DEFLECTIONS OF THE VERTICAL FROM GRAVITY ANOMALIES

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With the increasing number of surveys based on astronomical coordinates and with loran stations established on oceanic islands far from the mainland and its geodetic control, there has been an increasing interest in deflections of the vertical.

Some of those who have a professional interest in deflections of the vertical but who have not analyzed the underlying conceptions seem to have the vague idea that there is something in nature that might properly be called *the* deflection of the vertical. In the present state of geodesy there is nothing that may properly be called *the* deflection of the vertical at a given point. All existing numerical values of the deflection depend on the assumed geodetic datum, and geodetic datum might conceivably be anything whatever, though in practice the choice is confined within rather narrow limits.

The usual geodetic datum is essentially arbitrary. Two geodesists with the same general background and the same mass of geodetic material before them might hit upon approximately the same choice of a datum and then again they might not. And since the choice of a datum is arbitrary, the deflections, dependent as they are on the datum, are arbitrary also.

We diminish the arbitrariness of choice and, what is more important, we get an intrinsically better geodetic datum, if we require that the center of the ellipsoid of reference shall coincide with the center of gravity of the Earth and that the axis of the ellipsoid shall coincide with the axis of rotation of the Earth.

As a matter of fact, our United States Datum, the predecessor of the North American Datum, was not adopted as the result of any special study. At first it just grew. Geodetic positions based on a supposedly temporary datum used in the eastern part of the country were found to agree satisfactorily with the astronomical positions further west, so the geodetic positions based on the eastern datum were accepted and the United States Datum was redefined in terms of a geodetic station in Kansas. Indeed, two of the items mentioned above were not explicitly considered when the decision was made; the position of the center of the ellipsoid of reference, and the direction of its axis. These are undetermined to this day. It is precisely these two items that will be emphasized in this paper.

The word "ellipsoid" is used throughout this note in the sense of an ellipsoid of revolution. The more general term "spheroid" is avoided. It is not difficult to adapt the formula for theoretical gravity to a predetermined spheroid that is not an exact ellipsoid of revolution, but the computation of triangulation on anything except an ellipsoid of revolution requires intricate numerical calculations; whereas this note implies throughout a comparison between astronomico-geodetic deflections and gravimetric deflections. Hence, for convenience, we are obliged to stick to the ellipsoid and base our gravity anomalies on an appropriate formula for theoretical gravity.

About 100 years ago Stokes did some pioneer theoretical work that may eventually enable us to refer our geodetic measurements to an ellipsoid thus ideally situated; its center at the center of gravity of the Earth and its axis coincident with the Earth's axis of rotation. If we insist that our geodetic datum shall be based on an ideally-situated ellipsoid of this sort, the only elements of our datum left undetermined are the size and shape of the ellipsoid. If we specify these. then everything is determinate and we may properly speak of *the* deflection of the vertical at a given point. The problem then becomes one of computation.

Stokes' mathematical developments in their original form do not give the deflections directly. The formula that he gave enables us to express the elevation of the geoid at a given point above the ellipsoid of reference in terms of the gravity anomalies. A gravity anomaly and a deflection may seem rather different in kind, but the gravity anomaly is an anomaly in the vertical component of the gravitational attraction; a deflection is an anomaly in the horizontal component. The nature of the force of which both are manifestations and the ingenuity of the mathematician make the connection between them. The ellipsoid is the particular ellipsoid already referred to, properly centered and properly oriented. The formula for the elevation of the geoid above this ellipsoid looks complicated, not to say perverse. No term of it has any obvious physical meaning. Stokes would probably have said that he was summing a series of spherical harmonics by an ingenious device. The more modern mathematician might say that he was solving an integral equation (Idelson, 1933).

In practical application we need a contour map of gravity anomalies. From it we can deduce the elevation of the geoid at various points. From enough such elevations we could draw contour lines of the geoid elevations referred to our chosen ellipsoid. These would be height contours, exactly similar to ordinary topographic contours. If we have enough such contours, we can estimate the slope from the distance between neighboring contour lines. The slope of the geoid is simply another term for the deflection of the vertical in a direction perpendicular to the contours. Stokes himself suggested this.

An analytic process at once more elegant and more accurate was developed by Vening Meinesz (1928) and De Graaff Hunter (1935). We need, as before, a contour map of gravity anomalies. The formula looks complicated, as before, but this time the leading term, the important term when we are dealing with areas near the deflection station, is of the same form as the expression used in computing topographic deflections or in the isostatic reduction of deflections of the vertical. There is a simple reason for this, which need not be discussed here. The regions near the deflection station are highly important in computing deflections, the remoter regions much less so. In computing geoid elevations the remoter regions are relatively more important. This may help to explain why we have more computations of the deflection than of geoid elevations. Because of the lack of well-distributed gravity data we have only a few computations of either kind.

It is interesting to consider how we might use gravity data to improve current processes for the computation of ordinary geodetic triangulation. If we knew the elevations of the geoid above the ellipsoid of reference, we could reduce the lengths of our measured bases to the level of that ellipsoid. At present, for lack of adequate information, they are reduced to the level of the geoid, which, for the various bases, has various positions with respect to the ellipsoid. Again, if we knew the deflections of the vertical, we could apply the proper corrections to our observed horizontal directions. These corrections may be quite appreciable, especially for inclined lines of sight at stations where the deflection is large, but for the present these corrections are almost universally ignored because the necessary data are lacking. A possible use of the deflections of the vertical determined from gravity anomalies to improve the adopted semi-axes of the spheroid of reference will be mentioned later in connection with the work of Kasansky (1935).

Let us briefly review recent work on geoid elevations and deflections of the vertical based on actual gravity anomalies.

For the geoid elevation we have the pioneer work of Hirvonen (1934); it necessarily involved much guessing at gravity anomalies in unexplored areas, but the results are interesting. Hirvonen's work predicted a rather surprising rise of the geoid in the Malay Peninsula, a rise later confirmed by observation. Hirvonen's work would seem to show that for India the proper ideal ellipsoid of reference should lie some 60 metres (Gulatee, 1940, p. 116) below the geoid at the origin of the datum in order that the center of the ellipsoid may coincide with the center of the Earth. The not-so-ideal ellipsoids commonly used by geodesists keep fairly close to the geoid in the particular area under consideration, only to depart all the more widely from the geoid in remoter areas.

Work on deflections of the vertical from gravity anomalies has been done by Vening Meinesz in the Netherlands, by the « Survey of India » (Gulatee, 1940, pp. 97-101), and by Kasansky, in Russia. Of the work of Vening Meinesz we have few details. He makes the general statement that deflections were computed from gravity anomalies for three stations in the Netherlands and that the results agreed satisfactorily with those obtained from astronomico-geodetic work. The « Survey of India » also finds good agreement in a few test cases.

The work of Kasansky was more extensive than either of the foregoing and deserves careful study, not only for the results obtained but also for the methods employed. The region studied is near Moscow and covers an area of about three degrees in latitude and four degrees in longitude. This area contained a number of gravity stations so that the gravity contours could be drawn with fair accuracy; it also contained for comparison a considerable number of astronomical stations. Kasansky's astronomico-geodetic deflections and his gravimetric deflections are on two different bases, the first on an arbitrary system, the second on an ideal system, but the *relative* deflections may be compared and errors estimated. Let δa and δg denote, respectively, the astronomico-geodetic deflection and the gravimetric deflection (note that the subscript g is for gravimetric, not geodetic), either in the meridian or the prime vertical, for the average results to be quoted include both, and form the difference $\delta a - \delta g$. The mean error of this difference was found by Kasansky to be about 1".0 (Kasansky used *mean* errors throughout; to find " probable " errors apply the factor 2/3). After allowing for the error of δa, then the mean error of δg, the gravimetric deflection by itself, comes out to be 0".8. This arises from the errors of the observational data and of the contour lines based on them and on the inadequacy or absence of data for regions distant from Moscow.

Kasansky tests several groups of astronomico-geodetic stations. In one group the deflections were based on an origin at Pulkowa Observatory, about 400 miles northwest of Moscow; this was the origin of Struve's survey, one of the early Russian geodetic surveys. Bessel's ellipsoid was used; this is known to have its radial dimensions some 700 metres smaller than the ellipsoid best adapted to Russia. This error in radius was manifest in the astronomico-geodetic deflections δa, which changed systematically from northwest to southeast, even within the comparatively small area studied. The gravimetric defflections, on the other hand, are practically unaffected by the error in the dimensions of the ellipsoid. The error of the conventional astronomico-geodetic deflection in latitude for the center of the Moscow region is thus built up to about 2".5.

The method of gravimetric deflections can be applied anywhere in the world, provided the gravity observations are available. We could determine from gravity observations a proper datum for South America and it would be on the same basis as a datum similarly determined for North America, even before those two continents are connected by triangulation. Gravimetric data could also be used to great advantage in the proposed readjustment of the triangulation of Europe. As mentioned in a preceding paragraph, the bases could be reduced to the ellipsoid, the observed directions could be corrected for deflection and perhaps the dimensions of the adopted ellipsoid could be improved.

The Section of Gravity and Astronomy of the United States Coast and Geodetic Survey has undertaken the determination of the elevations of the geoid and the deflections of the vertical in Missouri. For Missouri itself the work will be based on a gravimetric map of the state published by the State Geological Survey. The gravity determinations were made with gravimeters and tied in with pendulum determinations of the Coast and Geodetic Survey.

The map gives contours for the Bouguer anomaly. It was necessary to redraw the map entirely in order to have contours for the free-air anomaly. It has been assumed that, in spite of the mildly rugged topography in the Ozark area, the results from the uncorrected free-air anomalies will be reasonably satisfactory. The question of the best method of reducing to sea level, whether by an isostatic reduction, by condensation, by inversion, or by no reduction at all except the free-air reduction for elevation, has not yet been settled to the entire satisfaction of all geodesists and is too large a question to be discussed here.

This study was undertaken partly for the information that it is expected to yield, partly for the training it will afford. For instance, it will be interesting to see what we find for the position of the ellipsoid with respect to the geoid in the area of our experimental work, namely Missouri. Hirvonen's results for the United States are rather curious, differing rather markedly from accepted ideas. His results are, of course, subject to substantial modification when additional gravity data are taken into account. At the end of the job we of the Section of Gravity and Astronomy should be familiar with the various necessary techniques of what seems to be the coming method of determining deflections of the vertical, deflections less arbitrary and more scientific than those in use hitherto, deflections that may be computed even for points isolated from existing blocks of geodetic triangulation.

We shall have also a scientific basis for the choice of a geodetic datum, a datum as free as possible from arbitrary elements. The center of our ellipsoid of reference will be where it should be; the direction of the axis of the ellipsoid will be what it should be. Only the linear dimensions will be open to choice, and a comparison of the gravimetric deflections at various points with the corresponding astronomico-geodetic deflections will give us a good hold on the proper dimensions to choose. This is as near as we can come to *the* deflection of the vertical at a point.

As a by-product of the study we shall have rather full and accurate tables of four functions useful in the computation of gravimetric deflections. Defining Stokes' function f (ψ) by the formula :

$$f(\psi) = \frac{1}{2}\left[\operatorname{cosec}\frac{\psi}{2} + 1 - 6\sin\frac{\psi}{2} - 5\cos\psi - 3\cos\psi\log_e\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right)\right]$$

we find it convenient to have, for computing deflection, tables of the values of

$$\frac{d\,f(\psi)}{d\,\psi} = \frac{1}{2}\left[-\frac{\cos\frac{\psi}{2}}{2\sin^2\frac{\psi}{2}} - 3\cos\frac{\psi}{2} + 5\sin\psi + 3\sin\psi\log_e\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right)\ldots\right.$$

$$\left.\ldots - \frac{3}{2}\cos\psi\cot\frac{\psi}{2}\,\frac{1 + 2\sin\frac{\psi}{2}}{1 + \sin\frac{\psi}{2}}\right]$$

$$\frac{d\,f(\psi)}{d\,\psi}\sin\psi = \frac{1}{2}\left[-\frac{\cos^2\frac{\psi}{3}}{\sin\frac{\psi}{2}} - 3\sin\psi\cos\frac{\psi}{2} + 5\sin^2\psi + 3\sin^2\psi\log_e\ldots\right.$$

$$\left.\ldots\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right) - 3\cos\psi\left(1 + \sin\frac{\psi}{2} - 2\sin^2\frac{\psi}{2}\right)\right]$$

Accordingly tables have been computed for $\frac{d\,f(\psi)}{d\,\psi}$, $\frac{d\,f(\psi)}{d\,\psi}$ sin ψ also for $\int_{\alpha}^{\psi}\frac{d\,f}{d\psi}\sin\psi\,d\,\psi$.

The indicated integration cannot be performed in terms of the elementary functions, so it was necessary to resort to expansion in power series and to mechanical quadrature. The value of α, the lower limit of integration, was arbitrarily taken as 10 metres of great-circle arc, or approximately one centesimal second, since a zero value would have given rise to a logarithmic infinity. The values of f (ψ) itself, used in computing geoid elevations referred to the spheroid, are given in Special Publication No. 199 of the U.S. Coast and Geodetic Survey (Lambert and Darling, 1936, pp. 114-117).

It will be fairly easy to interpolate in the new tables in order to form special tables for whatever arrangement of zones may be deemed convenient for map reading and computation. The tables will be published in a forthcoming issue of this journal.

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