THE APPLICABILITY OF LAPLACE'S DIFFERENTIAL EQUATIONS OF THE TIDES

by J. PROUDMAN, Department of Oceanography, University of Liverpool.

(Read at the General Assembly of the International Association of Physical Oceanography at Oslo, August 1948).

1. **Introduction.**—The general differential equations for the tides, as given in Laplace's *Mécanique Céleste*, have been used by all the chief workers on the dynamical theory of the tides. But in 1933 V. Bjerknes, J. Bjerknes, H. Solberg and T. Bergeron, in their *Physikalische Hydrodynamik*, claimed that these equations are quite inadequate to deal with diurnal constituents of the tides, and this claim was maintained by Solberg in 1936. In 1933 M. Brillouin and J. Coulomb pointed out that the equations did not appear to be valid for the semidiurnal constituent K_2, whose period is half a sidereal day, in the Polar Sea.

Laplace's differential equations are based on the hypothesis that the vertical component of acceleration, both geostrophic and relative to the earth, may be neglected, so that the pressure may be calculated on the principles of hydrostatics. This makes the equations inadequate for the treatment of internal cellular oscillations in homogeneous water.

Laplace's hypothesis may be tested by taking the more general differential equations, in which the vertical components of acceleration are not neglected, by solving definite problems on the basis of these equations, and then comparing the solutions with the solutions of the corresponding problems on the basis of Laplace's equations. In a paper published in 1941 I considered motions in basins of homogeneous water on a spherical earth generated by the tidal forces of the moon and sun. Special features of such problems are that the tide-generating forces are practically independent of depth in the ocean, that periods are prescribed, and, in certain types of problem, that wave-lengths are also prescribed. I showed that :—

(I) Laplace's celebrated solution for the diurnal constituent K_1 in an ocean of uniform depth covering the whole earth is completely valid;

(II) Laplace's differential equations give a very good approximation for all constituents in a narrow ocean of uniform depth bounded either by meridians or by parallels of latitude;

(III) Laplace's differential equations give a good approximation for all constituents, other than the semidiurnal ones, in a polar sea of uniform depth bounded by a parallel of latitude; and for all constituents, other than the long period ones. in a broad equatorial channel of uniform depth;

(IV) Laplace's differential equations give a good approximation to the K_2 and long period constituents in an ocean covering the whole earth whose depth varies with latitude but not with longitude, except near the poles in the case of the K_2 constituent and near the equator in the case of the long period constituents.

My general conclusions were that Laplace's equations are adequate for the treatment of all constituents in the actual oceans.

The actual oceans, however, are not of homogeneous water but are stratified in density, and in 1939 and 1943 E. A. Hylleraas showed that, for water which is sufficiently stratified, Laplace's differential equations of the tides are valid without the restrictions which have to be made in the case of homogeneous water.

The object of the present paper is to confirm the conclusions of Hylleraas and of myself as quoted above, using a mathematical argument which is much simpler than those either of us have used hitherto. I consider *free standing* oscillations with horizontal crests in uniformly stratified water in *polar* and *equatorial* basins of uniform depth. In each case there are *ordinary* tidal oscillations in which the currents are practically the same from sea-surface to sea-bottom, and there are also *internal* tidal oscillations in which the currents undergo large changes between sea-surface and sea-bottom. I show that Laplace's equations always give a good approximation to the relationships of the ordinary tidal oscillations, but that they only give a good approximation to those of the internal tidal oscillations when the vertical gradient of density is greater than a certain quantity depending on the period of the motion. In the special case of homogeneous water, Laplace's equations do not give any internal oscillations at all.

It thus appears that the inadequacy of Laplace's equations is only of importance in the case of internal oscillations in water which is very nearly homogeneous.

The bearing of these results on the question of forced tides arises through the possibility of resonance. Only when there is resonance between the tide-generating forces and a particular internal oscillation in water which is nearly homogeneous will Laplace's equations fail to account for the forced tides. In this connection we note that, of the particular free oscillations considered in this paper, only the semidiurnal oscillations of the polar sea and the long period oscillations of the equatorial sea have wave-lengths which are large compared with the depth.

2. **Notation and continuity.**—We shall use Cartesian coordinates x, y, z so that the sea-surface and sea-bottom are given by $z = 0$ and $z = -h$ respectively, and the corresponding components of current by u, v, w. We shall denote by ζ the elevation of the water at any point in the sea, and by ζ_0 the elevation of the sea-surface. Then we have

$$w = \frac{\delta \zeta}{\delta t} \tag{2.1}$$

t denoting the time.

We consider a sea in which, in equilibrium, the density is given by

$$\rho = \rho_0 \left(1 - \frac{z}{h_1} \right) \tag{2.2}$$

where ρ_0, h_1 are constants and h_1/h is large. The special case of homogeneous water is obtained by making h_1 infinite. As is usual in the treatment of ocean tides, we shall suppose that the motion is small and shall neglect products of small quantities.

The equation of continuity of mass is

$$\frac{\delta \rho}{\delta t} + w \frac{\delta \rho}{\delta z} = 0 \tag{2.3}$$

and on using (2.1) and (2.2) this becomes

$$\frac{\delta \rho}{\delta t} = \frac{\rho_0}{h_1} \frac{\delta \zeta}{\delta t}$$

on the integration of which we obtain

$$\rho = \rho_0 \left(1 + \frac{\zeta - z}{h_1} \right) \tag{2.4}$$

For simplicity we shall suppose that the motion is independent of y, and shall take, in all cases

$$\zeta_0 = H \cos \frac{2 \pi x}{\lambda} \cos \frac{2 \pi t}{T} \tag{2.5}$$

Where H, λ are constant lengths and T is the period. From (2.1) and the condition $w = 0$ at $z = -h$, we deduce that

$$w = - \frac{2 \pi H}{T} \frac{\sin \frac{2\pi}{d} (h + z)}{\sin \frac{2 \pi h}{d}} \cos \frac{2 \pi x}{\lambda} \sin \frac{2 \pi t}{T} \tag{2.6}$$

$$\zeta = H \frac{\sin \frac{2\pi}{d}(h+z)}{\sin \frac{2\pi h}{d}} \cos \frac{2\pi x}{\lambda} \cos \frac{2\pi t}{T} \tag{2.7}$$

The equation of continuity of volume is

$$\frac{\delta u}{\delta x} + \frac{\delta w}{\delta z} = o \tag{2.8}$$

and from this and (2.6) we deduce that

$$u = \frac{2\pi H}{T} \frac{\lambda}{d} \frac{\cos \frac{2\pi}{d}(h+z)}{\sin \frac{2\pi h}{d}} \sin \frac{2\pi x}{\lambda} \sin \frac{2\pi t}{T} \tag{2.9}$$

3. **Laplace's equations.**—When the vertical component of acceleration and also the geostrophic acceleration due to the vertical component of current are neglected, the equations of motion take the form

$$\left.\begin{aligned} \frac{\delta u}{\delta t} - \frac{2\pi}{Tp} v &= -\frac{1}{\rho}\frac{\delta p}{\delta x} \\ \frac{\delta v}{\delta t} + \frac{2\pi}{Tp} u &= o \\ o &= -\frac{1}{\rho}\frac{\delta p}{\delta z} - g \end{aligned}\right\} \tag{3.1}$$

Here Tp is half a pendulum-day, so that

$$Tp = \frac{Ts}{\sin \varphi}$$

where Ts is half a sidereal day and φ is the north latitude; at the equator Tp is infinite.

From (2.9) and the second of equations (3.1) we deduce

$$v = \frac{2\pi H}{Tp} \frac{\lambda}{d} \frac{\cos \frac{2\pi}{d}(h+z)}{\sin \frac{2\pi h}{d}} \sin \frac{2\pi x}{\lambda} \cos \frac{2\pi t}{T} \tag{3.2}$$

For an equatorial sea, $v = o$.

From (2.4), (2.7) and the third of equations (3.1) we deduce

$$\frac{\delta p}{\delta z} = -g\rho_o\left(1 + \frac{\zeta - z}{h_1}\right)$$

$$= -g\rho_o\left(1 - \frac{z}{h_1}\right) - \frac{g\rho_o H}{h_1} \frac{\sin \frac{2\pi}{d}(h+z)}{\sin \frac{2\pi h}{d}} \cos \frac{2\pi x}{\lambda} \cos \frac{2\pi t}{T}$$

so that, on integrating,

$$p = p_o - g\rho_o\left(z - \frac{1}{2}\frac{z^2}{h_1}\right) +$$

$$+ \frac{g\rho_o H}{2\pi} \frac{d}{h_1} \left\{ \frac{\cos \frac{2\pi}{d}(h+z)}{\sin \frac{2\pi h}{d}} + A \right\} \cos \frac{2\pi x}{\lambda} \cos \frac{2\pi t}{T} \tag{3.3}$$

where p_0 is the atmospheric pressure, assumed uniform, and A is a constant of integration.

On substituting from (2.9), (3.2) and (3.3) into the first of equations (3.1) we deduce that A = o and also that

$$\frac{d^2}{\lambda^2} = \frac{4\,\pi^2\,h_1}{g}\left(\frac{1}{T^2} - \frac{1}{Tp^2}\right) \tag{3.4}$$

We notice that d/λ will be small when T is nearly equal to Tp.

On z = o, we have

$$\begin{aligned} p &= p_0 + g\,\rho_0\,\zeta_0 \\ &= p_0 + g\,\rho_0\,H\cos\frac{2\,\pi\,x}{\lambda}\cos\frac{2\,\pi\,t}{T} \end{aligned}$$

and on comparing this with (3.3) we deduce that

$$\frac{2\,\pi\,h}{d}\tan\frac{2\,\pi\,h}{d} = \frac{h}{h_1} \tag{3.5}$$

Since the right hand side of (3.5) is very small, the smallest value of h/d is given approximately by

$$\left(\frac{2\,\pi\,h}{d}\right)^2 = \frac{h}{h_1} \tag{3.6}$$

The other values of h/d are given approximately by

$$\frac{2\,\pi\,h}{d} = n\,\pi + \frac{1}{n\pi}\,\frac{h}{h_1} \tag{3.7}$$

where n is any integer.

4. **Polar sea.**—When the vertical component of acceleration is not neglected, the equations of motion are

$$\left.\begin{aligned} \frac{\delta u}{\delta t} - \frac{2\pi}{Ts}\,v &= -\frac{1}{\rho}\frac{\delta p}{\delta x} \\ \frac{\delta v}{\delta t} + \frac{2\pi}{Ts}\,u &= o \\ \frac{\delta w}{\delta t} &= -\frac{1}{\rho}\frac{\delta p}{\delta z} - g \end{aligned}\right\} \tag{4.1}$$

In this case Tp = Ts and the only difference from Laplace's equations (3.1) is the presence of the term $\delta w/\delta t$.

From (2.9) and the second of (4.1) we deduce

$$v = \frac{2\,\pi\,H}{Ts}\,\frac{\lambda}{d}\,\frac{\cos\frac{2\pi}{d}(h+z)}{\sin\frac{2\,\pi\,h}{d}}\,\sin\frac{2\,\pi\,x}{\lambda}\cos\frac{2\,\pi\,t}{T} \tag{4.2}$$

which agrees with (3.2).

From (2.4), (2.6), (2.7) and the third of (4.1) we deduce that

$$\begin{aligned} \frac{\delta p}{\delta z} &= -\,g\,\rho_0\left(1 - \frac{z}{h_1}\right) \\ &+ \left(\frac{4\,\pi^2}{T^2} - \frac{g}{h_1}\right)\rho_0\,H\,\frac{\sin\frac{2\pi}{d}(h+z)}{\sin\frac{2\,\pi\,h}{d}}\cos\frac{2\,\pi\,x}{\lambda}\cos\frac{2\,\pi\,t}{T} \end{aligned} \tag{4.3}$$

and then we deduce, exactly as in § 3, that

$$\frac{d^2}{\lambda^2} = \frac{1 - \dfrac{T^2}{Ts^2}}{\dfrac{g\,T^2}{4\,\pi^2 h_1} - 1} \tag{4.4}$$

and

$$\frac{2\,\pi\,h}{d} \tan \frac{2\,\pi\,h}{d} = \frac{h}{h_1} - \frac{4\,\pi^2 h}{g\,T^2} \tag{4.5}$$

We notice that (3.4) and (3.5) differ from (4.4) and (4.5) respectively by the neglect of $4\,\pi^2\,h_1/gT^2$. Also that d/λ will be small when T is nearly equal to Ts.

Since the right hand side of (4.5) is very small, the smallest value of h/d is given approximately

$$\left(\frac{2\,\pi\,h}{d}\right)^2 = \frac{h}{h_1} - \frac{4\,\pi^2 h}{g\,T^2} \tag{4.6}$$

The other values of h/d are given approximately by

$$\frac{2\,h}{d} = n + \frac{1}{n}\left(\frac{1}{\pi^2}\,\frac{h}{h_1} - \frac{4\,h}{g\,T^2}\right) \tag{4.7}$$

where n is any integer.

5. **Equatorial sea.**—Taking the x direction to lie in the meridian, the general equations of motion are

$$\left.\begin{aligned} &\frac{\delta u}{\delta t} = -\frac{1}{\rho}\,\frac{\delta p}{\delta x} \\ &\frac{\delta v}{\delta t} - \frac{2\pi}{Ts}\,w = 0 \\ &\frac{\delta w}{\delta t} + \frac{2\pi}{Ts}\,v = -\frac{1}{\rho}\,\frac{\delta p}{\delta z} - g \end{aligned}\right\} \tag{5.1}$$

These differ from Laplace's equations by the presence of $\delta w/\delta t$ and the two terms containing Ts.

From (2.6) and the second of (5.1) we deduce

$$v = \frac{2\,\pi\,H}{Ts}\,\frac{\sin \dfrac{2\pi}{d}(h + z)}{\sin \dfrac{2\,\pi\,h}{d}}\,\cos\frac{2\,\pi\,x}{\lambda}\,\cos\frac{2\,\pi\,t}{T} \tag{5.2}$$

This is of the same order as the vertical component of current w. The corresponding part of Laplace's solution of § 3 is $v = 0$.

From (2.4), (2.6), (2.7), (5.2) and the third of (5.1) we deduce that

$$\frac{\delta p}{\delta z} = -g\,\rho_0\left(1 - \frac{z}{h_1}\right)$$
$$+\left(\frac{4\,\pi^2}{T^2} - \frac{4\,\pi^2}{Ts^2} - \frac{g}{h_1}\right)\rho_0 H\,\frac{\sin \dfrac{2\pi}{d}(h + z)}{\sin \dfrac{2\,\pi\,h}{d}}\,\cos\frac{2\,\pi\,x}{\lambda}\,\cos\frac{2\,\pi\,t}{T} \tag{5.3}$$

and then we deduce, exactly as in § 3, that

$$\frac{\lambda^2}{d^2} = \frac{g T^2}{4 \pi^2 h_1} + \frac{T^2}{Ts^2} - 1 \qquad (5.4)$$

and

$$\frac{2 \pi h}{d} \tan \frac{2 \pi h}{d} = \frac{h}{h_1} - \frac{4 \pi^2 h}{g} \left(\frac{1}{T^2} - \frac{1}{Ts^2}\right) \qquad (5.5)$$

We notice that (3.4) and (3.5) differ from (5.4) and (5.5) respectively by the neglect of

$$\frac{4 \pi^2 h_1}{g} \left(\frac{1}{T^2} - \frac{1}{Ts^2}\right)$$

Also, that λ/d will be large when T is large.

Since the right hand side of (5.5) is very small, the smallest value of h/d is given approximately by

$$\left(\frac{2 \pi h}{d}\right)^2 = \frac{h}{h_1} - \frac{4 \pi^2 h}{g} \left(\frac{1}{T^2} - \frac{1}{Ts^2}\right) \qquad (5.6)$$

The other values of h/d are given approximately by

$$\frac{2h}{d} = n + \frac{1}{n \pi^2} \frac{h}{h_1} - \frac{4h}{ng} \left(\frac{1}{T^2} - \frac{1}{Ts^2}\right) \qquad (5.7)$$

where n is any integer.

6. **Ordinary and Internal Tides.**—From (2.5) and each of the pairs of equations (3.3) and (3.5), (4.3) and (4.5), (5.3) and (5.5) we deduce that

$$p = p_0 - g \rho_0 \left(z - \frac{1}{2} \frac{z^2}{h_1}\right) + \frac{\cos \frac{2\pi}{d} (h + z)}{\cos \frac{2 \pi h}{d}} g \rho_0 \zeta_0 \qquad (6.1)$$

The equations (3.6), (4.6), (5.6) show that there are always small values of h/d, though the actual values differ in the three cases. The corresponding first approximations to the equations (2.7), (2.9) and (6.1) are respectively

$$\left.\begin{aligned} \zeta &= H \left(1 + \frac{z}{h}\right) \cos \frac{2 \pi x}{\lambda} \cos \frac{2 \pi t}{T} \\ u &= \frac{H}{T} \frac{\lambda}{d} \sin \frac{2 \pi x}{\lambda} \sin \frac{2 \pi t}{T} \\ p &= p_0 - g \rho_0 \left(z - \frac{1}{2} \frac{z^2}{h_1}\right) + g \rho_0 \zeta_0 \end{aligned}\right\} \qquad (6.2)$$

Also, when the particular small values of h/d are substituted respectively into (3.4), (4.4), (5.4) we find

$$\frac{g h}{\lambda^2} = \frac{1}{T^2} - \frac{1}{Ts^2} \qquad (6.3)$$

in all three cases. These are the formulae for *ordinary* tidal oscillations, and Laplace's equations give the same first approximations as the more general equations. In the differential equations for the polar and equatorial seas

$$\frac{4 \pi^2 H}{g T^2} \quad \text{and} \quad \frac{4 \pi^2 H}{g} \left(\frac{1}{T^2} - \frac{1}{Ts^2}\right) \qquad (6.4)$$

respectively are neglected. Taking

$$H = 1 \text{m.}, \qquad T = 12.4 \text{ hr}, \qquad Ts = 12.0 \text{ hr},$$

the two quantities (6.4) are 2.1×10^{-9} and $- 1.4 \times 10^{-10}$ respectively.

The equations (3.7), (4.7), (5.7) give other values of h/d. and on writing

$$H^1 = \frac{(-1)^n H}{\sin \frac{2 \pi h}{d}}$$

the corresponding first approximations to (2.7), (2.9) and (6.1) are

$$\left.\begin{aligned} \zeta &= H^1 \sin \frac{n \pi z}{h} \cos \frac{2 \pi x}{\lambda} \cos \frac{2 \pi t}{T} \\ u &= \frac{n \pi H^1}{T} \frac{\lambda}{h} \cos \frac{n \pi z}{h} \sin \frac{2 \pi x}{\lambda} \sin \frac{2 \pi t}{T} \\ p &= p_0 - g \rho_0 \left(z - \frac{1}{2} \frac{z^2}{h_1} \right) + g \rho_0 H^1 \cos \frac{n \pi z}{h} \cos \frac{2 \pi x}{\lambda} \cos \frac{2 \pi t}{T} \end{aligned}\right\} \quad (6.5)$$

where, for Laplace's equations,

$$\lambda^2 = \frac{\mu}{n \pi} \frac{g h}{1/T^2 - 1/Tp^2}, \qquad n \pi \mu = \frac{h}{h_1} :$$

for the polar sea,

$$\lambda^2 = \frac{\mu}{n \pi} \frac{g h}{1/T^2 - 1/Ts^2}, \qquad n \pi \mu = \frac{h}{h_1} - \frac{4 \pi^2 h}{g T^2} ;$$

and for the equatorial sea,

$$\lambda^2 = \frac{\mu}{n \pi} g h T^2, \qquad n \pi \mu = \frac{h}{h_1} - \frac{4 \pi^2 h}{g} \left(\frac{1}{T^2} - \frac{1}{Ts^2} \right)$$

These are the formulae for *internal* tidal oscillations and Laplace's equations give an approximation so long as

$$\frac{4 \pi^2 h_1}{g T^2} \quad \text{and} \quad \frac{4 \pi^2 h_1}{g} \left(\frac{1}{T^2} - \frac{1}{Ts^2} \right) \qquad (6.6)$$

are small for the polar and equatorial seas respectively. Taking

$$h_1 = 10^6 m, \qquad T = 12.4 \text{ hr}, \qquad Ts = 12.0 \text{ hr},$$

the two quantities (6.6) are 2.1×10^{-3} and -1.4×10^{-4} respectively. When h_1 is infinite, so that the water is homogeneous, $\mu = 0$, $\lambda = 0$ for Laplace's equations, and these equations do not give any internal oscillations, though the more general equations do.

REFERENCES :

1933. BRILLOUIN M. and COULOMB J., « Inst. Mécan. Fluides », Paris.
1939. HYLLERAAS E. A., « Astrophys. Norvég. », 3., 139-164. Oslo.
1941. PROUDMAN J., « Proc. Royal. Soc. », 179, 261-288. London.
1943. HYLLERAAS E. A., « Geofys. Pub. », 13 (10). Oslo.