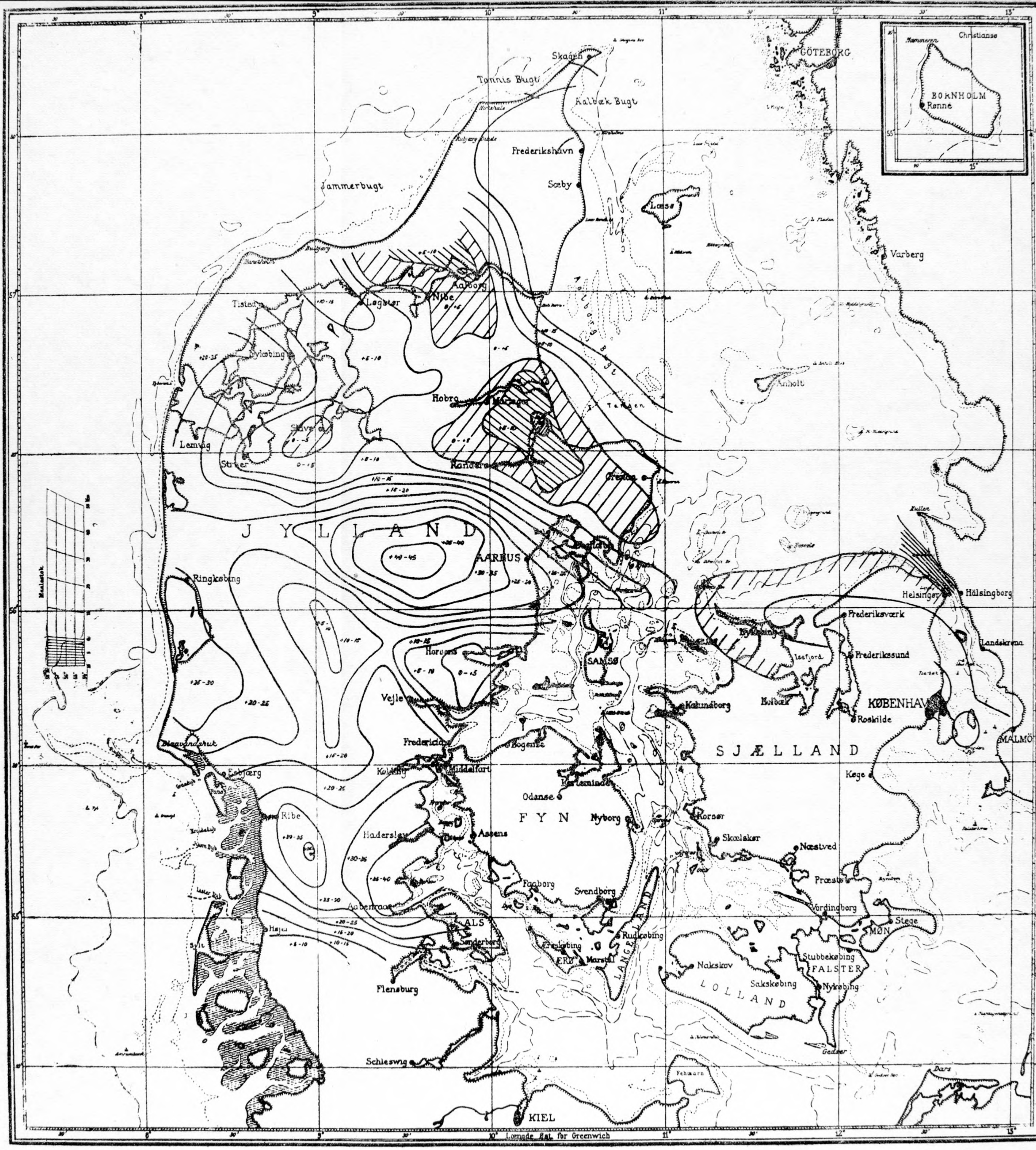
It is thus possible by moving the pendulum to measure the gravity difference from point to point. The observations are, however, still rather protracted and cumbersome, as the period of oscillation must be observed with great accuracy (which requires the application of the coincidence principle, radio time signals, chronograph or photographic recording), and there are several corrections to be made for temperature, air pressure, moisture and co-oscillation.

The relative gravity measurements have become much simpler and speedier since the development of the static-gravimeters on the spring balance or torsion thread principle. I shall not enlarge upon the construction of the different instruments, but only call attention to the fact that such an instrument, by means of which the gravity difference is measured from point to point, frequently over long distances, must be as portable, sensitive and stable as possible, i.e. constructed in such a way as to be as far as possible unaffected by shocks in transportation, electricity, magnetism, changes in temperature, air pressure and humidity, so as to have no (or only very slight) " rate ".

By these instruments the observations of relative gravity have been facilitated to such an extent that a single observation may be made in a few minutes, and the instruments have a very great sensitivity (reading accuracy 2-3 decimal of a mgal).

As mentioned before, only few absolute gravity determinations have been made in the world. The oldest of the absolute gravity determinations still in use is the so-called Potsdam System, in which the absolute gravity was determined by F. Kühnen and Th. Furtwängler in 1898-1904 in the pendulum hall of the geodetic institute at Potsdam (52°22',86 N. - 13°04',06 E., 87 metres above sea level) by means of 5 reversible pendulums of different weight.

The result was g = 981,274 ± 0.003 gal.



Skagen
GÖTEBORG
Christianse
BORNHOLM
Rønne
Tannis Bugt
Aalbæk Bugt
Frederikshavn
Sæby
Jammerbugt
Læsø
Varberg
Aalborg
Nibe
Løgstør
Tisted
Nykøbing
Anholt
Hobro
Mariager
Skive
Lemvig
Struer
Randers
Grenaa
J Y L L A N D
AARHUS
Ringkøbing
Helsingør
Hälsingborg
Frederiksværk
Landskrona
Nykøbing
Isefjord
Frederikssund
Horsens
SAMSØ
Vejle
Kalundborg
KØBENHAVN
Roskilde
MALMÖ
Blaavandshuk
Fredericia
Bogense
S J Æ L L A N D
Esbjerg
Kolding
Middelfart
Køge
Odense
Kerteminde
Ribe
F Y N
Nyborg
Korsør
Haderslev
Assens
Skælskør
Næstved
Præstø
Faaborg
Svendborg
Aabenraa
Vordingborg
ALS
Stege
MØN
Sønderborg
Rudkøbing
Sylt
ÆRØ
Marstal
LANGELAND
Nakskov
Stubbekøbing
FALSTER
Sakskøbing
Nykøbing
Flensburg
L O L L A N D
Gedser
Schleswig
Fehmarn
KIEL
Dars
Længde Øst for Greenwich

Fig. 7

The observational mean error given of ± 3 mgal is an " internal mean error " ascertained in the different observation series with different pendulums.

Denmark still adheres to that system, the gravity on the reference pillar in the basement of the geodetic institute being determined by a transfer from Potsdam at g = 981,558 gal.

A more recent determination was made at Washington (38°56'30",143 N. 77°03'56",893 W., 94.75 metres above sea level) from September 1934 to July 1935 by the National Bureau of Standards on account of the fact that the gravity transferred relatively to that place from various stations in Europe and Canada varied from 980,112 to 980,121 gal, in other words, showed a difference of almost 10 mgal.

The determination was made by means of 4 reversible quartz pendulums at pressure of only 0.1 mm. with 70 individual measurements and gave the following result :—

$$g = 980{,}080 \pm 0.003 \text{ gal.}$$

The most recent absolute gravity determination was made by the National Physical Laboratory at Teddington near London (51°25'14" N. - 0°20'21" W., 10 metres above sea level) from July 1936 to February 1938. The determination was made in 9 series, and for the first time the linear length of the quartz reversion pendulums was determined by means of a lightwave interference comparator with a mean error of ± 0.00025 mm. The result of the determination was :—

$$g = 981{,}1815 \pm 0.0014 \text{ gal,}$$

whereas a relative gravity transfer from Potsdam gave g = 981,193 gal.

If conversely we transfer the gravity from Washington and Teddington to Potsdam, we obtain a value which is about 20 mgal smaller than that observed by Kühnen and Furtwängler in 1898-1904, and this may be due either to the fact that they could not then measure the absolute gravity with the same accuracy as now, or to one-sided errors in the different observations, or to the fact that in the course of the years the absolute gravity at Potsdam has changed, so that the actual potential surface (which is " alive ") through the pendulum hall at the geodetic institute at Potsdam now has a smaller normal vector than then.

All the absolute or relative gravity measurements made cannot be compared immediately; they are made on the physical Earth surface, i.e. at a great number of different, actual levels and in order to make it possible to compare them, they must all be reduced to the same level, and it would then be natural to reduce them to " mean sea level ", which with great approximation coincides with the geoid.

This reduction, the so-called "Bouguer Reduction ", consists of three different parts, and is named after Bouguer, who in 1749 in his publication *La Figure de la Terre* developed its second part.

I am not going to enlarge upon the derivation of the formula and the various corrections and their calculation, but only wish to call attention to the fact that in this case also there is a series of approximations. It is " assumed " that the density of the Earth below the point of observation down to the geoid is identical with that ascertained on the physical Earth surface, and regard has only been paid to the main part of the formula for the gravity at sea level (Clairaut's Theorem): it is " assumed " that the actual gravity lines run along straight lines down to the geoid, and of course, it is impossible to calculate on other assumptions without knowing the density function of the mass of the subsoil.

It is therefore, in geodetical respect, a great advantage for a country to be a low country situated by the sea, so that the physical Earth surface will not be far removed from the geoid.

The reduced, actual gravity (g''_0) will run into very high figures (in Denmark about 981480 - 981750 mgal), but is of no real interest. On the other hand, great interest is attached to the deviation of this value from normal, i.e. the gravity anomaly or the difference between the value of the reduced, actual gravity at sea level (g''_0) and the value (γ_0) which the gravity would have if the geoid was idealized into a spheroid and all irregularities in the mass distribution of the strata disappeared. If the spheroid assumes the shape of the international reference spheroid, the normal (idealized) gravity value will be given by the international formula :—

$\gamma_0 = 978{,}049\ (1 + 0.005\,2884 \sin^2 \varphi - 0.000\,0059 \sin^2 2\varphi)$ gal so that the normal value of gravity may be calculated when the idealized latitude φ is known.

A graphical reproduction of the gravity anomaly for Denmark is shown in the illustration.

When we consider such a graphic reproduction of the gravity anomaly ($g''_0 - \gamma_0$), it would be natural to think that it depicts the shape of the geoid in relation to the reference spheroid; if, for argument's sake, we imagine ourselves moving around on the geoid, we will always have the actual, reduced gravity (g''_0) provided that the reduction made is " correct ", and if we imagine ourselves moving around on the reference spheroid (which is reproduced by means of the chart plane), we will always have the idealized (normal) gravity γ_0 provided that the international gravity formula is " correct ". Thus, the gravity anomaly indicates the difference in gravity (i.e. the distance) between corresponding points on the geoid and reference spheroid, respectively, orientated with a common axis and equatorial plane, and expressed in the physical unit mgal.

Just as we speak of geometrical and astronomical levelling, we may term the gravity observations the gravimetrical levelling.

Along the curve in which the gravity anomaly is zero, the two globes, geoid and spheroid, will intersect and have normal vectors of the same numerical value (linear length), but possibly of different direction, and at the points at which the gravity anomaly is at maximum, the geoid will be distant a relative, maximum length from the reference spheroid, and there the normal vectors of the two planes will have the same direction, but possibly a different numerical value.

From such a chart of gravity conditions it is possible to read the plumb line deflection direct, as its direction must be at right angles to the " gravity level curves " (isoanomaly curves, i.e. curves through points having the same gravity anomaly) and its value in inverse proportion to the distance between the curves.

In the waters between Elsinore and Hälsingborg, where the curves are very closely packed in a NW-SE direction, the plumb line deflection will be in direction SW and attain its maximum value (about 20") within Danish territory.

This is in good conformity with the rather few NS-components of the relative plumb line deflection so far observed in Denmark.

Such a chart not only shows geologists that in the subsoil in Mid-Jutland, W of Aarhus, where the gravity anomaly has a maximum, there is an increase in density of the masses of the Earth, and that in the subsoil at Skive where the gravity anomaly has a minimum, there is a decrease in the density of the masses, but at the same time it shows geodesists that at each of these places there is a point at which the plumb line deflection is zero (i.e. where the plumb line is " vertical ").

At the same time it is, however, obvious that the said graphical reproduction of gravity conditions must not be taken to be gravity charts, but only considered as gravity maps.

They cannot be taken to conform to similar gravity maps prepared by other countries, as their geographical grids do not conform to the Danish grid, and as they may have used another datum for the absolute gravity. The idealized latitudes which have been " assumed " in the calculation of γ_0 were taken from our conical-orthomorphic projection chart plane, which is calculated on a globe (the reference ellipsoid of the Danish degree measurement) entirely different from the international reference spheroid, on the basis of which the international gravity formula is calculated.

The conical-orthomorphic system has become so much of a dogma to the Danish Geodetic Institute that it is considered faultless, so that they one-sidedly let the gravity anomaly be a function of the coordinates of the chart plane of that system, rather than using the gravity observations to show the deviation of the geoid from the reference spheroid or the deflection of the actual plumb line from that of the idealized plumb line, and then correcting all the other observations (terrestrial and astronomical) for one-sided errors.

The relative, terrestrial observations (base line measurements, horizontal direction—or angle measurements and the geometrical levelling) are made in the actual horizontal plane (horizontal direction); gravity measurements (gravimetrical levelling) are made in the actual plumb line (vertical direction), whereas astronomical observations are made in all directions.

A plumb line deflection (a small " slant " in the position of the plumb line) will therefore only slightly influence the relative, terrestrial observations and conversely a small one-sided error of a few seconds of arc in the idealized, geographical coordinates will only slightly influence the gravity anomaly, so that the maps (pictures) produced from the conical orthomorphic system and the deduced gravity maps will on the whole be nearly correct.

If we imagine that Denmark had used the most recent absolute gravity determination at Teddington as a datum instead of the absolute gravity determination at Potsdam, we will see that this will cause a shifting of the gravity map in a vertical direction; the iso-anomaly curve which now has the value of + 11.5 mgal will instead have the value zero. The geoid under Denmark will, on the whole, prove to approximate to the international reference spheroid more closely than appears from the present gravity map. The basic, absolute gravity determination has the character of an orientation constant positioning the reference spheroid (and thus the chart plane) in vertical direction.

If it were possible to consider the geoid to be a plane, the numerical value of the reduced gravity (the length of the normal vector) would everywhere on this plane be ∞. As it is possible by the absolute gravity measurements to ascertain that the gravity (the normal vector) is not infinite, but varies between about 978,049 gal at the equator and about 983,221 gal at the poles, we see that the absolute gravity observations are able to distinguish the geoid from a plane.

As the numerical value of gravity (the linear length of the normal vector) is large, it is difficult to determine it, and the mean error in the determination must be taken to be in excess of the internal mean error of $\pm$ 1.4 — $\pm$ 3 mgal found, which will appear when we transfer the gravity relatively between the three different absolute determinations. A one-sided error of $\pm$ 0.001 mm. in the determination of the pendulum lengths will cause a one-sided error throughout the determination.

The difference between the gravities at the different points may be determined with much greater relative accuracy by means of the relative gravity measuring instruments, as in this instance the values are much smaller; as previously mentioned, the most recent, static, relative gravity measuring instruments (gravimeters) have a sensitivity (reading accuracy) $\sim$ 2-3 decimals of a mgal.

It is even possible to show that the gravity at the same point does not remain a constant, which is only natural, as it is the normal vector of a "living" or "breathing" potential surface. In 1844 Professor C.A.V. Peters showed theoretically that the attraction of the Moon may give a maximum change in gravity at the same point of $\pm$ 0.11 mgal, and the attraction of the Sun a change of maximum $\pm$ 0.05 mgal. Only on the idealized and "dead" computation globe (reference spheroid) the gravity (γ_0) is constant at each individual point.

If the geoid could be considered a sphere, the numerical value of the reduced gravity would be a constant everywhere on this surface. As the gravity in Denmark varies from about 981.500 mgal in the southern part to about 981.750 mgal in the northern part, or a variation of about 250 mgal, and far more than the uncertainty of observation, we see that gravimetrical levelling extended over Danish territory will be plainly capable of distinguishing the geoid from a sphere.

If the geoid were a spheroid, the reduced gravity would be constant in the same parallel of latitude; as, however, a variation of several mgal of the reduced gravity in the same latitude may be measured, i.e. far more than the observational uncertainty over distances similar to an ordinary first order side, we see that the gravimetrical levelling at such distances is plainly able to distinguish between the geoid and the reference spheroid.

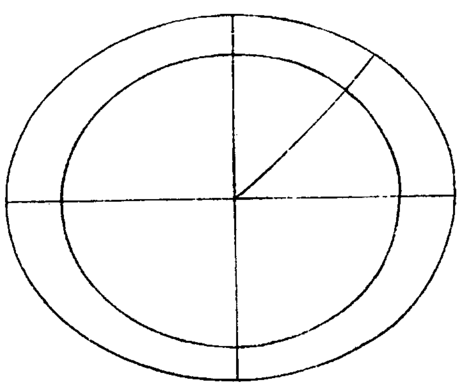


Fig. 8

As the gravimetrical levelling is sensitive enough to distinguish, even within relatively short distances, between the geoid and the reference spheroid, it would be natural to use it in the determination of the absolute plumb line deflection.

For the purpose of making a more detailed investigation of the various actual and idealized potential surfaces, we may first consider the idealized globe and potential surface, the spheroid. The fact that the spheroid is a potential surface will mean *that the work to be expended* in moving a particle from the centre of the spheroid to any point of its surface against the vector field produced by its attraction and its rotation is *constant* and independent of the way the particle is taken (the path of integration). If the particle is taken to one of the poles, only the attraction is to be overcome, and the centrifugal force (created by the rotation of the system) will have no influence. If, on the other hand, the particle is taken to the equator, its movement will receive the maximum aid from the centrifugal force, and the particle may therefore be taken a longer way before the same constant amount of work has been expended, and the radius vector to the equator will therefore be longer than the radius vector to either of the poles.

If, for argument's sake, we calculate how much water it will be necessary to supply to this idealized globe in order to make the level rise 1 m. everywhere on the surface, and we supply this amount of water to the idealized globe, the water will not actually rise 1 m. everywhere, but on account of the centrifugal force it will rise a little less than 1 m. at the poles and a little more than 1 m. at the equator.

This shows that the surface elements of the different idealized potential surfaces on the same gravity line are not parallel; only at the poles and at the equator where the idealized gravity lines are straight lines, all surface elements will be parallel, at all other places they will converge slightly towards the poles.

In the ordinary geometrical levelling it is "assumed" that all the actual level surface elements on the same plumb line are parallel, i.e. that all gravity lines are straight lines, or that the Earth is spherical; only in the precision levelling regard is had—by the so-called "orthometrical correction"—to the fact that this is not the case, and it is "assumed" that the actual level surface elements converge towards the poles to the same extent as the idealized level surface elements.

It further shows that the linear value of the mgal, i.e. the distance it is necessary to move upwards or downwards on the idealized plumb line in order to change the idealized gravity 1 mgal, is not a constant, but a function of the position; it is smaller at the poles than at the equator, as $\gamma_0 (\varphi)$ dz everywhere on the idealized globe must be a constant. Just as the linear length of a minute of are of the meridian *increases* with the idealized latitude φ, so that at the equator it is only about 1842.92 m, whereas at the poles it is about 1861.67 m., so the linear length of a mgal *decreases* with an increasing φ.

Besides being a function of the idealized latitude φ, the linear value of a mgal depends upon whether the movement is outwards in space or inwards inside the globe, as will be seen from Bouguer's correction.

If we move along any idealized gravity line from the infinitely remote (or the potential surface with potential zero) where γ is zero, γ will be continuously increasing till it reaches its maximum γ_0 on the surface of the spheroid and will then change and decrease continuously to zero in the centre of the spheroid.

If we next consider the mathematical Earth : the geoid, on which besides the normal attraction and centrifugal force, also irregularly distributed increases and decreases of the density of the surface layers occur, it will be evident that an increase of the density around a point will attract our particle so that it may be taken over a longer distance before the same amount of normal work has been expended, and thus increase the normal radius vector or cause a rise in the geoid from the normal spheroid. Conversely, a decrease in the density around a point will cause a lowering of the geoid in relation to the normal spheroid.

Coordination of the different observations.

The problem now at hand is to establish conformity between a number of observations differing in principle: astronomical, gravimetrical, terrestrial and tidal observations, for the purpose of producing a chart with the maximum absolute accuracy and the maximum economy as regards observations and calculations.

As mentioned above, of all observations the absolute gravity determination demands the greatest amount of work, next come the astronomical latitude, longitude and azimuth observations, the base line measurements, and, finally, those demanding the least work: the relative gravity measurements (gravimetrical levelling) and the other relative, terrestrial observations (angle or direction measurements, geometrical levelling), and the tidal observations.

For a rational, scientifically working geodetic institute which *does not* have the map as *its goal*, but for which the map projection is *only* a means to —

(1) show graphically in a plane the relative location of points which are naturally distributed in the three-dimensional space, and

(2) simplify the calculations from spheroidic trigonometry to plane trigonometry,

and for which the absolute location of the different points on the common, internationally adopted, idealized computation globe is the essential, a method of survey entirely different from the terrestrial (trigonometrical) method, would be the natural form.

This natural, geodetical method would be, firstly by astronomical observations to determine the astronomical latitude, longitude and azimuth of an observation point (fixed point); then by means of the international observations for the determination of the polar movement to reduce these observations to the Earth equator plane and Earth axis fixed in the Earth (i.e. to physical Earth latitude, longitude and azimuth), and, finally, by means of a combination of relative terrestrial and gravimetrical observations not only at the point of observation, but in a small area around it, to determine the absolute plumb line deflection, so as to be able to reduce latitude, longitude and azimuth to the idealized globe; as it were, to make a gravity detail chart of the vicinity of the point of observaion on a scale of 1 : 1 and thence take the plumb line deflection.

To be true, such a gravity chart cannot be made without a datum at which the absolute gravity is determined and without knowledge of latitude and azimuth (which again will mean longitude). But as a usable first approximation to latitude and azimuth, the values of physical Earth latitude and azimuth, which are "correct" within a few seconds of arc, may be applied, so that on the basis of these approximate values an approximate value of the plumb line deflection may be calculated, whereby a better second approximation of the idealized latitude and azimuth may be obtained, and from these again presumably a sufficiently accurate value of the plumb line deflection.

Through each such point of observation passes one actual potential surface, which may be depicted or established by an ordinary, geometrical or hydrostatic levelling, and one idealized potential surface. In the parallel of latitude (φ constant) on the idealized potential surface through the point of observation the gravity will be constant; if that is not the case in direction E-W (the compass line) on the actual level surface, this must be due to the deviation of the actual potential surface from the idealized potential surface in this direction, i.e. the E-W component η of the plumb line deflection, which may thereby be found. Correspondingly, the N-S component ξ of the plumb line deflection may be found by a similar combination of geometrical or hydrostatic and gravimetric levelling in direction N-S, inasmuch as the variation in this direction of the idealized gravity on the spheroid is given by the international gravity formula.

I shall not go into detail with regard to the way in which these observations may best be made in practice (how large the "vicinity area" should be, how densely the observations should be distributed, and whether observations should be made in several directions or should be confined to NS and EW, etc.) in order to obtain sufficient accuracy; these points can probably only be established by a geodetic institute possessing the necessary instruments and a qualified staff. My only object in this connection is to indicate that it is possible in this way to find the two components ξ and η of the plumb line deflection.

On this principle a pair of curved s-axes (gravity sections) are laid in direction N-S and E-W and the components ξ and η determined by numerical differentiation to be $\frac{d(g-\gamma)}{ds}$ in these directions. With the great sensibility of static gravimeters, I should think it possible to determine the components of the plumb line deflection with the same accuracy as astronomical latitude and longitude (inaccuracy in the second decimal of a second of arc), as for this purpose at most three correct digits only will be required.

If the polar movement is subject to a similar inaccuracy, the location on the idealized globe may be determined, in accordance with the law of accumulation of errors with an uncertainty of about $1 \times \sqrt{3}$ m., which it would be impossible to distinguish in a chart on a scale of 1 : 10 000.

The same principle is well known to every navigator in a field which also deals with attraction and potential, namely in the determination of the deviation of the compass by deflector in misty weather. By deflector the difference of the adjusting power of the compass from the normal is determined, i.e. the difference between the horizontal component of the Earth's magnetism (the normal) and the horizontal component of the ship's magnetism (the abnormal) on various courses, and thence the deviation of the compass is calculated (apart from the constant deviation, which plays the same part as an orientation constant, as does the absolute gravity determination). At the courses on which the adjusting power of the compass has a maximum or minimum the deviation is zero (apart from the constant deviation).

Just as the deflector serves to determine the difference of the adjusting power of the compass from normal on the different courses and thus the deviation of the compass on these courses, so the gravimeter may be used to determine the difference of gravity from normal (the gravity anomaly) in different directions and thence the component of the plumb line deflection in these directions; at the points at which $\frac{d\,(g - \gamma)}{ds}$ is zero in *all* directions (i.e. when the gravity anomaly has a maximum or minimum e.g. at Skive and in Mid-Jutland W of Aarhus) the absolute plumb line deflection will be zero.

As regards the absolute gravity determination (orientation constant, constant deviation) to be used as a datum for the system, I think that in this country we ought to use the British, absolute determination made at Teddington in 1936-38, for the following reasons :—

1° It is the most recent determination ;

2° It has the smallest "internal mean error";

3° It lies between the German and the American determination;

4° It lies so close to Scandinavia that the gravity may relatively easily and with a great number of repetitions (possibly by air) be transferred to this country;

5° It lies closest (10 m.) to the level (mean sea level) to which it is to be reduced, whereas the American determination lies 95 m. and the German 87 m. from it, rather far inland on the continents;

6° It is in the near vicinity of the internationally used reference point for time and longitude (Greenwich Observatory).

If the location on the idealized globe of all the many thousands of coordinated points in Denmark were to be determined in this way, the work would be immense and absolutely impracticable; even if we confined ourselves to the 48 first order points in Denmark, the work would nevertheless be very great and impracticable.

The disadvantage of this method of surveying is that it requires a great amount of work and that if it is desired to adhere to the scientific principle generally used of including only the "certain" ciphers, it is impossible at the present stage of development of science (astronomical, instrumental and observational technique) to obtain the relative accuracy (calculation accuracy) of 1 centimetre desired by the detail survey institutions.

The latter difficulty may, however, easily be surmounted by committing a breach on the above-mentioned principle and include the fourth decimal of a second of arc in the adjustment of latitude, longitude and plumb line deflection observations, even if this fourth decimal has no actual value whatever.

When a calculation accuracy of 1 centimetre ($\sim$ 4 decimals of a second) has been fixed, this accuracy must be consistently adhered to.

As the method of surveying requires much work, it is necessary to restrict it as much as possible in view of observation economy. We may, for instance, consider using it at a geodetic datum, just as the "Round Tower" was used in the conical-orthomorphic system only for orientation of the relative terrestrial observations. Thereby the advantage will be derived of obviating at the actual point of departure the one-sided error formerly introduced by the assumption that the plumb line at that point was "vertical". It might also be possible instead to take one of the points (e.g. at Skive or W. of Aarhus) at which $\frac{d\,(g - \gamma)}{ds} = 0$ in all directions, i.e. where the gravity anomaly is at an extreme and the plumb line deflection consequently zero, as a geodetic datum; such a point will, however, presumably not be a first order point, as these points are chosen on the basis of quite different considerations (high location, long sighting lines, well-shaped triangles), and the work of observation would thus be increased by the work of combining this point of departure with the first order network of triangles.

Neither of these methods will, however, utilize the great advantage of the natural, geodetic method of surveying, namely : that no accumulation of errors takes place, the observations at two different points being entirely independent of one another.

In contradistinction to the relative terrestrial method in which the accumulation of errors in a chain of triangles will increase with the extension of the chain, such an accumulation of errors will not exist in the determination of the absolute location on the idealized globe of the two termini of the chain of triangles. The method is therefore advantageous "on a large scale", but highly disadvantageous "on a small scale", where the relative terrestrial methods have their advantage.

In my opinion, the method may therefore be successfully used if it is applied to the termini of such long chains of triangles as it will be practical to adjust together (about 10-15 triangles) and at one point on each of the smaller islands which are so far removed that they cannot be trigonometrically connected with the rest of the system (e.g. Bornholm and Anholt). We may call the points thus determined O-order points.

When the chart planes and their rectangular, plane coordinate systems are defined from the international reference spheroid, it will be possible to depict the geographical grid in the chart plane on the scale 1 : 1 with any desired degree of accuracy. This is a sheer mathematical problem, which in actual fact has nothing to do with surveying.

As regards the two projections mostly used nowadays, the orthomorphic transverse cylinder projection (also called the meridian strip or transverse Mercator projection) and the normal, orthomorphic cylinder projection (Mercator's projection), the geographical grid may be easily and simply plotted by means of the excellent tables of the international reference spheroid with an accuracy of 1 mm. and 1 cm. respectively.

The task now facing the surveyor in order to make a chart will be to locate the more or less perfect human work, the picture of the country, to the best advantage in this geographical grid.

The O-order points may also easily and simply be transferred to the chart plane with the accuracy with which they are defined, e.g. 1 mm. or 1 cm., or a far greater accuracy than that with which they may at present be determined. The O-order points of one transverse Mercator projection may likewise be transferred with the same accuracy to another transverse Mercator projection defined from another meridian. The O-order grid formed by these O-order points should, in my opinion, form the basis of the picture, and the network of triangles of first order must be made to conform to them by grid adjustment, the natural, geodetical method of surveying being more advantageous "on the large scale" than the relative, trigonometrical method which is subject to accumulation of errors. Exactly in the same way as a navigator after a protracted voyage will prefer an observed position to a position by dead reckoning.

In the course of the voyage the navigation has been subject to one-sided errors (errors in the log reading, errors in the observed deviation, errors in the observed drift and in the estimated set and rate of the current, etc.), just as the chain of triangles in its progress has been subject to one-sided errors (errors in the observed length of the standard metre prototype with consequent errors in the observed length of the base line, errors due to the latter not being reduced to the reference spheroid, and errors in the observed directions on account of the plumb line deflection, in short, errors due to the difference between the geoid and the reference spheroid).

Just as the natural method when a thermometer tube and a thermometer scale are to be fixed in relation to each other would be to start from some physically defined datum point (the freezing and boiling points of distilled water at a certain atmospheric pressure), thus the natural method when the picture and the geographical grid are to be located in relation to each other will be to start from some geodetically defined datum points (O-order points), around which, and within an area so small that the relative terrestrial method will have no appreciable accumulation of errors, conformity has been established between astronomical, gravimetrical and terrestrial observations.

As the gravity anomaly $g''_0 - \gamma_0$ at a point indicates the distance at that point between the geoid and the reference spheroid, it will be possible by means of gravity observations and levelling to reduce the base line measurements to the reference spheroid before the computations (grid adjustments) are made.

As $\frac{d\,(g - \gamma)}{ds}$ at a point indicates the component of the plumb line deflection at that point in the direction of the s-axis, we are also capable of reducing the astronomical and the relative terrestrial observations to the reference spheroid before the adjustment is made, provided that we have a sufficient number of gravity observations and levels in the vicinity or a gravity chart.

For the correction of the astronomical observations the components ξ and η must be known with a great accuracy (the same accuracy as the actual astronomical observations) as they enter at about their full value (ξ, $\eta \sec \varphi$, $\eta \tan \varphi$), i.e. that the gravity chart of the small area around the O-order point must be made out on a very large scale or be given only in figures, as an actual graphical reproduction will be impossible on a full scale.

For the reduction of the relative, terrestrial direction observations in the network of triangles the components ξ and η of the plumb line deflection need not be known with any great degree of accuracy, they may be taken from a gravity chart on a small scale, as on account of the factor tan h (in which h is a small angle: the angle of depression) they enter with only a small fraction of their value, and the correction will be smaller than the observational mean error.

In the computations (the grid adjustment) the base line measurements are not to be considered faultless, but may be subject to error. By considering the corrections to be applied to the observed and reduced base line measurements in order to obtain conformity throughout the system, it is possible to obtain an impression of the degree of conformity, and gradually as results become obtainable from many countries distributed all over the Earth it may be seen whether they are of a casual or a systematic character, and we may thereby obtain data for a determination of a better shape of the spheroid and thus a better formula for the idealized gravity γ_0.

In the trigonometrical method (by considering the relative observations as subject only to casual errors) a continuous *accumulation of errors* is made in the building up of the system (the map) over and beyond the unavoidable observational errors in the determination of the geographical coordinates of the initial point (geodetic datum). The accumulation of errors will increase towards the extremities of the system.

In the natural, geodetical method a *distribution* is made *within the system* of the unavoidable observational errors in the determination of the geographical coordinates of the O-order points in the extremities of the system between the hundreds of relative observations which previously, to the greatest extent possible have been freed from the causes of one-sided errors.

The relative base-line measurements are in actual fact superfluous determinations, which serve only to check the efficiency of all the observations and the conformity between our two definitions of the metre and of the idealized globe.

As the Danish charts are to constitute a bridge between the German, Swedish and Norwegian charts in our production of a chart of the entire Earth, a parable from the science of engineering may be used.

When a large bridge is to be built the problem is not the simple one of making a cantilever girder of sufficient length (chain of triangles); the girder must perhaps be so long that it cannot support itself and will assume different shapes according to the points at which it is supported. Instead it is necessary first to make a careful survey, investigation of soil and sub-soil, and to prepare a project, corresponding to the now completed degree measurement epoch with its results : picture of terrain (our conical orthomorphic system), gravity conditions and our knowledge of the shape "on the whole" of the Earth.

Next, bridge piers (O-order points) will have to be built on the best possible foundation, and not until then the various spans (chains of triangles) fitting between the piers may be manufactured. The bridge piers may possibly be located only with a margin of some cm. even if the individual girders of the spans have a tolerance in fractions of a mm.

If more work is expended on building many bridge piers, the work to be expended on the consequent shorter spans will be less. Likewise, the computations (grid adjustment) of the chains of triangles will be simpler and more comprehensible the more O-order points we have determined, with the consequent shorter trains of triangles.

For purposes of working economy the most advantageous way would, however, probably be that of limiting the number of O-order points (bridge piers) as much as possible, but other considerations may, of course, intervene : e.g. the sub-division of the first order net into suitable chains of triangles which may make it desirable to include a central point at which several chains of triangles meet; further, it may be expedient for two countries at points at which they may be connected trigonometrically to use common O-order points (bridge piers) and first order triangle chains (spans), so that for instance Denmark and Sweden by a practical, scientific Scandinavism, jointly determined the O-order points of Kullen and St. Møllehøj

(Stevns) or Kongsbjerg (Møen) and the first order triangle chain between them; thereby complete conformity would be obtained between the charts of the two countries, which would be of great practical importance to hydrographic surveys and shipping.

On an estimate it will be necessary to use as many O-order points as we have hitherto used base line measurements, thus, in the Danish grid 5. Topographical conditions in the vicinity of the station may, of course, be of some importance in the choice of the points.

If a quantity is to be measured, this is done by comparing it with another quantity of known magnitude.

This can, of course, best be done if the two quantities have approximately equal magnitudes. If we are to measure the length of a large room, it will not be expedient to use a small foot-rule, even if it is very carefully calibrated and accurate; it is more advantageous to use a tape or rule of approximately the same length as the room and then determine only the small difference from the end wall of the room to the nearest calibration on the long rule by means of the small, finely calibrated rule.

On this ideal method the Danish standard metre prototype has been compared with the world prototype at Sèvres. This very comparison is subject to an inaccuracy in the first or second decimal of μ (μ = 1 micron = 0.001 mm.). In the comparison of the Danish prototype with the standard metre of the Danish geodetic institute, the uncertainty will at least be of the same magnitude. But when this standard metre by less ideal methods is being compared with 24 m. long base measuring tapes, the uncertainty will be correspondingly greater, and when these tapes are again compared with a base line several kilometres long, the uncertainty becomes appreciable, and the internal mean error of the observation need not give a true picture of the uncertainty to which the observation is subject and will do so only if all the preceding comparisons (observations) are not subject to one-sided errors. When thereupon this base line through angle or direction measurements (which may all be subject to one-sided errors) in the base grid is enlarged to a base side and this base side through a long chain of triangles (in which the observations may likewise be subject to one-sided error) is enlarged so as to cover a distance as, for instance, from the Danish-German frontier to the Scaw or from Esbjerg to Copenhagen (about 300 kilometres), it will be evident that centimetres will be of no actual value whatever on these distances. The distance (in centimetres) contains 8 digits, whereas perhaps only 7 digits of the standard metre of the Geodetic Institute are correct; the linear uncertainty must be presumed to be at least in tens of metres.

But what is understood by the term of a *known* magnitude ?

The magnitude of a quantity may be known through observation, but it must then be presumed to be known only subject to a certain uncertainty, as, for instance, the standard metre of the Geodetic Institute. Actually known are only such magnitudes as are fixed by definition as, for instance, the metre which is defined by our world prototype at Sèvres or our idealized globe (the international reference spheroid), which is defined by the international gravity formula or by $a = 6.378.388$ m and $\alpha = 1:297.0$; by this latter definition the geographical grid on the reference spheroid or in charts defined from the reference spheroid has been established.

Originally, the metre was defined as a one-tenth millionth of the Earth quadrant. As this definition proved unusable, a new definition was established. Now we have by definition fixed the idealized Earth, but then, in my opinion, we should take the full consequences of this definition and embark on new methods and working hypotheses and measure (compare) the geoid (the mathematical Earth surface) by means of the international reference spheroid which is of approximately the same size. These methods have at the same time been made possible by the development of the observational technique for absolute gravity measurements and by the development of the static gravimeters.

Perhaps, when sufficient data have become available for a judgment, a systematic anomaly will appear between our definition of the metre and our definition of the idealized globe. A new and better definition may then be given of the idealized globe (the shape of the Earth "on the whole"), and we will thus have a chance of getting a step nearer to the truth. "Science advances rather by providing a succession of approximations to the truth, each more accurate than the last, but each capable of endless degrees of higher accuracy" (Sir James Jeans : *The Universe around us*). The development of science consists in a constant struggle to eliminate one-sided errors by converting them into systematic errors the effects of which may be taken into account and to obtain better and better approximations. When a better approximation is obtained, it is taken into use in a new working hypothesis, and gradually the discrepancies remaining in the system will emerge and the causes of the one-sided errors will be found. The new one-sided errors will again by a new working hypothesis be converted into systematic errors, determined by special observations, they are corrected for, and the possibility of determining a new and better approximation arises.

The object of geodesy is to determine the mathematical Earth surface (the geoid) and this may be done by determining its normal vector at a sufficient number of points, it being known from potential theory that it has a continuous shape and is differentiable at all points.

By using the gravity observations we find g_0", the numerical value of the actual gravity acceleration vector at mean sea level given in the physical unit mgal. By comparing it with the numerical value γ_0 of the corresponding gravity acceleration vector of the reference spheroid, we will be able to ascertain the shape of the geoid in relation to the reference spheroid at the individual points and thus to reduce all the astronomical and terrestrial observations to the idealized, fixed plumb lines of the reference spheroid (i.e. correct them for systematic errors) before making the adjustment either on the reference spheroid or, what is considerably easier, in a chart plane defined from the reference spheroid. It is no longer our little standard metre which gives the dimensions, but the normal vector in the defined international reference spheroid itself.

In this method we *do not* work on the hypothesis that the geoid is considered to coincide with the reference spheroid, but we first reduce our observations to the computation globe and thence to the chart plane (planes) before we commence our adjustment.

On the other hand, we use as a new working hypothesis that mean sea level and the geoid are considered to coincide and like the actual gravity and the actual plumb line are considered fixed and unchangeable. The sea level is common to all open coasts on the Earth and its mean level may be determined by observations.

The computations (grid adjustment) will be relatively simple, comprehensible and easy, far simpler than the so-called Laplace adjustment used in Germany and Finland, inasmuch as the absolute gravity measurements and the astronomical latitude, longitude and azimuth observations after station adjustment and correction for plumb line deflection are considered to be without error. This is of course in full conformity with the adjustment principle normally used, in which two points are connected ("double"—determined) by two different methods of surveying :—

If the relative positions of the two points are "double"—determined by both first order triangulation and detail triangulation or by detail triangulation and traverse measurement, the method having the least accumulation of errors will dominate.

Just as a land surveyor will avoid blind traverses as far as possible, i.e. traverses the termini of which are not both determined by triangulation and would only think of using such traverses with at most one or two lines, thus a geodesist should, in my opinion, avoid blind triangle chains, i.e. a triangle chain, the termini of which are not both O-order points, and only think of using such chains on one or two triangles, e.g. on small isolated islands like Bornholm and Anholt.

The greatest possible care should be exercised in the determination of the O-order points on which the whole chart, the picture in relation to the geographical grid, is based. The azimuth should in this case be determined from as many observations as the horizontal direction observations at the first order points. Likewise the greatest possible care should be used in the transfer of the absolute gravity from the datum at Teddington in England to the O-order points in Denmark. This transfer should be made with many repetitions, as quickly as possible, and with Sun and Moon in as varying positions as possible, in order that a really valid mean value may be obtained, even if this will increase the "internal mean error", the actual gravity varying slightly at all times.

It will probably never be possible to obtain an absolute accuracy in terms of centimetres (4 "correct" decimals of a second of arc in the geographical coordinates), as this would require an immense development within all fields of science, more especially in the field of astronomy (better star places and polar movement coordinates) and in instrument technique (more sensitive spirit levels). At the present time our finest verniers and spirit levels can —with many repetitions—give us only an accuracy within the second decimal of a second of arc, so for the time being the absolute accuracy cannot possibly be greater than that. The second decimal of a second of arc alone requires the use of 8-place trigonometrical tables, whereas the present coordinates in Denmark are calculated only with 6-place trigonometrical tables corresponding to an accuracy within a second of arc.

The actual users of charts (science and shipping) do not therefore expect to have the absolute geographical coordinates given correctly with 4 decimals of a second of arc ($\sim$cm. accuracy) but are satisfied if the accuracy is great enough for errors to be indistinguishable on charts on the scales generally used (i.e. in charts on the scale of 1 : 10000 an uncertainty of a couple of metres $\sim$ uncertainty in the second decimal of second of arc).

Whether it will be possible to attain this accuracy cannot be said. The true value of an absolute quantity determined by observation and the true error on a series of observations of that quantity are unknown, only the most probable system of errors with the consequent internal mean error may be determined.

With the primitive methods used for taking astronomical observations on board, a ship will normally be unable to ascertain errors in the chart (the picture in relation to the geographical grid) as the sextant may be read only with an accuracy of 0.5 minutes of arc, as the same degree of accuracy is being used in the calculations, and as refraction and dip anomalies may occur together with the plumb line deflections of the same magnitude besides the polar movement. As soon as a ship is in sight of land, the navigator will therefore abandon the astronomical observation methods and will trust to terrestrial observations, i.e. assume that the chart is correct.

" The navigator may only be able to determine his position to the nearest mile or so, but he is entitled to expect that the geographical positions of points on the chart, as scaled off from the graduation, are free from any appreciable error " (*Admiralty Manual of Hydrographic Surveying,* London, 1938).

This requirement is not fulfilled as far as our charts are concerned, as the basis (the old reference ellipsoid) is obsolete and one-sided stress has been placed on the relative accuracy and, apart from the astronomical observations at the point of departure, only relative, terrestrial observations which are subject to an accumulation of one-sided errors have been used.

It will in no case be possible to determine the absolute position on the idealized globe absolutely orientated with a higher degree of accuracy than that with which the absolute plumb line deflection may be determined. An (unknown) N-S component of a plumb line deflection of 6-7" will thus give rise to an error in the idealized geographical latitude of 6-7", or about 200 metres, i.e. it will only become imperceptible in charts on a scale of less than 1 : 1 000 000. As shown there is a great difference between relative accuracy and absolute accuracy.

Somewhat similar considerations apply to levels.

Some users of levels require to know the height above mean sea level and the height which at various times may be anticipated between the immediate sea level and mean sea level. These users are mostly scientists (who have to reduce different observations to sea level), civil engineers (working out projects for draining and reclamation work, dykes, etc.) and shipping (moving on the surface of the sea). These absolute levels can never be determined with a greater degree of accuracy than that with which the mean level of the sea may be determined (inaccuracy in centimetres), and the farther we pass from the sea, where mean sea level may be observed, inland, the greater the inaccuracy with which the absolute levels are determined. Other users (inspectors of roads and surveyors, architects, housebuilders, etc.), require only level differences with great relative accuracy (mm) in order to ascertain the gradient of the terrain or in order to undertake a local levelling and thus work on the basis created by the geodetic institute, whereas the actual level of the place above mean sea level is of no interest to them.

It would be a misunderstanding to think that these last-mentioned users require greater accuracy than the others. It is only a question of another kind of accuracy, a relative accuracy instead of an absolute accuracy.

If we look more closely at the Danish list of levels, the curious fact will emerge that some fixed points used in the levelling have a negative level, even if they are actually on dry land without protection of dykes or the like. Just as the Geodetic Institute has one-sidedly let the absolute geographical coordinates (the geographical grid) be functions of the relative coordinates (the picture or map), thus they have also one-sidedly let the absolute levels (the level above mean sea level) be functions of the geometrical and hydrostatic levelling based on the Danish standard zero point in the Cathedral of Aarhus, rather than letting them be functions of the different tidal observations.

The relative geometrical levelling is presumably also subject to a (slight) accumulation of one-sided errors.

In my opinion, the mean sea level indicated by the tide gauges should be used as O-order fixed points in the levelling instead of the Danish standard zero point.

As mentioned, considerable discrepancies between the charts of the different countries may be ascertained. Last summer the R.A.F. in coorperation with the Geodetic Institute in Denmark and the Norwegian Geographical Survey made trigonometrical observations of parachute flares dropped over the Skagerrack in order to ascertain the discrepancies between the Danish and the Norwegian charts. The discrepancies amounted to 150-180 metres in direction N-S and E-W. I have been told that the discrepancies between the Spanish, Norwegian and Finnish charts at the northern frontier are considerably greater.

In the Hydrographic Office in England I was shown that the discrepancies between Greek, Italian and Turkish charts at Cyprus amounted to about 5-6000 metres.

Even in making hydrographic surveys of the Sound it is necessary to use geodetic control either on one side or on the other, as the Danish and Swedish geographical coordinates do not agree.

Before the invasion of Normandy it was necessary in England to convert the French chain of triangles from Calais-Dover (where contact may be established) to Normandy from the French reference ellipsoid to that used in England (Airy's) in order to be able to utilize the accuracy which the Gee and Decca stations established in England might give.

From the point of view of geodesy each country is at present actually lying on its own globe. The project prepared by the Decca-Navigation Co. for a Scandinavian chain cannot be carried out at present, as the relative positions of Hirtshals in Denmark and Gothenburg in Sweden are not known with sufficient accuracy.

Therefore, in my opinion the adoption of the international reference spheroid is of immense and epoch-making importance. The object of geodesy is now quite different from that of former days, when the idealized Earth, which is now fixed by definition, was not known. Plumb line deflections have become absolute quantities which may and should be measured. The object is now to compare the faintly "alive" geoid with the international spheroid and to transfer the mean positions of the points to this "dead" computation globe before the relative observations are adjusted.

The calculation accuracy is something fixed or adopted. In Germany they used calculation accuracy in terms of millimetres ($\sim$ 5th decimal of a second of arc in the geographical coordinates); although it cannot be claimed that the German charts had greater absolute accuracy than the Danish ones.

If the calculation accuracy is to assume any actual importance, endeavour should be made before the adjustment to correct for all the systematic sources of error that may apply to the calculation accuracy, irrespective of whether the corrections are less than the normal mean error of the category of observation used; for instance for the plumb line deflection in the first order network of triangles as 0"06 at a distance of 70 kilometres will mean 2 centimetres for the level of the distant point, and in astronomical latitude observations for the level of the point of observation, as 0"01 in latitude will mean about 30 centimetres.

The absolute accuracy is something that only the geodetic institutes may strive to impart to the system during its elaboration by endeavouring to find the cohesion between the phenomena, by endeavouring to seek the truth by counteracting accumlation errors, correcting for as many sources of error as possible, and by using sufficiently efficient working hypotheses, and the absolute accuracy can never be greater than that with which the Geodetic Institute can determine the position of a single point on the international, scientifically used reference spheroid orientated with its axis and equatorial plane coinciding with those of the Earth.

A chart is more than a map; it is a map orientated in a geographical grid.

By the relative, terrestrial methods of surveying (land surveys) it is possible only to create maps.

By a combination of terrestrial and astronomical observations it may be possible to create a chart, but not with greater absolute accuracy than that of the plumb line deflection at the point of departure.

Only by a combination of terrestrial, astronomical and gravimetrical observations will it—in my opinion—be possible to produce a chart which will presumably satisfy science and shipping as a scientific instrument and at the same time contribute towards the solution of the geodetical problem: the determination of the shape and size of the geoid and the spheroid.

"The various sciences can no longer be treated as distinct; scientific discovery advances along a continuous front, which extends unbroken from electrons of a fraction of a millionth of a millionth of an inch in diameter to nebulae whose diameters are measured in hundreds of thousands of millions of millions of miles. A gain of astronomical knowledge may add to our knowledge of physics and chemistry and vice versa". (Sir James Jeans : *The Universe around us.*)

Astronomy, physics and instrument technique have advanced greatly since the present basis of our charts was established a hundred years ago. Geodesy has fallen behind the other sciences and stands (at any rate as far as our charts are concerned) where it stood then. Also the other sciences require the scientific instrument constituted by a modern chart.

One of the disadvantages of the Decca system is that the rate of propagation of the radio waves used is not as yet known with sufficient accuracy. At present the figure of 2.9925×10^5 km/sec. is used, and this mean figure has been obtained by letting aircraft fly around Decca bases with a Decca receiver for the purpose of ascertaining how many hyperbolae they contained. Actually the base line is measured with half the comparison wave-length in question as a unit, and the rate of propagation is calculated there-from. It is the task of geodesy to provide the measure for these long base lines, which should preferably lie across water, and it is not sufficient to have the base length given with a relative accuracy in terms of centimetres when perhaps they are uncertain by tens or hundreds of metres.

The base lines, the foundation of hydrographic surveys, are always on land. The farther we get away from land, the less the accuracy will be, but at the same time the navigator's demands for accuracy will be less. Charts on a scale of 1 : 10000 with a reading accuracy of a couple of metres is not required in mid-Atlantic; they are needed only in difficult fairways.

The new methods of obtaining radio fixes afford hydrographic surveying a chance of getting much farther away from land with much less losses of accuracy than did the old methods. But at the same time they make much greater demands as regards absolute accuracy and the geodetical foundation, now that it is possible to reach across the sea from shore to shore.

In my opinion the Scandinavian Hydrographic Offices should therefore apply to the respective geodetic institutes and suggest a mutual cooperation, so that at least the Scandinavian countries can all use the same globe, and so that our charts can obtain both the requisite relative accuracy and the maximum absolute accuracy attainable at the present time.