METHOD OF COMPUTATION OF THE HOMOFOCAL GRID FOR HYPERBOLIC NAVIGATION CHARTS.

In Volume XXIV of the *International Hydrographic Review*, Monaco, 1947, pages 176 and 177, some information was given on the method of computing the hyperbolic lattice grid to be plotted on charts for the use of the "DECCA Navigator".

The International Hydrographic Bureau recently received a photolithographic reproduction of a preliminary paper entitled "The Computation of DECCA Lattices" compiled in 1947 by H. M. Nautical Almanac Office and giving a description of the methods used by that Office for the computation of the DECCA lattice.

The same methods and technique are naturally applicable, to within a few limitations, to all other systems of hyperbolic navigation such as Loran and Gee. The sole purpose of the calculation is to provide the numerical data required by the cartographer and draughtsman to plot the curves on the fair proof of the chart, by means of splines, and previous to subsequent reproduction for publication, by joining a series of points the co-ordinates of which are provided by the computer. It would be possible to represent those lattices by plane hyperbolae for considerable ranges (up to 100 miles on small scales) if they could be drawn on charts of constant scale. The general use of the Mercator projection, the scale variations resulting therefrom and the variation of their boundaries to fit the irregularities of the coast line imply, however, a few necessary precautions.

The pamphlet is divided into several Sections. The Introductory Section gives a general account of the problem in all its aspects.

Geometrically, the accuracy of the determination of position is conditioned by the "sensitivity" (i. e. the reciprocal of the lane width) of the lattices and upon the angle of cut. On the base-line the width of a lane is half the wave-length and it increases outwards ; the sensitivity is zero on the base-line extensions. The angle of cut between two lattices at a point some distance from the stations can be estimated from the angle subtended by the lines joining the point to the mid-points of the two base-lines concerned.

Other uncertainties arise from :

(*i*) Instrumental limitations.

(*ii*) The variability of the speed of transmission of radio waves according to the terrain over which they pass.

(*iii*) Errors in the determination of the positions of the stations, and in the geodetic connection between different triangulation systems.

The fundamental geodetic data for any survey are :

(*i*) The latitude and longitude of a point of origin.

(*ii*) The orientation of the co-ordinate system used.

(*iii*) The constants for an accepted figure of the Earth.

(*iv*) The standard of linear measurement.

In respect of these data the Ordnance Survey differs from the surveys of the countries on the continent of Europe. Therefore, in order to plot correctly, on charts of the French, Belgian, Netherlands, etc., coasts, the computed latitudes and longitudes which determine the lattice lines, it is essential to bring the geographical positions of points on those coasts into accord with the Ordnance Survey system. The data of the national surveys, and of the connections between them, will determine corrections to the graticules of the foreign charts. These corrections must be applied before plotting the computed values defining the lattice lines.

The correction to be applied depends mainly on the error in the position of origin : this error is a constant one for all the charts under consideration, and corresponds to a translation of all positions in the direction of the error in position of the origin. The second factor, the error in orientation of the co-ordinate system, gives rise to a rotational correction which is superimposed on the main (constant) effect already mentioned. The remaining errors are comparatively small, and the resultant corrections can again be superimposed : thus correction curves can be drawn across the charts under consideration, in order to facilitate the amendment of the points which determine the lattice lines.

Even in the British Isles, the charts on which the latticing is to be performed rarely correspond to the figure of the Earth adopted for the survey of the appropriate area. Here, however, the point of origin is not in error, so that any correction which is to be applied must necessarily be small.

For the numerical calculations it is necessary to retain a precision higher than that attainable in the physical system concerned, but not so much higher that unnecessary work is involved. In this case, there are two alternative standards of accuracy which we can use depending on :

(*i*) the overall accuracy of the system, expressed in distance on the Earth's surface,

(*ii*) the accuracy to which the lattices can actually be plotted on the chart.

The greatest difficulty occurs relatively close to the stations, where the system might justly be expected to attain its highest accuracy. It is doubtful if a nominal accuracy of 20 feet can be achieved with the present equipment, but allowance must be made for technical advances, and a convenient working unit would appear to be 0.001 lanes which, with the present frequencies, corresponds to about 2 feet on the base-line, increasing outwards. If the lane numbers are required with an error of less than 0.001, the working unit in distance must be one foot, and this is the unit used in the Nautical Almanac Office. It probably represents too great an accuracy and a yard or metre would probably be more realistic.

It is generally agreed that the lattices cannot be impressed upon charts more accurately than to 0.01 inches. The following table shows the linear distances corresponding to this quantity on various scales of chart :

Reciprocal of scale...............	10,000	50,000	100,000	250,000
Distance in feet (approx.).......	8	40	80	200

The draughtsman must be provided with the co-ordinates of points on the lattices, preferably with the longitudes on parallels or the latitudes on meridians. There is (except in the neighbourhood of the stations) no practicable direct method of calculating one geographical co-ordinate as a function of lane number and the other co-ordinate. Although a method is described for obtaining the geographical position corresponding to the "DECCA co-ordinates" (or a pair of lane numbers) at a point, this method is not suitable for systematic computation and is only used to a relatively low degree of accuracy. It is therefore necessary, in general, to calculate the lane numbers (which are simply linear functions of the difference in distance from the master station and from the slave station) at a series of points with known geographical co-ordinates, and by the arithmetical process known as "inverse interpolation" to deduce the geographical co-ordinates in the form required.

At moderate distances from the stations, lane numbers can be calculated at wide intervals of longitude or latitude (at least 5') and the inverse interpolation done without appreciable error. The lane number, however, has singularities at each of the stations and it can only be interpolated at an interval which rapidly diminishes as the station is approached. If this method is employed, it therefore requires the use of a disproportionately large number of points near to the stations.

The areas in the vicinity of the stations can, however, be catered for by the "polar method"; the general plan is therefore to calculate lane numbers for a longitude and latitude lattice on the Mercator projection at a spacing of 5', and to use other methods near the stations.

The errors in applying the standard method increase as the station is approached, while the errors due to the Polar method roughly increase with distance from the station. The two methods of computation are thus complementary, and all areas can be latticed to a satisfactory degree of accuracy.

The polar method is unsatisfactory near the base-line extension, since adjacent radii from the station will not intersect a given lattice line at sufficiently close intervals. However, normally it is not required to lattice such an area — the cut-out area. If coverage is desired for special requirements (e. g. survey work or a two-slave system), another method, mentioned below, the "Rectangular Method" can be utilized.

Conventions and Enumeration of Lanes.

Strictly speaking, a lane is the space between two successive lattice lines, along each of which the phase difference between the harmonics of the two transmission frequencies, equal to the comparison frequency, vanishes. The term "lane-number" is, however, used to denote the bounding lattice line itself, in the sense that the lane numbers increase from zero at the Master Station at intervals of one unit for every lane. To avoid confusion, the numbering of each set of lattice lines is modified by the constant addition of some convenient multiple of 100, so that one lane number corresponds to a lattice line unique in the whole chain. Lanes are grouped into zones, each of which is denoted by a letter. The number of lanes in a zone is equal to the comparison frequency divided by the basic frequency. Within each zone the lanes are numbered consecutively, so that any lane is identified by a letter and a number. The computational work, however, is always done directly in terms of lane number, which will be used throughout this account.

For calculating the co-ordinates of points on the DECCA hyperbolic lattice, the Nautical Almanac Office makes use of a special equipment which is described in Section III of the pamphlet.

It consists of calculating machines, a printing adding-machine, mathematical and auxiliary tables and a punched-card outfit.

Calculating machines.

Any modern calculating machine — hand or electric — will suffice for most of the computations. For certain stages (inverse interpolation and the direct calculation of position from DECCA co-ordinates) a minimum capacity of 10 figures on the keyboard (or levers) is essential. If possible, both hand

and fast automatic electric machines should be used — the former for the difficult calculations where judgment is required, and the latter for the routine work where speed, without fatigue, is desirable.

For one special operation, a twin machine is recommended — but it is not essential. A twin machine consists of two identical machines so connected that all operations of multiplication performed on one machine can, at the will of the operator, be performed positively, negatively or not at all on the second machine. Such machines are the Brunsviga Twin and the Twin Marchant hand-machines.

The machines recommended are the Brunsviga 20 of capacity $12 \times 11 = 20$ for the hand machine and the Marchant ACT-1OM of capacity $10 \times 10 = 20$ for the fast electric machine ; the automatic Marchant cannot easily be used for inverse interpolation, and if only one machine is adopted, it should not be this one.

The printing-adding machine is the National Accounting Machine [1] with six adding mechanisms or registers each with 12-figure capacity ; two of these registers (Nos 1 and 3) can subtract directly as well as add, but subtraction in the other four must be performed by the addition of complements. A number set on the 12-bank keyboard can be added (including subtraction in registers 1 and 3) to any combination of the six registers ; the contents of any selected register can be transferred, with (a total or "T" operation) or without (a sub-total or "ST" operation) zeroising the register selected ; in each case the number added or transferred is printed.

The moving carriage carries a removable "form-bar" on which stops can be easily placed in any desired position and order : it normally "tabulates" or moves from one stop to the next, after each operation, automatically feeding the paper and returning to the first position after reaching the end of a line.

A large part of the systematic computation can be done in a straightforward manner on the National Machine. The machine is particularly suited for :

1) : Adding and subtracting ;
2) : Integration of functions from their differences ;
3) : Differencing known functions ;
4) : Subtabulating ;
5) : Producing printed copy.

The following tables are used at various stages of the work ; not all are essential, but their availability is a great convenience.

ANDOYER.... *Nouvelles Tables Trigonométriques*, Paris, 1915. Gives 15 figure values of all trigonometrical functions at intervals of 10".

BARLOW ..... *Tables of Squares, Cubes, Square Roots, Cube Roots and Reciprocals*, 4th Edition, London, 1941.

LOHSE......... *Tafeln für numerisches Rechnen mit Maschinen*, 2nd Edition, Leipzig, 1935.
Gives 5 figure values of all six trigonometrical functions for each 0°.01.

(1) "Inverse Interpolation and Scientific Applications of the National Accounting Machine", by L. J. COMRIE, *Journal of the Royal Statistical Society* (Supplement), Vol. III, 87, 1936.

TABLE III

φ	48°		49°		50°		φ
′	a cos u	c sin u	a cos u	c sin u	a cos u	c sin u	′
00	14 026 557	15 474 155	13 753 343	15 715 874	13 475 905	15 952 829	00
01	14 022 038	15 478 223	13 748 754	15 719 863	13 471 245	15 956 738	01
02	14 017 519	15 482 289	13 744 163	15 723 850	13 466 585	15 960 645	02
03	14 012 998	15 486 353	13 739 571	15 727 836	13 461 923	15 964 551	03
04	14 008 476	15 490 417	13 734 978	15 731 820	13 457 260	15 968 455	04
05	14 003 952	15 494 479	13 730 384	15 735 803	13 452 597	15 972 358	05
06	13 999 428	15 498 540	13 725 788	15 739 785	13 447 932	15 976 260	06
07	13 994 902	15 502 599	13 721 192	15 743 766	13 443 265	15 980 160	07
08	13 990 375	15 506 657	13 716 594	15 747 745	13 438 598	15 984 059	08
09	13 985 847	15 510 714	13 711 995	15 751 723	13 433 930	15 987 957	09
10	13 981 318	15 514 770	13 707 395	15 755 700	13 429 260	15 991 853	10
11	13 976 788	15 518 824	13 702 793	15 759 675	13 424 589	15 995 748	11
12	13 972 256	15 522 877	13 698 191	15 763 649	13 419 917	15 999 642	12
13	13 967 723	15 526 928	13 693 587	15 767 621	13 415 244	16 003 534	13
14	13 963 189	15 530 979	13 688 982	15 771 593	13 410 570	16 007 425	14
15	13 958 654	15 535 028	13 684 376	15 775 563	13 405 894	16 011 315	15
16	13 954 118	15 539 075	13 679 769	15 779 531	13 401 218	16 015 203	16
17	13 949 580	15 543 122	13 675 161	15 783 498	13 396 540	16 019 090	17
18	13 945 041	15 547 167	13 670 552	15 787 464	13 391 862	16 022 976	18
19	13 940 501	15 551 211	13 665 941	15 791 429	13 387 182	16 026 860	19
20	13 935 960	15 555 253	13 661 329	15 795 392	13 382 500	16 030 743	20
21	13 931 418	15 559 294	13 656 716	15 799 354	13 377 818	16 034 624	21
22	13 926 874	15 563 334	13 652 102	15 803 315	13 373 135	16 038 504	22
23	13 922 330	15 567 373	13 647 487	15 807 274	13 368 450	16 042 383	23
24	13 917 784	15 571 410	13 642 870	15 811 232	13 363 765	16 046 261	24
25	13 913 237	15 575 446	13 638 253	15 815 189	13 359 078	16 050 137	25
26	13 908 688	15 579 480	13 633 634	15 819 144	13 354 390	16 054 011	26
27	13 904 139	15 583 514	13 629 014	15 823 098	13 349 701	16 057 885	27
28	13 899 588	15 587 546	13 624 393	15 827 051	13 345 011	16 061 757	28
29	13 895 036	15 591 576	13 619 770	15 831 002	13 340 319	16 065 628	29
30	13 890 483	15 595 606	13 615 147	15 834 952	13 335 627	16 069 497	30
31	13 885 929	15 599 634	13 610 522	15 838 900	13 330 933	16 073 365	31
32	13 881 374	15 603 660	13 605 897	15 842 848	13 326 238	16 077 232	32
33	13 876 817	15 607 686	13 601 270	15 846 794	13 321 543	16 081 097	33
34	13 872 259	15 611 710	13 596 641	15 850 738	13 316 845	16 084 961	34
35	13 867 700	15 615 733	13 592 012	15 854 682	13 312 147	16 088 823	35
36	13 863 140	15 619 754	13 587 382	15 858 624	13 307 448	16 092 685	36
37	13 858 579	15 623 774	13 582 750	15 862 564	13 302 747	16 096 544	37
38	13 854 017	15 627 793	13 578 117	15 866 504	13 298 046	16 100 403	38
39	13 849 453	15 631 811	13 573 484	15 870 441	13 293 343	16 104 260	39
40	13 844 888	15 635 827	13 568 849	15 874 378	13 288 639	16 108 116	40
41	13 840 322	15 639 842	13 564 212	15 878 313	13 283 934	16 111 970	41
42	13 835 755	15 643 855	13 559 575	15 882 247	13 279 228	16 115 823	42
43	13 831 186	15 647 868	13 554 936	15 886 180	13 274 521	16 119 675	43
44	13 826 617	15 651 879	13 550 297	15 890 111	13 269 812	16 123 525	44
45	13 822 046	15 655 888	13 545 656	15 894 041	13 265 103	16 127 374	45
46	13 817 474	15 659 897	13 541 014	15 897 970	13 260 392	16 131 222	46
47	13 812 901	15 663 904	13 536 371	15 901 897	13 255 680	16 135 068	47
48	13 808 327	15 667 909	13 531 726	15 905 823	13 250 968	16 138 913	48
49	13 803 751	15 671 914	13 527 081	15 909 747	13 246 253	16 142 757	49
50	13 799 175	15 675 917	13 522 434	15 913 671	13 241 538	16 146 599	50
51	13 794 597	15 679 918	13 517 786	15 917 593	13 236 822	16 150 440	51
52	13 790 018	15 683 919	13 513 138	15 921 513	13 232 104	16 154 279	52
53	13 785 437	15 687 918	13 508 487	15 925 432	13 227 386	16 158 117	53
54	13 780 856	15 691 916	13 503 836	15 929 350	13 222 666	16 161 954	54
55	13 776 274	15 695 912	13 499 184	15 933 267	13 217 945	16 165 790	55
56	13 771 690	15 699 907	13 494 530	15 937 182	13 213 223	16 169 624	56
57	13 767 105	15 703 901	13 489 876	15 941 096	13 208 500	16 173 456	57
58	13 762 519	15 707 893	13 485 220	15 945 008	13 203 776	16 177 288	58
59	13 757 932	15 711 884	13 480 563	15 948 919	13 199 051	16 181 117	59
60	13 753 343	15 715 874	13 475 905	15 952 829	13 194 324	16 184 946	60

PETERS........ *Siebenstellige Werte der Trigonometrischen Funktionen*, Leipzig, 1918 : Reprinted as *Seven-place Values of Trigonometrical Functions* by Van Nostrand, New York, 1942.
The interval is $0^{o}.001$.

PETERS........ *Achtstellige Tafeln*, Leipzig, 1911 : Reprinted Reich Survey Office, Berlin, 1939.
The interval is 1".

PETERS........ *Interpolation and Allied Tables*, H. M. Stationery Office, 1936. Reprinted 1947.

Apart from Barlow's *Tables* and the Interpolation tables, these are mainly used in the preparatory work or for the preparation of *auxiliary tables*, such as *Table III* showing a cos *u* and c sin u in feet with argument φ ; *Table IV* — *h* and Δh ; *Table V* — ρ_0 ρ_1 and ρ_2 ; *Table VI* — η_0 η_1 η_2.

Many of the stages of the systematic calculation are well-suited, either in the present form or slightly modified, for punched-card machines. Generally, these machines would do all the stages of the work at present done on the National machine, more or less as on that machine, and all other systematic computation, with the exception of the inverse interpolation.

A description of the various methods of calculating the co-ordinates of points on the DECCA hyperbolic lattices is given in Section IV.

There are four main methods :

a) the standard method ;
b) the polar method ;
c) the rectangular method ;
d) the direct method.

The Standard Method consists of calculating lane numbers at points on a graticule of longitude and latitude, deducing the intersections of the required lanes with meridians or parallels by inverse interpolation, and preparing copy by subtabulation.

The calculation of lane numbers is performed by using formulae referred to pages 176 and 177 on *International Hydrographic Review*, Vol. XXIV, 1947, and auxiliary table (Table III) giving values in feet of a cos u, c sin u with argument φ.

For the latitudes required, values are taken directly from the Table and differenced to the first or second difference ; on the National machine, each of the differences

$$a \cos u - a \cos u_B, \qquad c \sin u - c \sin u_B$$

is formed for the master (A) and slave (B, C, D) stations by feeding in the precomputed differences. The values of a cos u_B, etc., are found by interpolation in Table III.

For each latitude and station are formed :

$$P_B = 10^{-6} (a \cos u - a \cos u_B)^2 + 10^{-6} (c \sin u - c \sin u_B)^2$$
to two decimals

$$Q_B = 10^{-6} \times 2 a^2 \cos u \cos u_B$$
to the nearest unit.

For each longitude and station are formed :

$$R_B = 10^8 \times (1 - \cos(\lambda - \lambda_B))$$

to two decimals ; R_B is formed (*i*) by direct interpolation from Andoyer's tables, (*ii*) by finding 2 $\sin^2 1/2\ (\lambda - \lambda_B)$ from Peter's tables, or (*iii*) by systematic interpolation on the National machine from the differences of $\cos\ \lambda$ at intervals of 5'.

These values are checked by differencing.

For each latitude, the series of values

$$h_B^2 = 10^6 P_B + Q_B R_B$$

is now built up on the National machine from multiples of the known differences of R_B.

h_B is formed from h_B^2 by extracting the approximate square root to four or five figures by means of Barlow's tables, dividing this value into h_B^2 on a fast automatic calculating machine, and then taking the mean of quotient and divisor. The result should be checked by squaring.

Δh_B is found from a critical table, with argument h_B ; it is always positive. (A copy of the full tables may be obtained on request from H. M. Nautical Almanac Office).

l_B can now be found immediately from formula

$$l_B = k_B\ (r_A - r_B + t_B) \quad \text{in the form :}$$

$$l_B = k_B\ (t_B + h_A + \Delta h_A - h_B - \Delta h_B)$$

The quantity inside the bracket can conveniently be formed on the National machine, leaving the multiplication by k_B to be effected on a fast automatic machine.

It is important to note that all the above operations could be performed with any calculating machine, without calling upon the National.

If punched-card equipment is available, consideration must be given to forming h_B^2 directly from the formula ; r_B would of course be formed directly from h_B^2 by means of a punched-card table with consequent simplification in the just-above formula.

Then for the *Inverse Interpolation* a very careful survey must be made of the meridians and parallels along which it is desired to give the points of intersection with a particular lattice. It is clear that the angle of cut determines the precision with which any intersection can be calculated, and that the lane will be defined equally precisely whatever system of co-ordinates is used ; but it is very much more convenient, for computers and draughtsmen alike, to choose directions which cut the hyperbolae at the largest possible angle. The choice must be guided, however, by the actual requirements for particular charts, and the desirability of preserving uniformity.

The lane numbers are differenced (by hand or on the National machine) in the direction in which inverse interpolation is required, and their consistency is thus checked. In *Interpolation and Allied Tables*, the following notation is used :

Arg. n	*Function.*		*Differences.*		
0	f_0		Δ_0''		Δ_0^{IV}
		$\Delta'_{1/2}$		$\Delta'''_{1/2}$	
1	f_1		Δ_1''		Δ_1^{IV}

Inverse interpolation is performed by finding the interpolating factor, n, such that the corresponding function takes the known value f_n. Now the ordinary modified-Bessel interpolation formula is :

$$f_n = f_0 + n\,\Delta'_{1/2} + B_n''\,(M_0'' + M_1'') + B_n'''\ \Delta_{1/2}''' + \ldots$$

where

$$M'' = \Delta'' - 0.184\,\Delta^{IV}$$

and B_n'' and B_n''' are the Besselian interpolation coefficients. This formula can be used provided Δ^{IV} is less than 1000 ; it is most unlikely that in this case Δ^{IV} is significant.

The solution of this polynomial equation is performed by successive approximations as in Interpolation and Allied Tables, but one machine only is used, leading to greater speed and certainly at the expense of some complication. It is first noted that in inverse interpolation the value f_n required is usually some value ending in zeros. The second-last-above equation is rewritten and solved in the form :

$$f_n - (B_n'''\ \Delta'''_{1/2}) = f_0 + n\,\Delta'_{1/2} + B_n''\,(M_0'' + M_1'') + \ldots$$

The quantity $(M_0'' + M_1'')$, previously formed by hand from the differences, is rounded off by one figure if necessary and checked by differencing.

Details of operation depend on the exact machine available.

The interpolating factor n, corresponding to a predetermined range of lane numbers f_n, is subsequently converted to latitude or longitude as the case may be.

The latitudes and longitudes obtained by inverse interpolation are subtabulated to fifths, tenths or other intervals on the National machine by methods similar to those devised by Dr. L. J. Comrie in : Inverse Interpolation and Scientific Applications of the National Accounting Machine" *Journal of the Royal Statistical Society* (Supplement) Vol. III, 87, 1936.

THE POLAR METHOD.

The latticing of an area near a station can be effected more simply by using the polar and rectangular methods :

s = Rhumb Line Distance from A to P.

S = Rhumb Line Bearing of P from A.

δ_B = Natural lattice co-ordinate

$$= 1 - \frac{1}{k_B\ \alpha_B}\ (l_B - L_B)$$

A = the Azimuth between the meridian and the geodetic joining the Master station to the Slave.

It is possible to find a simplified relationship between s, S and δ by ignoring in the formulae all but the leading terms; this is equivalent to neglecting the variability of scale on the Mercator chart, and also the effects due to the curvature of the Earth. This last-named cause of error is always negligible compared with the first.

By adopting this rather drastic approximation, bearings and distances from the station are taken directly from the Earth's surface and transferred to the chart to be latticed. The lattice lines will then appear as true plane hyperbolae, and can be drawn by a simple geometric method. Alternatively, and preferably, simple tables could be prepared for a number of radii, originating at the station, and the distance along these radii, in minutes of longitude, to correspond with any lane number could be found by looking up the appropriate entry and multiplying by a constant depending on the dimensions of the system for which computations are required.

The relationship between s, S and δ can be expressed in the form :

$$s/K = \rho_0 + (B_1 \rho_1 + B_2 \rho_2) + \dots$$

where

$$K = \frac{\alpha}{a \cos u \sin 1'}$$

$$10^{-2} B_1 = \frac{\alpha \sin \varphi}{a \cos u} \cos A$$

$$10^{-2} B_2 = \frac{\alpha \sin \varphi}{a \cos u} \sin A$$

and

$$\rho_0 = \frac{1 - \delta^2}{2 \ [\delta + \cos (S - A)]}$$

$$10^2 \rho_1 = \frac{1}{2} \rho_0^2 \left[\cos (S - A) - \frac{\sin^2 (S - A)}{\delta + \cos (S - A)} \right]$$

$$10^2 \rho_2 = \frac{1}{2} \rho_0^2 \left[\sin (S - A) + \frac{\sin (S - A) \cos (S - A)}{\delta + \cos (S - A)} \right]$$

The first term ρ_0 has the effect of reproducing distance unaltered on to the Mercator chart : the two second order terms allow the greater part of the variability of scale of the chart near the station to be taken into consideration. B_1 and B_2 are constants for the particular station and chain under consideration, and K is a scale factor so chosen that *s* is expressed in minutes of longitude.

Clearly, for fixed values of S — A, the quantities ρ_0, ρ_1 and ρ_2 can be tabulated as functions of z. Auxiliary tables have been computed for S— A = ± 5° (5°) to ± 135°.

With the aid of these tables, interpolating where necessary, radial distances from the station on the chart, s, can be determined for intersections of any lattice line (specified by z) with fifty-five radii vectores ; if the station is on the chart, distances are measured from it along the radii by means of the longitude scale ; if not, intersections of the radii with convenient meridians or parallels can be calculated from the relations :

$$s \cos S = \Delta M \qquad\qquad s \sin S = \Delta \lambda$$

and distances can be measured from these as false origins. ΔM can be found from $\Delta \varphi$ by using a Meridional Parts table for the appropriate figure of the Earth.

The polar method cannot easily be employed near the base-line extension. In this region the *Rectangular Method* should be applied.

Rectangular axes on the Mercator chart, with origin at the station, are chosen, the direction of the ξ — axis making an angle A with the meridian : the positive direction of the η — axis is $90^\circ + A$. The ξ — axis is not the base-line extension ; the latter is a curve on the Mercator chart and the ξ — axis is the tangent to this curve at the station. The co-ordinates (ξ, η) of any point on the chart can be expressed in terms of its bearing and distance from the station by the following relationships :

$$\xi = \frac{1}{a_0} s \cos (S - A) \qquad\qquad \eta = \frac{1}{a_0} s \sin (S - A)$$

where

$$a_0 = \frac{\alpha}{a \cos u} = K \sin 1'$$

The following expression for η in terms of ξ and δ may be found

$$\eta = \eta_0 (C_1 \eta_1 + C_2 \eta_2) + \dots$$

where

$$10^{-2} C_1 = \frac{1}{2} a_0 \sin \varphi \cos A \qquad\qquad 10^{-2} C_2 = \frac{1}{2} a_0 \sin \varphi \sin A$$

C_1 and C_2 are related to B_1, B_2 by the formulae

$$C_1 = \frac{1}{2} B_1 \qquad\qquad C_2 = \frac{1}{2} B_2$$

Also

$$\eta_0 = \pm (1 - \delta^2)^{\frac{1}{2}} \left[(\xi - \frac{1}{2})^2 / \delta^2 - \frac{1}{4} \right]^{\frac{1}{2}}$$

$$10^2 . \eta_1 = 2 \xi \eta_0 - \frac{1 - \delta^2}{\delta^2} . \frac{\xi - \frac{1}{2}}{\eta_0} (\xi^2 - \eta_0{}^2)$$

$$10^2 \eta_2 = (\xi^2 - \eta_0^2) + \frac{1 - \delta^2}{\delta^2} . \frac{\xi - \frac{1}{2}}{\eta_0} . 2 \xi \eta_0$$

The quantities C_1 and C_2 are constants, and can be precomputed for the particular station and pattern under consideration.

The quantities η_0, η_1 and η_2 are functions of ξ and δ alone, and for given values of ξ, they can be tabulated as functions of $z = 1 - \delta$. The two signs of η_0 and η_1 correspond to the two regions on either side of the base-line extension.

It is possible to cover the base-line extension area quite adequately by giving 13 tables, for $\xi = 0.00$ (—0.05) — 0.60. The three quantities η_0 η_1 and η_2 are tabulated in units of the fifth decimal.

THE DIRECT METHOD.

If the lane numbers for two station pairs at a point are known, the great circle bearing θ and distance r of that point from the master station can be found by means of two pairs of equations each of the form

$$\frac{r}{\alpha} = \frac{(1 - \delta^2)}{2\,(-\delta + \cos(\theta - A - E))}$$

$$E = \frac{1}{6}\,\frac{\alpha\, r}{a^2}\,\sin\,(\theta - A)$$

The latitude and longitude of the point can then be deduced by using equations

$$\Delta\,\varphi = F\frac{r}{a}\left(\cos\theta + f_1\frac{r}{a} + f_2\left(\frac{r}{a}\right)^2 + \ldots\right)$$

$$\Delta\,\lambda = G\frac{r}{a}\sin\theta\left(1 + g_1\frac{r}{a} + g_2\left(\frac{r}{a}\right)^2 + \ldots\right)$$

where

$$F = 3437.75\;\frac{\sin^2\varphi\,\cos\varphi}{\sin^2 u\,\cos u}$$

$$f_1 = -\frac{1}{2}\tan\varphi\;3e^2\cos^2\varphi + \sin^2\theta\left[1 - \frac{1}{2}e^2\,(1 + 5\cos^2\varphi)\right]$$

$$f_2 = -\frac{1}{6}\sin^2\theta\,\cos\theta\,(1 + 3\tan^2\varphi)$$

$$G = 3437.75\,\sec u$$

$$g_1 = \sec u\,\sin\varphi\,\cos\theta$$

$$g_2 = \frac{1}{3}\sec^2 u\,(3 - 2\cos^2\varphi) - (4 - 3\cos^2\varphi)\sin^2\theta$$

Since two lanes of different systems may intersect in two points, there may be an ambiguity: this can, however, usually be resolved by other methods.

The application of the method is limited, since the method of calculation is not adaptable to systematic computation. There is, however, a demand for the direct method; the derivation of isolated geographical positions from DECCA co-ordinates is of value in experimental work in the early stages of setting up a chain of stations.