TWO-DIMENSIONAL OCEAN CURRENTS

by

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The method for calculating ocean current in a given direction, derived by W. WERENSKIOLD (1), is indeed much simpler than the usual computations of dynamical height of isobaric surfaces. His method, which applies only to one dimensional currents, can be extended so as to become applicable also to two-dimensional currents, of which the direction is not given before hand.

WERENSKIOLD's formula is

$$v = \frac{g\delta\Sigma \tan j}{2\rho\omega \sin\varphi}, \tag{1}$$

where v is the velocity, or the velocity component, at some level, perpendicular to a vertical hydrographic section, which passes through the station, and j is the angle of inclination, measured in clockwise direction, of each isopycnal line drawn on the vertical section with a constant step, δ, in density in situ, ρ, while g, ω and φ have the usual meaning. The summation must be ranged from a base-level, $z = z_0$, where there is no considerable current, to the higher level at which the speed of current is calculated.

The shortest way to obtain the formula is, perhaps, somewhat like as follows (2) :—

In a left-hand system of rectangular coordinates (x, y, z), let z-axis be directed vertically downwards and let the vertical section be parallel to z-x-plane. Then, the hydrodynamic equations for steady non-turbulent ocean currents are

$$2v\omega \sin\varphi = \frac{1}{\rho}\frac{\delta p}{\delta x} \quad \text{et} \quad g = \frac{1}{\rho}\frac{\delta p}{\delta z}, \tag{2}$$

where p is the pressure and v the y-component of velocity, while the latitude φ is measured positive toward north. Integrating the second equation in (2) from the sea surface $z = z_s$ to some level we get

$$p = g\int_{z_s}^{z} \rho dz + g\rho_s\eta_s + p_s, \tag{3}$$

where η_s and ρ_s are respectively the elevation and the density of water at the sea surface, and the atmospheric pressure, p_s, is assumably constant. Inserting (3) into the first equation of (2), and putting $\rho = \rho_1$ at level z_1, we have

$$2v\rho_1\omega \sin\varphi = g\frac{\delta}{\delta x}\left(\int_{z_s}^{z_1} \rho dz + \rho_s\eta_s\right),$$

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(1) WERENSKIOLD, W.: Coastal currents. *Geofys. Publ.* 10 (13), 1935, 1-14; Die Berechnung von Meeresströmungen. *Ann. d. Hydrogr. u. Marit. Meteor.*, 65 (2), 1937, 68-72; Berechnung der Geschwindigkeit an der Wasseroberfläche. *Ann. d. Hydrogr.* u.s.w., 65 (4) 1937, 185-186.

(2) *see the remark on page* 4 *below.*

and specially
$$o = g\frac{\delta}{\delta x}\left(\int_{z_s}^{z_0} \rho dz + \rho_s \eta_s\right).$$

By side-by-side substraction, we obtain

$$v = \frac{-g}{2\rho_1 \omega \sin\varphi} \cdot \frac{\delta}{\delta x}\int_{z_1}^{z_0} \rho dz. \tag{4}$$

Now, by integration by parts, we can put

$$\int_{z_1}^{z_0} \rho dz = \rho_0 z_0 - \rho_1 z_1 - \int_{p_1}^{p_0} z d\rho,$$

where ρ_0 denotes the density at level z_0. In the last term, z is considered as a function of ρ and x, i.e. as the depth of iso-pycnal line, while the limits of integration must be considered as functions of x and z, z representing a fixed level. By differentiation with respect to x, we obtain

$$\frac{\delta}{\delta x}\int_{z_1}^{z_0} \rho dz = \frac{\delta}{\delta x}(\rho_0 z_0 - \rho_1 z_1) - \frac{\delta \rho_0}{\delta x} z_0 + \frac{\delta \rho_1}{\delta x} z_1 - \int_{p_1}^{p_0} \frac{\delta z}{\delta x} d\rho = -\int_{p_1}^{p_0} \frac{\delta z}{\delta x} d\rho. \tag{5}$$

On the other hand, by definition of j, we can write $(\delta z/\delta x) = \tan j$. Hence, (4) becomes

$$v = \frac{g}{2\rho_1 \omega \sin\varphi}\int_{p_1}^{p_0} \tan j d\rho, \tag{6}$$

which is nothing but the formula (1).

The extension to two-dimensional cases is performable by just the similar way, starting from the fundamental equations,

$$2v\,\omega \sin\varphi = \frac{1}{\rho}\frac{\delta p}{\delta x}, \quad -2u\,\omega \sin\varphi = \frac{1}{\rho}\frac{\delta p}{\delta y} \quad \text{et} \quad g = \frac{1}{\rho}\frac{\delta p}{\delta z}.$$

We obtain similarly as (4) and (5),

$$v = \frac{g}{2\rho_1 \omega \sin\varphi}\int_{p_1}^{p_0} \frac{\delta z}{\delta x} dp \quad \text{et} \quad u = \frac{-g}{2\rho_1 \omega \sin\varphi}\int_{p_1}^{p_0} \frac{\delta z}{\delta y} d\rho,$$

or denoting the averaged value of ρ_1 by $\bar{\rho}_1$, and putting

$$\psi = \frac{-g}{2\bar{\rho}_1 \omega \sin\varphi}\left(\int_{p_1}^{p_0} z d\rho - \rho_0 z_0 + \rho_1 z_1\right), \tag{7}$$

we can write for sufficient approximation $u = \frac{\delta\psi}{\delta y}$, and $v = -\frac{\delta\psi}{\delta x}$ which show that ψ is a stream function.

In the form like as (1), the stream function (7) can be written as

$$\psi = \frac{-g}{2\bar{\rho}_1 \omega \sin\varphi}\left(\delta \cdot \sum_{p=p_1}^{p_0} z - \rho_0 z_0 + \rho_1 z_1\right), \tag{8}$$

of which the numerical computations are very easy with any hydrographic chart. Now, the stream function ψ can have an arbitrary additive constant. Hence, in the first term of (8), depth of each iso-pycnal surface z can be substituted by the difference, $\zeta = z - z'$ from a suitable constant level z', chosen each for each iso-pycnal surface, provided that the value of ρ for that iso-pycnal surface lies between the maximum density ρ_m at level z_1 and the minimum density ρ_{m0} at level z_0, or, in other words, provided that the iso-pycnal surface passes through all stations in the domain at some level between z_1 and z_0 (Fig. 1). For iso-pycnal surfaces, of which ρ is greater than ρ_{m0}, let $z = z_0 + \zeta$ and at the same time let $\rho_0 = \rho_{m0} + n_0\delta$, then

$$\delta.\sum_{p=p_{m0}}^{p_0}(z_0+\zeta)=n_0\delta.z_0+\delta.\sum_{p=p_{m0}}^{p_0}\zeta,\quad \text{et}\quad \rho_0 z_0=\rho_{m0}z_0+n_0\delta.z_0,$$

where $\rho_{m0}z_0$ is constant. Hence, if we substitute ζ to z, we can drop the second term in (8). Similarly, if we measure the depth of each iso-pycnal, of which ρ is less than ρ_m, from the level z_1, and denote it by ζ, we can drop the third term in (8). Thus, we finally obtain :—

$$\psi=\frac{-g}{2\rho_1\omega\sin\varphi}\delta.\sum\zeta, \tag{9}$$

where $\zeta = z - z'$, provided that $z' = z_1$ for $\rho \leqq \rho_m$ and that $z' = z_1$ for $\rho \geqq \rho_{m0}$.

The last restriction for the choice of z' disables the formula from being employed, if there exists such an iso-pycnal surface, which passes through both of the end levels z_1 and z_0, Some device may be possible to remove this difficulty. But, in such a case it is perhaps to resort to the formula (8), in which the third term can be dropped by putting $z_1 = 0$.

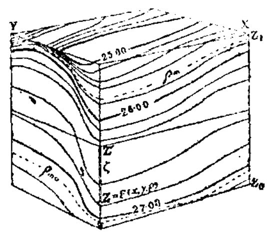

Fig. 1

It is a matter of course, that ψ = const. is a stream line, and the resultant velocity V is given by the maximum gradient of ψ , which again clearly coincides with (1), if we take the vertical section perpendicularly to the stream line. In practicing the numerical calculations, meter-ton (10^6 g)-second system of unit is employed for convenience' sake.

The local difference in heigh η_1 of an isobaric surface at level z_1 can also be estimated readily from (8). Since we have assumed that there is no current at level z_0, the isobaric surface at level z_0 is horizontal. Then, for the difference in pressure between two isobaric surfaces, mentioned just above, we have

$$p_0-p_1=g\int_z^{z_0}\rho dz+g\rho_1\eta_1$$

$$=g\left[\rho_0 z_0-\rho_1 z_1-\int_{\rho_1}^{\rho_0} z d\rho\right]+g\rho_1\eta_1$$

Or, for a sufficient approximation,

$$\eta_1=-\frac{2\omega\psi\sin\varphi}{g}+\frac{p_0-p_1}{g\rho_1}, \tag{10}$$

provided ψ is calculated by (8). To the last term in (10), we can substitute the difference z_0-z_1, for further approximation. If we calculate ψ by (9), then we have

$$\eta_1=-\frac{2\omega\psi\sin\varphi}{g}+\text{const.}$$

and we can still know the relative variation in η_1.

A *remark* to the derivation of WERENSKIOLD's formula should be given on this occasion

The relation between the inclination of iso-pycnal, tan j, and isobaric, tan i, surfaces can be derived as follows without aid of the distinctly stratified layers :—

In Fig. 2, $\tan i = (\delta p/\delta x)/(\delta p/\delta z)$ and $\tan j = -(\delta \rho/\delta x)/(\delta \rho/\delta z)$.

or $$(\delta p/\delta x) = \tan i.(\delta p/\delta z) \qquad (11)$$

and $$(\delta \rho/\delta x) = -\tan j.(\delta \rho/\delta x) \qquad (12)$$

but, by (2), $$(\delta p/\delta z) = g\rho, \qquad (13)$$

Hence, from (11), $$(\delta p/\delta x) = \tan i.g\rho. \qquad (14)$$

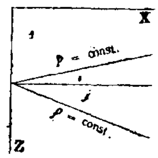

Fig. 2.

Differentiating (13), with respect to x and (14) with respect to z and equating, we get $$(\delta \rho/\delta x) = \delta(\rho \tan i)/\delta z. \qquad (15)$$

From (12) and (15), we obtain :—

$$\delta(\rho \tan i)/\delta z = -\tan j.\delta \rho/\delta z). \qquad (16)$$

On the other hand, we have, from (2) and (14),

$$2v\rho\omega \sin \varphi = (\delta p/\delta x) = \rho g \tan i. \qquad (17)$$

Differentiating both sides of (17) with respect to z, and remembering (16),

$$2\omega \sin \varphi\delta(v\rho/\delta z) = -g \tan j.(\delta \rho/\delta z). \qquad (18)$$

Finally, integrating both sides of (18) from level z to level z_0, we obtain :—

$$2v\rho\omega \sin \varphi = g\int_{p}^{p_0} \tan j.d\rho.$$

This is the same as (6) and is equivalent to the formula (1).