HISTORICAL NOTE ON THE TREATISE "MODERN NAVIGATION"

by

Captain JOHN B. GRAMBOW, Berlin-Adlershof.

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With the introductory work of Dr. Fritz CONRAD on "Modern Navigation" in Vol. 8-1937, pages 267-272 of the *Seewarte* an urgent appeal has happily been made to all who are interested in navigation, whether mariners, aviators or scientists, to cooperate fully in the further development of navigation; i.e. to discuss and develop the present methods, to propose new methods and to report on the experience gained with new instruments, calculating machines etc. In connection with the above it is immaterial whether these reports are based on the results of practical experience or theoretical investigations, since both contribute in an equal measure towards the attainment of the desired goal.

One problem that now occupies the centre of the stage and towards the solution of which much time and energy has been expended, is the simplification of position finding from two stars in general and the calculation of the altitude in particular.

In this connection the present writer wishes to elaborate the matter somewhat further. We shall not treat here however, such questions as the introduction of the 400 grad division in place of the 360°, whether to employ time or angular measure, decimal division of the degree, grad time-keepers, aeronautical clocks, the Aeronautical edition of the Nautical Almanac etc., all of which have to do more or less with the calculation of the altitude of the heavenly body. These particular questions have all been treated so thoroughly in the work of Dr. FREIESLEBEN that nothing of importance can be added and we have to await the final judgment afforded by practical experience.

On the other hand the questions pertaining directly to the computation of the altitude and the method to be pursued to obtain a simplification is still so acute and urgent that a general discussion of the problem is advisable.

To attain this goal various different and widely divergent paths have been followed. Mathematical, tabular, graphic, nomographical and mechanical methods have been proposed and developed; but up to the present none of these have shown any appreciable simplification over the older methods generally employed by mariners, so that one involuntarily asks whether such simplification in the calculation of altitude is actually possible.

Therefore in this connection it seems of interest to investigate how the altitude computation has developed from the most primitive time up to the present day methods and what hope can be entertained for a further simplification. In order to avoid excessive material we shall deal with the purely mathematical procedure and take the graphical methods into consideration only in so far as they are essential towards the comprehension of the mathematical, reserving the treatment of the graphical, nomographical, tabular and mechanical solutions for another article.

The fact that astronomy was rather highly developed amongst the ancient cultured races, the Chinese, Egyptians and Babylonians is well known. However, we do not know whether they possessed any methods with the aid of which the altitude of a star could be determined by means of the factors φ, δ and t.

It has often been assumed — and writings of some of the ancient Greeks vouch for the fact — that they were aware of graphical methods which permitted a solution of the problem. The nucleus of the method employed in that case probably consisted of the orthogonal projection of the sphere on the three mutually perpendicular planes; the meridian, the vertical and the horizontal circle, as described by the Greek astronomer Claudius PTOLEMAEUS (200 A.D.) in his "Analemma" (i.e. as an auxiliary figure).

The orthographic and also the orthogonal or parallel projection, i.e. that projection in which the eye of the observed is supposed to be at infinity, is used today principally for the representation of the surface of the moon, the sun and the planets, since these can generally be regarded as being at an infinite distance. Also in the older text-books they are often described and used; for instance, in RÜMKERS well known "Handbuch der Schiffahrtskunde" Hamburg, fifth edition of 1850, which we shall have occasion to mention later. For those who are not familiar with the orthographic projection we shall give here the basic equation for its representation, in order to facilitate the comprehension of what follows :

$$x = r.\cos\lambda.\cos\varphi \qquad \qquad (1)$$
$$y = r.\sin\varphi$$

We shall then turn to the methods described by PTOLEMAEUS in his "Analemma" for the determination of h from φ, δ and t.

With a radius equal to r we describe about O the circle S.E.Z.N.Na. which represents the local meridian.

The straight line Z O Na represents the prime vertical and the perpendicular to it through O, or S O N is the horizon. From O N the angle N O P equal to φ is laid off which gives the straight line P O P', or the world axis, and at right angles thereto passing

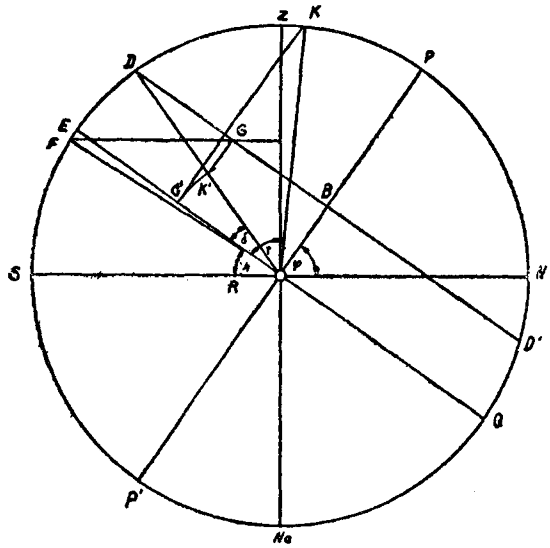

FIG. 1.

through O the celestial equator, E O Q. If we lay off from O to E O the angle E O D equal to δ , then the line drawn through D parallel to E O Q represents the declination parallel. In order to determine the point on this parallel occupied by the star at the instant "t" the following construction is necessary. From E O we lay off the angle E O K equal to "t" and drop a perpendicular from K on the E O Q, which gives the point G'. i.e. the intersection of the hour angle circle with the equator. If from O we describe a circle with radius O G', which intersects the straight line O D at K', then by means of a perpendicular from D D' dropped on to K' the point G is determined, or, the locus of the star at the instant "t". A line drawn through G parallel to S O N intersects the local meridian at F. The angle F O S then gives the required altitude of the star. The proof of the correctness of this construction is easily found from the basic equations representing the projection (1).

The first application of the direct mathematical determination of the altitude from the magnitudes φ, δ and t comes from the Indians. These people, who were as talented in mathematics as the Greeks and were also gifted with an equally good foundation in geometry understood how to derive a mathematical method from the above described graphical method given in the "Analemma".

In the most ancient work from the Indians, now extant, the Surya-Siddhânta, or the "certain Truth, revealed by the sun" (about 400 A.D.) there is treated in verses 34, 35 and 36 the problem of determining the altitude of the sun at any desired hour of the day by means of the declination and the elevation of the pole. The rule reads : "When we augment the radius by the sine of the ascension difference, in the case of north declination, or diminish in the case of southerly declination, then we obtain the measure of the day; this then diminished by the versed sine of the hour angle, then multiplied by the radius of the daily arc and divided by the radius (of the sphere) gives the divisor; the latter is then multiplied by the sine of the complement of the latitude and divided by the radius which gives the sine of the altitude".

This rule can easily be proven by means of the orthogonal projection (Fig. 2).

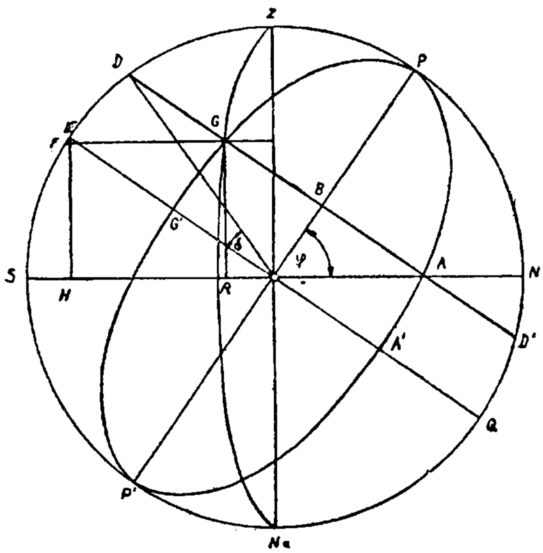

FIG. 2.

By construction we have :—

$$\Delta\ GRA \sim EHO$$

therefore

$$\frac{GR}{AG} = \frac{EH}{OE}$$

further

$$\sin h = GR$$

and also

$$\sin h = \frac{AG \,.\, EH}{OE}$$

As expression for A G we obtain :

$$AG = \frac{A'G' \,.\, BD}{OE}$$

and as we obtain directly from Fig. 1 :—

$$EG' = 1 - \cos t = \text{versin } t$$

and therefore

$$AG = \frac{(\text{versine } t_0 - \text{versine } t)\ BD}{r}$$

from Fig. 2, further

$$E\,H = \sin(90^\circ - \varphi)$$

consequently

$$\sin h = \frac{(\text{versin } t_0 - \text{versin } t) \, . \, \sin(90^0 - \varphi) \, . \, B\,D}{r \, . \, r} \quad (2)$$

which formula corresponds exactly to the reading of the verse quoted. If we put B D equal to cos δ , r equal to 1 and consider that $-\cos t_0 = \text{tang } \varphi \text{ tang } \delta$ then after a simple reduction, we obtain :

$$\sin h = \sin \varphi . \sin \delta + \cos \varphi . \cos \delta . \cos t.$$

This shows that even amongst the anciens Indians there were indications of the beginnings of the science of nautical astronomy (1).

The celebrated Arabian scientist AL-BATTANI devised a similar rule in his work "On the Stars" (about 880 A.D.) with the aid of the projection method which, written in mathematical form is as follows :

$$\sin h = \frac{(\text{versin } t_0 - \text{versin } t) . \sin(90^0 - [\varphi - \delta])}{\text{versin } t_0} \quad (3)$$

which, as well as formula (2), can be directly derived from the Fig. 2.

The same method served AL-BATTANI for the solution of a number of other problems, of which we shall quote one, namely the determination of the sun's azimuth from the declination, altitude and latitude. The rules laid down by him give us the following formula in our notation :

$$\sin(90^\circ - \alpha) = \frac{\dfrac{r . \sin(90^\circ - \delta)}{\sin(90^\circ - \varphi)} - \dfrac{\sin h . \sin \varphi}{\sin(90^\circ - \varphi)}}{\sin(90^\circ - h)} \quad (4)$$

a formula, which with $r = 1$ and the introduction of the cosines passes directly into our spherical cosine formula. Regarding this BRAUNMÜHL (2) : "We have here the most ancient place in the literature available to us in which our second fundamental law of spherical trigonometry appears in its integral form. But we do not believe that we are therefore entitled to acclaim AL-BATTANI as the discoverer of this law, as has been done by others. Then, just as the Indians, AL-BATTANI had no idea that with this projection method he had discovered a trigonometrical law which would be applicable to any desired triangle".

This determination was made only several centuries later by the German scholar Joh. MÜLLER, better known under the name of REGIOMONTAN (1436-1475). The words in which REGIOMONTAN clothed his rule are expressed by the following nautical-astronomical formula which applies to the basic triangle P Z G :

$$\begin{aligned} &\text{versin } t : (\text{versin } z. - \text{versin } [\varphi - \delta]) \\ &= r : \sin(90^\circ - \varphi). \sin(90^\circ - \delta). \end{aligned} \quad (5)$$

It is of interest to know that this same formula was given about 400 years later in a nautical manual. In the above mentioned "Handbuch der Schiffahrtskunde" of RÜMKER we find on page 39 under the methods described, for the calculation of the angle of the spherical triangle from the three sides, the following rule:— From the natural versed

(1) *According to* H. ZEUTHEN, *math. I.* 1900 — p. 20-27 *the mathematical treatment here ascribed to the Indians can be found in principle in the Analemma of Ptolemaeus.*

(2) Von BRAUNMÜHL, *Vorlesungen über Geschichte der Trigonometrie I. TeTil S.* 53 Leipzig 1900.

sine of the side opposite the angle sought, substract the natural versed sine of the difference between the sides bounding the angle sought: to the logarithm of the balance and the log cosecant of the bounding sides, the sum of these three logarithms is the log of the rising time of the angle sought.

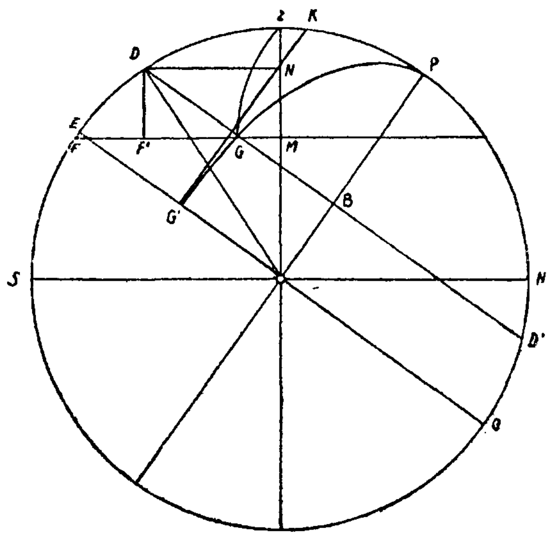

FIG. 3.

In the same manner as REGIOMONTAN, RÜMLER uses the orthographical projection for the proof of this formula as well as other nautical problems given in his Manual. The proof of these rules by means of this method is given below.

In the triangle Z P G (Fig. 3) we have the three given sides Z G, P G, Z P; let Z P G be the angle sought, which is measured by the arc O G'. We shall imagine the semi-circle E O Q, which is perpendicular to E P Q, turned through 90° about the common axis E Q so that it coincides with the latter; then G' will fall at K and KG' becomes the sine, G' E the versine of the arc EK = arc G' E = angle E P G'.

$$\sphericalangle\ F'DG = \sphericalangle\ EOZ = \text{arc}\ ZE = 90^\circ - PZ$$

$$BD = \sin PD = \sin PG$$

$$DF' = ZM - ZN = \text{versin}\ ZC - \text{versin}\ (ZD = PG - ZP)$$

$$\text{rad} : DF' = \sec F'DG : DG$$

$$BD : \text{rad} = DG : EG'$$

$$BD : DF' = \sec F'DG : EG'.$$

from this it follows by the substitution of the above values :

$$\sin PG : \text{versin}\ ZG - \text{versin}\ (PG - PZ)$$

$$= \text{cosec}\ PZ : \text{versin}\ ZPG$$

or :

$$\text{versin}\ ZPG = \tag{6}$$

$$(\text{versin}\ ZG - \text{versin}\ [PG - PZ].\ \text{cosec}\ PG.\ \text{cosec}\ PZ$$

which, as may readily be seen, is identical with the REGIOMONTAN formula.

By introducing the relations in the basic nautical astronomical triangle and by transformation of the equation we obtain :

$$\text{versin}\ z = \text{versin}\ (\varphi - \delta) + \cos\varphi .\ \cos\delta .\ \text{versin}\ t$$

or :

$$\cos \varphi . \cos \delta . \text{ versin } t = D\ F'$$
$$\text{versin } (\varphi - \delta) + D\ F' = \text{covers } h$$

in which form it appears on page 188 in the RÜMLER Manual and is recommended for use for obtaining the altitude. We see that it is practically identical with the formula used at sea today viz.:—

$$\begin{aligned} \text{hav } y &= \cos \varphi . \cos \delta . \text{ hav } t \\ \text{hav } z &= \text{hav } (\varphi - \delta) + \text{hav } y \end{aligned} \tag{7}$$

From this it is evident that the most modern equation used for the determination of the altitude of the heavenly body can be traced back 500 years. It is noteworthy that REGIOMONTAN used his formula simply for the determination of the angle from the three sides of the spherical triangle. For the calculation of the altitude from the magnitudes φ, δ and t (according to Prof. A. WEDEMEYER (3)) he made use of a formula which in accordance with present-day terms would be expressed as follows :—

$$\begin{aligned} \tan x &= \cos t . \text{ cotan } \varphi \\ \sin h &= \sin \varphi \quad \sec x . \cos (d - x) \\ d &= (90^\circ - \delta). \end{aligned} \tag{8}$$

If we investigate the previously known rules somewhat closer we find they are apparently not much more complicated in form and in content are about as comprehensive as our modern formulae. In those days the calculations were much more laborious and difficult since there were no logarithm tables or calculating machines available as auxiliaries and every multiplication and division had to be carried out in full as written. No wonder therefore that even in those days much trouble and pains were taken to discover some simpler solution.

The first efforts to obtain this were due to the Nüremburg pastor Johannes WERNER (1468-1528) through the invention of the so-called "prosthapharetic method" (4). This peculiar word, formed from the Greek words meaning to give to and to take away, ,means an addition and subtraction method:— It consists essentially in the use of the trigonometrical formulae :

$$\sin \alpha . \sin \beta = 1/2 \left(\cos [\alpha - \beta] - \cos (\alpha + \beta]\right)$$
$$\cos \alpha . \cos \beta = 1/2 \left(\cos [\alpha - \beta] + \cos (\alpha + \beta]\right)$$

If the product of two numbers (less than 1) is to be found, then from the trigonometrical tables of these numbers we find the angles α and β of which they were the sines or the cosines. Then from the sum or the difference of these angles one has again to find the equivalent cosines and to divide the sum (or difference) by two.

The process is much more cumbersome than the logarithmic calculation, but it is based on the same fundamental idea and is replaced today by the concept of trigonometrical functions as exponential functions with imaginary exponents for the logarithms with which it stands in direct relation.

With the aid of the "prosthapharetic method" the REGIOMONTAN cosine formula was transformed into the following form by WERNER.

$$\begin{aligned} \cos z &= \sin (90^\circ - \varphi + \delta) \\ &+ \text{versin } t \left(1/2 \sin [90 - \varphi + \delta] + 1/2 \sin [90 - \varphi - \delta] \right. \end{aligned} \tag{9}$$

where, with respect to the REGIOMONTAN formula there is only one place requiring three multiplication processes.

In order to avoid this last multiplication, the Swiss mathematician Jobst BÜRGI (1590) proposed the following :

$$\begin{aligned} \sin h = 1/2 \left(\cos[\varphi - \delta] - \cos [\varphi + \delta] + 1/2 [\cos [x - t] \right. \\ \left. + \cos [x + t]]\right) \end{aligned} \tag{10}$$

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(3) A. WEDEMEYER, "*Zur Höhenberechnung.*", *Ann. d. Hydr.*, 1903, S. 366.

(4) Von BRAUNMÜHL, "*Beitrag zur Geschichte der prosthaphäretischen Methode*". Bibliotheca Mathem. 1896, p. 105-108.

addition and substraction method:— It consists essentially in the use of the trigonometrical from which the auxiliary angle x is calculated from the following :—

$$\cos x = 1/2\ (\cos [\varphi - \delta] + \cos [\varphi + \delta])$$

The Italian mathematician, Giovanni Antonio MAGINI in 1609 gave the following form to the basic equation in his work which appeared in Bologna known as "*Primum mobile duo decim libris contentum*":

$$\begin{aligned} \text{versin } z &= \text{versin } (\varphi - \delta) \\ &+ \text{versin } t\ (1/2 \cos [\varphi - \delta] + \cos [\varphi + \delta]]) \end{aligned} \tag{11}$$

While the mathematician Bartholomacus PITISCUS, born in Silesia wrote the equation in his "Trigonometria" in 1900 as follows :—

$$\cos z = \cos (\varphi - \delta) + \text{versin } t\ (1/2\ [\cos [\varphi - \delta] + \cos [\varphi + \delta]]) \tag{12}$$

which agrees with the form already given it by WERNER.

Even after the invention of logarithms the prosthapharetic method remained in use for quite a long time and even in the nautical manuals the cosine law appeared frequently in its prosthapharetical form. Thus, for instance in 1903 there appeared in the Italian Maritime Journal "Rivista Marittima" the following formula by PESCI :—

$$\begin{aligned} \cos z = 1/2\ (\cos [\varphi - \delta] - \cos [\varphi + \delta] + \cos [\varphi - \delta] \\ + \cos [\varphi + \delta]) \cos t. \end{aligned} \tag{13}$$

A particularly elegant form was developed by EULER in the "Mémoires de l'Académie Royale des Sciences et Belles Lettres", Vol. IX, page 253, Berlin 1753. He writes :

$$\begin{aligned} \cos z = 1/2 \cos (\varphi - \delta) - 1/2 \cos (\varphi + \delta) + 1/2 \cos (\varphi + \delta + t) \\ + 1/4 \cos (\varphi + \delta - t) + 1/4 \cos (\varphi - \delta + t) + 1/2 \cos (\varphi - \delta - t) \end{aligned} \tag{14}$$

In this case, as with the BURGI formula, all multiplication is avoided.

In the form to which we are accustomed today the cosine law is first found written by the great French mathematician of the 16th century Franciscus VIETA. The rule which he proposed is written in our present day notation as follows (5):

$$(\cos \varphi . \cos \delta) : (\sin h - \sin \varphi . \sin \delta) = 1 : \cos t^{[5]} \tag{15}$$

Exactly as did REGIOMONTAN, VIETA used his formula for the calculation of the angle from the three given sides of the spherical triangle. For the calculation of the altitude he made use of, a new formula :

$$(\sec \varphi . \sec \delta) : (\cos t + \tan \varphi . \tan \delta) = 1 : \sin h. \tag{16}$$

which is readily derived from equation (15):

Unfortunately VIETA did not give any derivation of these formulae, but it may be assumed that with the aid of the stereometrical method invented by Nicolaus COPERNICUS, i.e. with the aid of the solid angles, as is done today, this result was achieved.

The 17th and the first half of the 18th century brought little that was new towards the development of the formula for altitude. The rules are given in general in the form of short sentences or analogies, although many scholars sought to introduce in these formulae a formal method of notation for the sake of brevity. This was done at one stroke however as soon as that most prolific of all mathematicians EULER began to busy himself with trigonometry. Even as early as 1729, in his treatise "Solutio prolematis astronomici etc." Comm. Ac. Petr. IV. EULER wrote the cosine formula for the triangle A B C as follows :—

$$\cos BC = \cos AB . \cos AC + \cos A . s AB . s AC$$

in a manner which no previous mathematician had adopted. In this formula there is only lacking the more convenient designation of the sides and the abbreviation "sin" in place of "s".

(5) *Opera,* p. 407, N° XV.

Twenty four years later, in the paper which EULER presented to the Berlin Academy entitled "Principes de la Trigonométrie sphérique, tirés de la méthode des plus grands et plus petits" 1753; there appeared all at once all the formulae, proportions etc. in the notation in use today. Here there appeared for the first time the basic formula which is so important for all nautical purposes in its usual form :

$$\sin h = \sin \varphi . \sin \delta + \cos \varphi . \cos \delta . \cos t$$

which has been preserved until today and which will continue to be used.

A development of more than one thousand years lies behind us in this formula. We have seen how the form of the equation was changed when the effort was made to avoid multiplication. With the invention of the logarithms multiplication lost its terrors. From then on efforts were directed towards giving the equation a form which would make the logarithmic process applicable and practical. These transformations and similar efforts to obtain greater simplification in the calculation of the altitude, will be treated later on.

Considerable progress was made in the simplification of the numerical calculations with the discovery of logarithms by John NEPER (Lord Napier of Merchiston 1550-1617) whose numerical logarithm Tables "Mirifici logarithmorum canonis descriptio" appeared in 1614 in Edinburgh. A few years earlier the Swiss mathematician Jobst BURGI, or BYRGI, (1552-1632), who lived mostly in Kassel, had prepared similar tables for his own personal use, but these were first published in Prag in 1620 under the title "Arithmetische und Geometrische Progresstabulen, sambt gründlichem unterricht, wie solche nützlich in allerley Rechnungen zu gebrauchen und verstanden werden sol". Napier's friend, Henry BRIGGS (1556-1630) improved the logarithms considerably by giving them the base ten of the decimal system. BRIGGS and his successor Adrian VLACQ calculated the logarithms of all numbers from 1 to 100,000 to ten and fourteen decimal places. (6)

With this discovery of logarithms the practical calculations involving multiplication and division received their final form. Although before this time all efforts had been directed towards the avoidance of all multiplication and division and the formulae transformed in such a manner that only addition and substraction need be carried out, as evidenced by the development of the prosthapharetic formula mentioned above, now however the idea was to transform the equation in such a manner that the plus and minus factors no longer appeared, and the equations thus be better adapted for logarithmic calculations.

In order that the altitude of a star should be determined by purely logarithmic calculations a transformation of the basic equation

$$\sin h = \sin \varphi . \sin \delta + \cos \varphi . \cos \delta . \cos t \qquad (17)$$

is not absolutely essential, since by the partition of the basic astronomical triangle into two right triangles a set of formulae can be derived which permit of pure logarithmic calculation. This is exemplified by the two systems which were already known to REGIOMONTAN:—

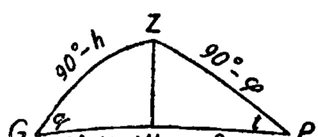

FIG. 4.

$$\left.\begin{aligned} \tan\theta &= \cot\varphi . \cos t \\ \sin h &= \sin\varphi . \sec\theta . \sin(\delta+\theta) \end{aligned}\right\} \qquad (18)$$

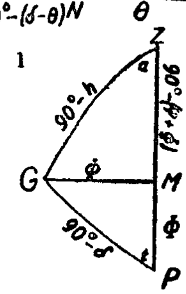

FIG. 5.

$$\left.\begin{aligned} \tan\theta &= \cot\delta . \cos t . \\ \sin h &= \sin\delta . \sec\Phi . \sin(\varphi+\Phi) \end{aligned}\right\} \qquad (19)$$

which can be deduced directly from the Fig. 4 and Fig. 5 reproduced herewith.

(6) H. BRIGGS, "*Logarithmorum Chilias prima*", *London* 1617; "*Arithmethica Logarithmica*", 1624 (14 place), *completed by the* 2nd. *Edition, Gouda* 1628 *of Vlacq* (10 place tables).

However, the calculation for the altitude of a star did not become of practical importance in applied navigation until towards the end of the 18th century when the tables of lunar distances finally attained sufficient accuracy and the sextants had been improved to such an extent that the measurement of lunar distances was feasible and an accuracy to within several minutes of longitude was obtainable. (7) Therefore it is not surprising that since that time the text-books on navigation have devoted considerable space to the calculation of the altitude of a star. In practice equations 18 and 19 described above have enjoyed great popularity and for a long time they served as the principal formulae. For instance, the manual on navigation brought out by the Imperial German Navy in 1879 (1st Edition, 1881, 2nd. Edition 1881 and 3rd. Edition 1891) contains simply the altitude formulae of group 18, which however is not obtained by the decomposition of the basic triangle but is deduced by analytical means from the basic equation, as shown by Euler (8).

In the reduction of the lunar distances the calculation of the altitude plays a subordinate role, since it can often be avoided by observation of the star's altitude. The importance of altitude calculations of the star first came to be realized when A. Blond de Marcq St. Hilaire, invented the line of position method which has been named after him. In this treatise entitled "Calcul du point observé" in the Revue Maritime et Coloniale, of July 1875, in which he expounded his method in more detail, on page 346, regarding the calculation of the altitude he recommends equation (19) supplemented by a third equation for the calculation of the azimuth. Viz :—

$$\begin{aligned} \tan \Phi &= \operatorname{cotan} \delta . \cos t \\ \sin h &= \sin \delta . \sec \Phi . \sin (\varphi + \Phi) \\ \cos a &= \tan h . \operatorname{cotan} (\varphi + \Phi) \end{aligned} \tag{20}$$

On page 363 he introduces also another formula for altitude

$$\sin h = \cos (\varphi - \delta) - \cos \varphi . \cos \delta . \operatorname{versin} t.$$

which we shall have to examine later in more detail. The system (20) was widely adopted in practice in America and Holland. In the Netherlands it is known as the "Netherlands Φ formula" and is written :—

$$\begin{aligned} \tan \Phi &= \operatorname{cotan} \delta . \cos t \\ \sin h &= \sin \delta . \sec \Phi . \sin (\varphi + \Phi) \\ \operatorname{cotan} a &= \operatorname{cotan} . t . \cos (\varphi + \Phi) . \operatorname{cosec} \Phi . \end{aligned} \tag{21}$$

As may be seen they employ a somewhat different formula for the azimuth computation. The same notation is employed in the American text-books such as J.H.C. Coffins, "Navigation and Nautical Astronomy" New York, 1890 and Muirs "A Treatise on Navigation and Nautical Astronomy" Annapolis 1911.

In the work of Villarceau et Aved de Magnac "Nouvelle astronomie Nautique" Paris 1877, the following group of formulae is given for the altitude computation.

$$\begin{aligned} \tan \Phi &= \operatorname{cotan} . \delta . \cos t \\ \tan a &= \tan t . \sec (\varphi + \Phi) . \sin \Phi \\ \tan h &= \tan (\varphi + \Phi) . \cos a \end{aligned} \tag{22}$$

which is used principally in the nautical manuals of France, Italy and Germany. The well-known text-book of Albrecht und Vierow (9) still retains this group of formula in its tenth edition brought out in Berlin in 1913.

In recent times this group of formulae has been recommended to aviators with a somewhat different notation. For instance, in the "Aeronautischen Tafeln zur astronomischen Ortsbestimmung", 1934, by H. Gadow which however, has not yet appeared in print, the

(7) *A table of pre-calculated lunar distances was first included in the Nautical Almanac of* 1767.

(8) Euler, *Hist. Mem. Ac.*, Berlin 1753.

(9) "*Lehrbuch der Navigation und ihrer Mathematischen Hilfswissenschaften*".

group of formulae is given as follows :—

$$\begin{aligned} \text{cotan}\ \Phi &= \text{cotan}\ \delta .\ \cos t. \\ \text{cotan}\ a &= \text{cotan}\ t.\ \sec \Phi .\ \sin b \\ \text{cotan}\ z &= \text{cotan}.\ b.\ \cos a \\ b &= \Phi' - \varphi \end{aligned} \qquad (23)$$

The "Aeronautische Tafel" 1935 which is being brought out by the Deutsche Seewarte, at the instigation of the Reichsluftfahrtministeriums (Reich Air Ministry) employs another transformation of the formula viz :—

$$\begin{aligned} \text{cotan}\ y &= -\ \text{cotan}\ \delta .\ \cos \tau \\ \text{cotan}\ a &= \text{cotan}\ \tau .\ \cos Y.\ \sec y \\ \text{cotan}\ h &= -\ \text{cotan}\ Y.\ \sec a. \end{aligned} \qquad (24)$$

Where $Y = 90 - \varphi + y$ when φ and δ are of the same name and $Y = 90° - \varphi - y$ when φ and δ are of opposite names. The systems (22) to (24) are very well adapted for the calculation of altitude and azimuth. An investigation which I made regarding the accuracy of system (24) showed that the calculations carried out with four-place logarithms would always yield an accuracy within 2'. The disadvantage is that the rules for the signs must be carefully regarded and that when near 90° an interpolation is necessary for y and a respectively to guarantee the accuracy of 2'. However, this interpolation can be made at a glance and consists in the fact that one takes the log sec. differently by as much as the log cot. differs from the next value given in the tables.

The American Naval Lieutenant A. Ageton (10) believes that by the use of the system:

$$\begin{aligned} \text{cosec}\ m &= \text{cosec}\ t.\ \sec \delta . \\ \text{cosec}\ y &= \text{cosec}\ \delta : \sec h \\ \text{cosec}\ h &= \sec (y - \varphi).\ \sec m. \\ \text{cosec}\ a &= \text{cosec}\ m : \sec h \end{aligned} \qquad (25)$$

the calculations for altitude and azimuth can be simplified, and he published the corresponding tables. An investigation of the accuracy showed however that this system is somewhat unfavorable since it is necessary to employ six-place logarithms if an accuracy of 2' is to be obtained under all conditions. Further, interpolations are necessary as soon as y approaches 90°.

However, with systems which had been derived up to now by a decomposition of the basic triangle into two right spherical triangles to facilitate the logarithmic solution of the altitude calculation, neither mathematicians, mariners nor astronomers were satisfied. The tendency was rather towards a transformation of the basic equation which would permit a purely logarithmic solution. This was obtained by the introduction of auxiliary factors. Several of these transformations, which are of historical interest will be examined here.

Johann Heinrich Lambert (11) a well known mathematician of the time of Frederick the Great, who always had in mind the practical application of a formula obtained a particularly beautiful transformation of the basic equation, by putting

$$\begin{aligned} \sin h &= \sin \varphi .\ \sin \delta + \cos \varphi .\ \cos \delta .\ \cos t. \\ \cos t &= 1 - 2 \sin^2 t/2 \end{aligned}$$

and

$$\sin^2 \theta/2 = \frac{\cos (\varphi - \delta)}{2 \cos \varphi .\ \cos \delta .}$$

(10) A. Ageton, "*Dead Reckoning Altitude and Azimuth Tables*", H.O. 211, Wash. 1934.

(11) J.H. Lambert, "*Beiträge zur Mathematik*", 1765, p. 415-417.

whereby he obtained the elegant formula :

$$\sin h = 2 . \cos \varphi . \cos \delta . \sin 1/2 (\theta + t) . 1/2 \sin (\theta - t) \qquad (26)$$

A somewhat different transformation was given by W. GROSWELL, instructor in navigation, in a rule printed by the American Academy of Arts and Sciences, II, 1780, published 1793, page 18-20.

In his work entitled "Cose trigonometriche", 1786, Antonio COGNOLI gives the formula :

$$\sin z/2 = \frac{\sin 1/2 (\varphi - \delta)}{\cos \theta} \qquad (27)$$

where

$$tg^2 \theta/2 = \frac{\cos \varphi . \cos \delta . \sin^2 t/2}{\sin^2 1/2 (\varphi - \delta)}$$

and

$$\sin z/2 = \sin 1/2 (\varphi - \delta) . \sqrt{1 + \frac{\cos \varphi . \cos \delta . \sin^2 t/2}{\sin^2 1/2 (\varphi - \delta)}}$$

which is also found in a large number of the older text-books. BOLTE also recommends it in his manual entitled "Neuen Handbuch der Schiffahrtskunde", Hamburg 1899, but puts however

$$\tan^2 w = \frac{\cos \varphi . \cos \delta . \sin^2 t/2}{\sin^2 1/2 (\varphi - \delta)} \qquad (28)$$

$$\sin^2 z/2 = \sin^2 1/2 (\varphi - \delta) . \sec^2 w \qquad \text{for } w < 45^0$$

$$\sin^2 z/2 = \sin^2 1/2 (\varphi - \delta) . \tan^2 w . \operatorname{cosec}^2 w . \qquad \text{for } w > 45$$

In RIDDLES "Treatise on Navigation", 1842 the formula :

$$\cos z/2 = \cos 1/2 (\varphi - x) . \cos x$$

where : (29)

$$\sin x = \sec 1/2 (\varphi - \delta) . \sqrt{\cos \varphi . \cos \delta . \sin^2 t/2}$$

is recommended for the altitude computation. It may readily be seen however that this formula is poorly chosen since the factor z/2 is determined by the cosine, which function changes very slowly at small angles.

RÜMKER in his "Handbuch der Schiffahrtskunde" 1844, gives the formula :

$$\sin h = 2 . \sin 1/2 [x + (\varphi - \delta)] . \sin 1/2 [x - (\varphi - \delta)] \qquad (30)$$

$$\cos x = \cos \varphi . \cos \delta . \operatorname{versin} t .$$

A similar equation is found in BREUSINGS "Kleine Steuermannskunst", Bremen 1852 :

$$\sin^2 z/2 = \cos [1/2 (\varphi + \delta) + x] . \cos [1/2 . (\varphi + \delta) - x]$$

where (31)

$$\sin x = \cos t/2 . \sqrt{\cos \delta . \cos \varphi .}$$

and in RAPERS, "Practice of Navigation" 1840 in which the altitude formula is written :

$$\text{hav. } z = \cos 1/2 (\varphi + \delta + x) . \cos 1/2 (\varphi + \delta - x)$$

$$\text{hav. } x = \cos \varphi . \cos \delta . \text{hav } t' \qquad (32)$$

$$t' = (12^h - t).$$

Among the altitude formulae contained in the older text-books there is one which must be mentioned, not because it allows of easy calculation by logarithms but because it was widely used in practice. We mean the DOUWES formula :

$$\sin h = \cos (\varphi - \delta) - \cos \varphi . \cos \delta . \operatorname{versin} t \qquad (33)$$

In his treatise: "Verhandling om buiten den Middag op zee de waare middags Breedte te vinden" Haarlem 1747, Cornelius DOUWES, who was appointed by the Statthalter of the Netherlands, Willem IV, Head of the newly founded "Zeemannscollege", gives a very clear derivation of this equation from the orthographic projection of the sphere.

In the Fig. 6, let Q Q' represent the equator, D D' a parallel of declination, D the position of the star at noon and G the position of the star at the time t. F G a parallel of altitude. By construction we have

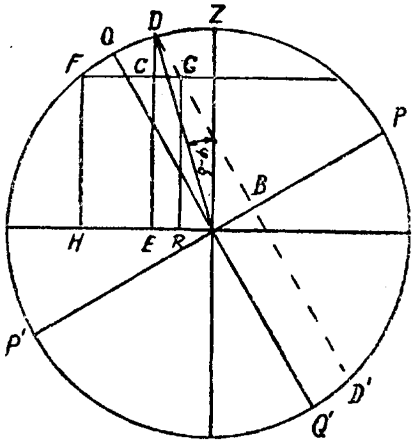

FIG. 6.

$$FH = GR = \sin h$$

and

$$FH = DE - DC.$$

Further, as may be seen from Fig. 6,

$$DE = \cos (\varphi - \delta)$$

and therefore :

$$\sin h = \cos (\varphi - \delta) - DC. \qquad (a)$$

By construction, further

$$DB = \cos \delta.$$

$$GB = \cos \delta. \quad \cos t,$$

therefore

$$DG = DB - GB$$

$$= \cos \delta. \text{ versin } t$$

But as may be seen from the Fig., we have also

$$DC = DG \,.\, \cos \varphi$$

and therefore

$$D\,C = \cos \varphi. \cos \delta. \text{ versin } t$$

By substituting this in formula (a) we obtain

$$\sin h = \cos (\varphi - \delta) - \cos \varphi. \cos \delta. \text{ versin } t$$

The derivation of a formula by means of the projection method which, as was shown in the first part of this treatise, was known to the Indians (about 400 A.D.) and probably also to the ancient Greeks, appears to us today a trifle cumbersome. However, in most manuals of navigation it was retained even long after the Euler principles had become known through his work "Principes de la Trigonométrie", Berlin, 1753 which gave the basis of the analytic treatment of the spherical trigonometrical problems, because of the clarity of the demonstration.

It is noteworthy also that in his writings where he solves the nautical problems DOUWES makes use of five-place logarithms. The log versine t he calls the "logarithmus rijzing" (Logarithmus Steigezeit), (logarithm of the rising time) a designation which has been retained in English text-books on nautical astronomy to the present day. (12)

(12) J.W. NORIE, "*A complete set of Nautical Tables*", London 1920.

Amongst the earliest known trigonometrical functions the versine was known to mathematicians as well as the sine. It is encountered in the most ancient work of the Indians now extant, namely the Sûrya-Siddhânta (about 400 A.D.) and the somewhat later work of *Arjabhata* (560 A.D.) who calculated a table of these functions. The Indians, who called the sine " jya " or " jiva " (chord) designated the versine as " utkramajya " or " uttrmadjya ", which means the reversed sine. The Arabs used the designation " sahem " (arrow) for this function, from which we have the latin " sagitta ", as the versine is known today in many text-books on trigonometry. According to R. WOLF, " Handbuch der Astronomie, ihre Geschichte und Litteratur ", Zürich, 1890-1893, the designation versine was introduced by *Apian*. Today this designation has disappeared and in its place we find the haversine (hav. x = 1/2 versine x).

The Douwes formula is well adapted for the calculation of the altitude. With the use of a five-place logarithm table it gives an accuracy of 2' without any interpolation from altitude to altitude up to 80°. It may be found in most text-books both in Germany and in most foreign countries. Albrecht und Vierow give it even as late as the tenth edition of their text-book in 1913. Further, a great number of nautical tables adapted to this particular formula have been published. Amongst others we cite here only the " Taboa Polytelica " of Jose Nunez da Matta, Lisbon 1906 and the " Zeevaartkundige Tafelen " by R. Peaux, Rotterdam 1912.

In recent times however, it has been replaced by other formulae, such as the so-called naval formula :

$$\begin{aligned} \text{hav. } x &= \cos\varphi \,.\, \cos\delta \,.\, \sec(\varphi-\delta)\ \text{hav } t \\ \sin h &= \cos(\varphi-\delta) \,.\, \cos t \end{aligned} \tag{34}$$

taken from the Bolte system (28) and from the formula :

$$\begin{aligned} \text{hav. } y &= \cos\varphi \,.\, \cos\delta \,.\, \text{hav. } t \\ \text{hav. } z &= \text{hav. } (\varphi-\delta) + \text{hav. } y. \end{aligned} \tag{35}$$

As I have already shown in the first part of this work, this formula (35) is to be found in the work of REGIOMONTAN and was first taken up again in the last two decades. Albrecht und Vierow bring it out first in their 1925 edition. In *Breusings* " Steuermannskunst ", it was first given in the 1932 edition. The English and American Mariners appear to have made use of it somewhat earlier, since in the tables published by Johnson in 1900, entitled " Short Tables and Rules for finding Latitude and Longitude " we encounter it in the form :

$$\begin{aligned} \text{versin } \theta &= \text{versin } t : \sec\varphi \,.\, \sec\delta \\ \text{versin } z &= \text{versin } (\varphi-\delta) + \text{versin } \theta \end{aligned} \tag{36}$$

Tables which are specially adapted for the solution of formula (34) are those of Davis, " Requisite Tables ", London 1905; those of G. PES, " La Rette di posizione ", Genoa 1921, the Tables of Yonemura, Tokio 1920, the Fulst Tables, widely used in Germany and the " Altitude Azimuth and Line of Position Tables ", H.O. N° 200, Washington, 1st Edition, 1913. All witness to the wellnigh international acceptance of this formula.

Professor WEDEMEYER called my attention to a similar formula which can be derived by a transformation of the Douwes formula. If we put

$$\begin{aligned} \sin h &= \cos(\varphi-\delta) - \cos\varphi \,.\, \cos\delta \,.\, \text{versin } t \\ \cos x &= \cos\varphi \,.\, \cos\delta \,.\, \text{versin } t \end{aligned}$$

then we have

$$\sin h = \cos(\varphi-\delta) - \cos x, \tag{37}$$

which gives a formula coinciding with the hav z formula in so far as form is concerned. I hope very shortly to be able to publish in *der Seewart* an investigation on the accuracy of this formula.

Another formula which has attained a certain importance in nautical astronomy, is the following, derived from the Delambre equation :

$$\sin^2 z/2 = \sin^2 1/2\,(\varphi-\delta).\ \cos^2 t/2 + \cos^2 1/2\,(\varphi+\delta).\ \sin^2 t/2, \tag{38}$$

to which WEDEMEYER called attention in 1903 on account of its great arithmetrical accuracy in calculation. From this time onwards there appeared quite a number of different arithmetrical methods in the various countries, based on the above cited equation. Thus we have the "Höhentafel" of SOEKEN in 1914, the "Taboas de Altura Para Calculo Da Recta Marcq. St. Hilaire" of BRAGA in 1924 which, in their arrangement and content, are very similar to the Soeken Tables. Further TEEGE in 1919 brought out a table (13) which was supposed to facilitate the solution of this equation, as did also CARIC in 1932 (14). Both make use of the addition logarithms. If in equation (38) we put the greater of the products on the right hand side equal to m, and the smaller equal to n, then we have

$$\sin^2 z/2 = m + n = m\left(1 + \frac{n}{m}\right)$$

or for :

$$\tan^2 x/2 = \frac{n}{m}$$

$$\sin^2 z/2 = m\,(1 + \tan x/2) = m.\sec^2 x/2.$$

The Teegesch Tables contain the four-place logarithms of sin $x/2$; cos $X/2$ from 0° to 180° and of $\sec^2 x/2$ and $\tan g^2 x/2$ from 0° to 90° progressively from minute to minute. CARIC uses the same method but writes :

$$\sin^2 z/2 = m + n = m\left(1 + \frac{1}{\frac{m}{n}}\right)$$

or :

$$\log^2 \sin z/2 = \log m + A,$$

i.e. here one employs the difference

$$\log m - \log n = B.$$

as argument to enter the addition logarithms A given on the page and a half of the table.

In England and America it appears that the work of WEDEMEYER has remained unknown, since as late as 1910, H.B. GOODWIN published in the Nautical Magazine and in the "Proceedings of the U.S. Naval Institute" an article entitled "The Haversine in Nautical Astronomy" in which the formula

$$\text{hav } z = \text{hav }(p - c) + [\text{hav }(p + c) - \text{hav }(p - c)]\,.\,\text{hav } t \qquad (39)$$

was deduced. In this GOODWIN thought that he had discovered a new altitude formula but, as may readily be seen, it is identical with the Delambre formula (38). Only a year later did GOODWIN come upon the identical Delambre formula in the equation

$$\text{hav } z = \text{hav }(p - c)\,.\,(1 - \text{hav } t) + \text{hav }(p + c)\text{ hav } t \qquad (40)$$

He wrote this discovery to the well-known Italian navigator Capt. ALESSIO who discussed it and praised it highly in an article published in November 1921 in the "Revista Marittima". In the year 1926 in the Nautical Magazine we find quite a number of articles dealing with the various methods of solving equation (40), among them being the proposed use of the addition logarithms. A nomogram, based on this formula was published by LITTLEHALES in the "Proceedings of the U.S. Naval Institute" in 1917. LITTLEHALES had the following to say about this formula :— "Because this formula (39) if successfully represented in graphical form, might provide the aerial navigator with the equivalent of a volume of Nautical Astronomy in a form simple enough to fulfil the instant needs of flight". This judgment is somewhat exaggerated because the nomogram on the scale drawn by LITTLEHALES was too small to permit of the accuracy desired for anything like exact navigation.

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(13) TEEGE, "*Vierstellige logarithmische Tafel zur Berechnung der Höhe eines Gestirns*", Reichs-Marine-Armt., 1919.

(14) CARIC, "*Tavole Nautiche*", Cattaro, Jugoslavia, 1924.

For the determination of the position by the altitude method it is not absolutely essential that the altiude at the dead-reckoning position in question be known. It is much more essential that the difference between the observed and computed altitudes at the D.R. position be known. In practice this will hardly ever exceed 30'. In the altitude problem therefore it is a question in reality of the calculation of not more than 30 different positive or negative numbers. FULST has shown in the Annalen der Hydrographie 1900 page 320 how this difference can be found without resorting to the calculation of the altitude proper. Later formulae were published by REUTER, WEDEMEYER, which also showed the manner of calculating this altitude difference directly. Several of these formulae are given herewith :—

FULST, Annalen der Hydrographie, 1900, page 320

$$S = \operatorname{colog} \cos^2 t/2 + \log \sec \varphi + \log \sec \delta + \log \cos 1/2\,(z_1 + \varphi + \delta) + \log \cos 1/2\,(z_1 - \varphi - \delta)$$

$$dh = \frac{S}{m}$$

in which $2\,m = D \pm d$, D is the minute difference of the log cos 1/2 $(z_1 + \varphi + \delta)$, d is that of log cos 1/2 $(z_1 - \varphi - \delta)$ The upper sign is valid when $z > \varphi + \delta$ and the lower when $z < \varphi + \delta$.

REUTER : Annalen der Hydrographie, 1902, p. 37

$$dh = \frac{2.\cos\varphi.\cos\delta.\sin^2 t/2 - 2.\sin(z_1 + \varphi - \delta).\sin(z_1 - \varphi + \delta)}{\sin z_1.\sin 1'}$$

WEDEMEYER : Annalen der Hydrographie, 1902, page 399.

$$dz' = k'\,d\,(\log \operatorname{hav} t).\sin 1/2\,(z + \varphi + \delta).\sin 1/2\,(z - \varphi + \delta).\operatorname{cosec} z,$$

where

$$k' = \frac{2.\sin 1'}{\text{mod}}$$

TEEGE : Annalen der Hydrographie, 1903, page 154.

$$dh = \frac{2.\cos\varphi.\cos\delta.\cos^2 t/2 + 2.\cos 1/2\,(z_1+\varphi+\delta)\cos 1/2\,(z_1-\varphi-\delta)}{\sin z_1.\sin 1'}$$

None of the above formulae have been adopted in the practice of navigation. The same may be said of the calculation of the altitude by means of the Mercator functions, as has been repeatedly proposed by mariners at the end of the last century and the beginning of the present century. The mercator function is defined by

$$f(\varphi) = \rho.y\,.$$

where $y = \int_0^{\varphi} \frac{d\varphi}{\cos\varphi} = 1\,n \tan(45^\circ + \varphi/2)$

and ρ is equal to the arc of a circle which is equal to the semi-diameter

or : $$\rho = \frac{180 \times 60}{\pi} = 3437'{\cdot}7$$

The numerical value of this function is found in the table of meridional parts or the tables of increased latitudes, given in every collection nautical tables. "BORGEN" (15) proposed the name "Mercator function" in memory of the great geographer Gerhard Mercator, the inventor of the Mercator Chart projection, As may readily be seen this table is simply a table of logarithmic tangents, calculated for the base *e* and multiplied by a constant. In the logarithmic calculation the base of the chosen system is of no importance and the multiplication by a constant factor remains without influence on the result.

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(15) BÖRGEN : "*Uber die Auflösung nautisch-astronomischer Aufgaben mit Hilfe der Tabelle der Meridionalteile*". *Aus dem Archiv der Deutschen Seewarte,* 1898.

Therefore it is not surprising that the problems of nautical astronomy can be solved with the aid of a table of meridional parts. The solution of similar problems with the aid of the tangents was also known to the Arabian mathematicians in the early Middle Ages. One formula, making use of the Mercator functions for the calculation of the altitude and azimuth reads as follows :—

$$\begin{aligned}
f\ (x_1) &= f\ (\varphi) + (\delta) \\
f\ (x_2) &= f\ (\varphi) - f\ (\delta) \\
\operatorname{cof}\ (a + q) &= -\operatorname{cof}\ (x_1) + \operatorname{cof}\ (t) \\
\operatorname{cof}\ (a - q) &= \operatorname{cof}\ (x_2) - \operatorname{cof}\ (t) \\
f\ (x') &= \operatorname{cof}\ (a) + \operatorname{cof}\ (q) \\
\operatorname{cof}\ (z) &= \operatorname{cof}\ (x') - \operatorname{cof}\ (\varphi + \delta)
\end{aligned}$$

in which q represents the parallactic angle. The fact that the attempt to introduce such complicated systems into practical navigation was doomed to failure could have been predicted. How much more then must we be astonished that despite this fact, these were later proposed for aerial navigation, for instance by MARCUSE in his book " Astronomischen Ortsbestimmung im Ballon ", Berlin 1909. It seems however that frequently, many systems which have been rejected by mariners as impracticable have now been proposed to aerial navigators as " elegant new methods ". Therefore a thorough study of nautical astronomy might also be recommended for aerial navigators.

Of the many transformations of the 1st cosine equation, the basic equation of nautical astronomy, I have given only a few, but these are the most important. An almost complete collection of the various transformations hitherto known may be found by those interested in the subject in the work of A. WEDEMEYER entitled " Zur Höhenberechnung " Ann. der Hydro. 1903, p. 211, page 248 and p. 263.

Now I should like to answer briefly one question :—

Which formula or system of formulae is best adapted for the calculation of the altitude ? In my opinion for the pure logarithmic calculation of the altitude and azimuth in one series of calculations, the best system is number (24).

$$\begin{aligned}
\cot y &= -\cot \delta \, . \, \cos \tau \\
\cot a &= \cot \tau \, . \, \cos Y \, . \, \sec y \\
\cot h &= -\cot Y \, . \, \sec a.
\end{aligned}$$

which forms the basis of the Seewarte Tables. This is probably the only system which permits of the calculation of altitude and azimuth with four-place logarithms which will insure an accuracy of 2' in every case. Further, the altitude can be calculated by the Delambre formula with four-place logarithms to within 2' by means of the Delambre equation (38)

$$\sin^2 z/2 = \sin^2 1/2\ (\varphi - \delta).\ \cos^2 t/2 + \cos^2 1/2\ (\varphi + \delta).\ \sin^2 t/2.$$

With five-place tables we recommend the following formulae :

$$\begin{aligned}
\text{hav. } y &= \cos \varphi \, . \, \cos \delta \, . \, \text{hav } t \\
\text{hav. } z &= \text{hav. } (\varphi - \delta) \, . + \text{hav } y
\end{aligned} \qquad (35)$$

and

$$\begin{aligned}
\cos x &= \cos \varphi \, . \, \cos \delta \, . \, \text{versin } t \\
\sin h &= \cos (\varphi - \delta) - \cos x.
\end{aligned} \qquad (37)$$

The favorite formula at the moment is probably (35) which, in the 1933 edition of the German Admiralty's Book, " Nachtrag zum Lehrbuch der Navigation " has been given preference and which well deserves the designation " international altitude formula ", because it is used to calculate the stars' altitude not only by the Germans but also by the naval and merchant marine officers in England, America, Italy, the Scandinavian countries, Japan and many other nations. The four formulae given above are the best and I doubt very much if with the mathematical aids available today we can find any which will better serve the purpose; i.e. we must be content with the progress we have made. The question then remains, " How can the calculation of the altitude be still further simplified ? ". For this there are several possibilities. Thus by means of tabular, graphical or mechanical aids we may partially or entirely avoid the arithmetical calculation. To what extent this is possible, I should like to discuss in a subsequent work. There remains, however, still another way; i.e. the use of calculating machines. This most modern of all aids to arithmetical calculation has practically replaced the use of logarithms on shore. On the other hand its aid in the solution of the nautical astronomical problems has hardly been sought. For the calculation of the altitude by means of a commercial calculating machine it would be necessary to transform the basic equation in such a manner that accurate results to five places can be obtained which would not only give the requisite accuracy for navigation but would insure a smooth and easy solution.