CONFORMAL PROJECTIONS OF THE ELLIPSOID

by

INGÉNIEUR HYDROGRAPHE GÉNÉRAL P. DE VANSSAY DE BLAVOUS,
DIRECTOR.

I.

In the Hydrographic Review, Vol. XVI, N° 1, of May 1939, page 24, attention has been invited to the fact that the very simple and well known definition of the stereographic projection of the sphere on to a plane may be extended to the terrestrial ellipsoid in various ways depending on whether one wishes especially to retain this or that property pertaining to this projection of the sphere, but it is not possible to preserve all of these properties at the same time. The same is true for other plane projections of ellipsoid, and in particular for the conformal projections of which we have spoken in the Hydrographic Review Vol. XVI, N° 2. Consequently it will not suffice to label these projections by the same name employed to designate the analogous projection of the sphere; it is indispensable that we should specify which property of the sphere we wish to retain or else to characterise it in some other manner. Further, it is not surprising that the formulae proposed by various authors should show some discrepancies and that they should not all produce exactly the same coordinates: one is preferable to another depending upon the principal aim in view.

We have also seen that the definition of the stereographic projection of the ellipsoid by means of geodetic lines, which might appear to be desirable, cannot lead to a conformal projection (1). The same is true in general for the other projections and in particular for the so-called Gauss projection, or transverse Mercator projection. The grid formed on the ellipsoid by the meridian of origin and the geodetic parallels to this meridian (obtained by laying off equal lengths on the geodetics normal to them), and by these geodetics, cannot be isometric. In order that this should be the case, it would be necessary that the geodetic parallels should have a constant geodetic curvature, and that consequently the ellipsoid should be one of revolution about the perpendicular dropped from its center on to the plane of the meridian of origin. This is however not the case of the terrestrial ellipsoid.

In the formulae which we have developed in order to avoid passing through the intermediary of the projection of the ellipsoid on to the sphere, we have always sought to preserve the condition of conformality. The

(1) Except in the case of the polar stereographic projection.

developments of the function of the Mercator complex coordinate insure this condition, regardless of the number of terms in the development employed; those which are a function of the lengths of the meridian of origin and of the parallel of the position point can only fulfill this condition if the number of terms of the development is sufficiently large so that the terms which are neglected represent lengths which are inappreciable with respect to the accuracy of the measurements.

These two systems of coordinates: lengths of meridian of origin and of the parallel and Mercator complex coordinate are furnished us by tables, the first of which gives us all the accuracy which may be desired. The second is constituted by Special Publication N° 21 of the International Hydrographic Bureau. It might be desirable, where great accuracy is sought, to employ the table of meridional parts to 6 decimal places instead of 5, although 5 generally suffices for all geodetic operations undertaken by the Hydrographic Services. But, in the higher latitudes, interpolation becomes difficult by reason of the very rapid increase in the scale on the Mercator projection. Further on we give the procedure to be employed to remedy this difficulty.

II.

Meanwhile one is forced to conclude that in the high latitudes, it would be interesting to employ another coordinate than that of Mercator. Captain L. Tonta has shown in the Hydrographic Review Vol. VI, N° 1, page 83 et seq., that to the north of latitude 36° 52' 12" one obtains a scale error which is less than the scale error obtained with the Mercator projection by using the polar conformal projection of Lambert, also called the central conformal projection or the polar stereographic projection. In the above cited issue of the Hydrographic Review, he gave tables of the radii of the parallels in this system of projection. But these tables were only calculated for the aims of cartography, and as such give the values of these radii to the nearest 5 metres, an accuracy which is quite insufficient for the geodetic calculations, which require 4 or at least 3 more decimals.

We give below the transformations which must be effected for the use of a new complex coordinate in place of that of Mercator. This is to be recommended for all the calculations in a system of projection in which the latitude of origin is greater than 36° 52' 12".

Let z_p be the polar stereographic complex coordinate. In the Hydrographic Review, Vol. XVI, N° 1, page 29 note 7 the formula connecting it to the Mercator complex coordinate $\zeta = l - l_0 + i\,G$ has been given

$$z_p = \frac{-2a}{\sqrt{1 - e^2}} \left(\frac{1 - e}{1 + e}\right)^{\frac{e}{2}} e^{-l_0}\, e^{-\zeta}$$

If D_0 be the radius of the parallel corresponding on that stereographic projection to L_0, the latitude of the origin of another projection, then :

$$D_0 = \frac{2\,a}{\sqrt{1-e^2}}\left(\frac{1-e}{1+e}\right)^{\frac{e}{2}} e^{-l_0} = \frac{2\,a}{\sqrt{1-e^2}}\left(\frac{1-e}{1+e}\right)^{\frac{e}{2}}\left(\frac{1+e\sin L_0}{1-e\sin L_0}\right)^{\frac{e}{2}}\frac{\cos L_0}{1+\sin L_0} = \frac{a\cos L_0}{\lambda_0};$$

By putting :

$$\lambda_0 = (1+e)^{\frac{1+e}{2}}(1-e)^{\frac{1-e}{2}}\left(\frac{1-e\sin L_0}{1+e\sin L_0}\right)^{\frac{e}{2}}\sin^2\left(45+\frac{L_0}{2}\right)$$

ζ and z_p are connected by the following relation :

$$(1)\quad z_p\, e^{\zeta} = -D_0$$

D being the radius of the circle corresponding to latitude L on the polar stereographic projection, then :

$$De^{l} = D_0\, e^{l_0} = \frac{2\,a}{(1-e)^{\frac{1-e}{2}}(1+e)^{\frac{1+e}{2}}}, \qquad z_p = -De^{-iG}.$$

For further expansion let us take as new complex coordinate :

$$(2)\quad \tau = \lambda_0\,\frac{D_0+z_p}{a},$$

this term reduces to zero at the origin of projection, and, being a function of z_p, that is to say of the complex coordinate of a conformal projection, it provides for a new conformal projection whatever may be the function or the number of terms in the development adopted.

From formulae (1) and (2) the following is derived :

$$e^{\zeta} = \frac{1}{1-\dfrac{a\,\tau}{\lambda_0\,D_0}} = \frac{1}{1-\dfrac{\tau}{\cos L_0}}$$

then, limiting to the terms of the 4th degree :

$$\zeta = -\operatorname{Log_n}\left(1-\frac{\tau}{\cos L_0}\right) = \frac{\tau}{\cos L_0}+\frac{\tau^2}{2\cos^2 L_0}+\frac{\tau^3}{3\cos^3 L_0}+\frac{\tau^4}{4\cos^4 L_0},$$

$$\zeta^2 = \frac{\tau^2}{\cos^2 L_0}+\frac{\tau^3}{\cos^3 L_0}+\frac{11}{12}\,\frac{\tau^4}{\cos^4 L_0},$$

$$\zeta^3 = \frac{\tau^3}{\cos^3 L_0}+\frac{3}{2}\,\frac{\tau^4}{\cos^4 L_0},$$

$$\zeta^4 = \frac{\tau^4}{\cos^4 L_0}.$$

In the Hydrographic Review, Vol. XVI, N° 2, page 35 a general expansion to 4 terms for conformal projection has been given

$$(3)\quad \frac{z}{N_0 \cos L_0} = \zeta - \frac{\zeta^2}{2}\sin L_0 + \frac{A}{12}\zeta^3 + \frac{B}{24}\zeta^4 \sin L_0.$$

By substituting the above values of the powers of ζ in this development we obtain, in terms of the complex variable τ, the general expression of the rigorously conformal projections limited to 4 terms

$$(4)\quad \frac{Z}{N_0} = \tau + \frac{\tau^2}{2}\operatorname{cotg}\left(45 + \frac{L_0}{2}\right) + \frac{\tau^3}{3\cos^2 L_0}\left(1 - \frac{3}{2}\sin L_0 + \frac{A}{4}\right)$$
$$+ \frac{\tau^4}{4\cos^3 L_0}\left(1 - \frac{11}{6}\sin L_0 + \frac{A}{2} + \frac{B}{6}\sin L_0\right).$$

Such defined projections benefit of the same properties as the above expansion in powers of ζ, though not being quite identical.

Let us use the value found on page 36 of the above quoted Review for $3A + B$ and substituting for p the expression $4\,\upsilon\,\frac{N_0}{\rho_0}\cos^2 L_0$ in which υ should have the value 0 or 1 according to whether preference is made for the 3rd order term of the linear modulus to be either in $e'^2 X^3$ or in $e'^2 XY^2$, (by putting $45 + \frac{L_0}{2} = L'_0$) the following is obtained :

$$(5)\quad \frac{Z}{N_0} = \tau + \frac{\tau^2}{2}\operatorname{cotg} L'_0 + \frac{\tau^3}{3\cos^2 L_0}\left(1 - \frac{3}{2}\sin L_0 + \frac{A}{4}\right) + \frac{\tau^4}{8\cos L_0}$$
$$\left[\frac{1 + \frac{A}{2}}{\sin^2 L'_0} - \frac{5}{3}\sin L_0 - \frac{N_0}{\rho_0}\frac{\sin L_0}{3}\left(1 - 4\,\upsilon\, e'^2 \cos^2 L_0\right)\right].$$

Thus the following expansion for the stereographic projection of L_0 origin is obtained :

$$\frac{Z}{N_0} = \tau + \frac{\tau^2}{2}\operatorname{cotg} L'_0 + \frac{\tau^3}{4}\operatorname{cotg}^2 L'_0 + \frac{\tau^4}{8}\operatorname{cotg}^3 L'_0 - \frac{e'^2\tau^3\cos L_0}{12}$$
$$\left[\cos L_0 + \tau\left(\frac{3}{2} - \sin L_0 - 2\,\upsilon\,\frac{N_0}{\rho_0}\sin L_0\right)\right].$$

For the Gauss projection, the expansion is :

$$\frac{Z}{N_0} = \tau + \frac{\tau^2}{2}\operatorname{cotg} L'_0 + \frac{\tau^3}{4}\left(\operatorname{cotg}^2 L'_0 - \frac{1}{3}\right) - \frac{\tau^4}{8}\operatorname{cotg} L'_0 \frac{\sin L_0}{\sin^2 L'_0}$$
$$- \frac{e'^2\tau^3\cos L_0}{12}\left[2\cos L_0 + \tau\left(3 - \frac{5}{2}\sin L_0 - 2\,\upsilon\,\frac{N_0}{\rho_0}\sin L_0\right)\right].$$

and, more generally speaking, for the Jung projections :

$$(6)\quad \frac{Z}{N_0} = \tau + \frac{\tau^2}{2}\operatorname{cotg} L'_0 + \frac{\tau^3}{4}\left(\operatorname{cotg}^2 L'_0 - \frac{n}{3}e^{-2i\theta}\right) + \frac{\tau^4}{8}\operatorname{cotg} L'_0$$

$$\left(\operatorname{cotg}^2 L'_0 - n e^{-2i\theta}\right) - \frac{e'^2 \tau^3 \cos L_0}{12}\left[\left(1 + n e^{-2i\theta}\right)\cos L_0 + \tau\left(\frac{3}{2} - \sin L_0 + 3 n e^{-2i\theta}\cos^2 L'_0 - 2\upsilon\frac{N_0}{\rho_0}\sin L_0\right)\right].$$

For calculating coordinates X and Y by means of these formulae, let us put:

$$(7) \qquad \tau = \lambda_0 \frac{D_0 - D e^{-iG}}{a} = d\, e^{i\delta};$$

δ and d being calculated by means of formulae :

$$\operatorname{tg}\delta = \frac{D \sin G}{D_0 - D\cos G}, \qquad d = \lambda_0 \frac{D_0 - D\cos G}{a\cos\delta} = \lambda_0 \frac{D \sin G}{a \sin\delta}$$

From formula (6) the following is obtained

$$(8)\left\{\begin{aligned}
\frac{X}{N_0} &= d\cos\delta + \frac{d^2}{2}\operatorname{cotg} L'_0 \cos 2\delta + \frac{d^3}{4}\left(\operatorname{cotg}^2 L'_0 - \frac{e'^2}{3}\cos^2 L_0\right)\cos 3\delta \\
&\quad - \frac{n d^3}{12}\frac{N_0}{\rho_0}\cos(3\delta - 2\theta) + \frac{d^4}{8}\operatorname{cotg} L'_0\left[\operatorname{cotg}^2 L'_0 - \frac{2}{3}e'^2(3 - 2\sin L_0)\right. \\
&\quad \left.\sin^2 L'_0 + \frac{8}{3}\frac{N_0}{\rho_0}\upsilon e'^2 \sin L_0 \sin^2 L'_0\right]\cos 4\delta - \frac{n d^4}{8}\frac{N_0}{\rho_0}\operatorname{cotg} L'_0 \cos(4\delta - 2\theta). \\[1em]
\frac{Y}{N_0} &= d\sin\delta + \frac{d^2}{2}\operatorname{cotg} L'_0 \sin 2\delta + \frac{d^3}{4}\left(\operatorname{cotg}^2 L'_0 - \frac{e'^2}{3}\cos^2 L_0\right)\sin 3\delta \\
&\quad - \frac{n d^3}{12}\frac{N_0}{\rho_0}\sin(3\delta - 2\theta) + \frac{d^4}{8}\operatorname{cotg} L'_0\left[\operatorname{cotg}^2 L'_0 - \frac{2}{3}e'^2(3 - 2\sin L_0)\right. \\
&\quad \left.\sin^2 L'_0 + \frac{8}{3}\frac{N_0}{\rho_0}\upsilon e'^2 \sin L_0 \sin^2 L'_0\right]\sin 4\delta - \frac{n d^4}{8}\frac{N_0}{\rho_0}\operatorname{cotg} L'_0 \sin(4\delta - 2\theta).
\end{aligned}\right.$$

III.

The calculation of the geographical position of a point as defined by its coordinates X and Y will be deduced from the reciprocal of formula (3), if meridional parts have been used :

$$\begin{aligned}
\zeta\cos L_0 &= \frac{Z}{N_0} + \frac{Z^2}{2N_0^2}\operatorname{tg} L_0 + \frac{Z^3}{12 N_0^3}\left(4\operatorname{tg}^2 L_0 + \frac{N_0}{\rho_0} + n\frac{N_0}{\rho_0}e^{-2i\theta}\right) \\
&\quad + \frac{Z^4}{24 N_0^4}\operatorname{tg} L_0\left(6\operatorname{tg}^2 L_0 + 3\frac{N_0}{\rho_0} - 4\upsilon\frac{N_0}{\rho_0}e'^2\cos^2 L_0 + 2n\frac{N_0}{\rho_0}e^{-2i\theta}\right) \\
&\quad + \frac{Z^5}{48 N_0^5}\left[10\operatorname{tg}^4 L_0 + \frac{N_0}{\rho_0}\operatorname{tg}^2 L_0\left(5 - n e^{-2i\theta} - 24\upsilon e'^2\cos^2 L_0\right)\right. \\
&\quad \left. + \frac{N_0^2}{\rho_0^2}\left(1 + n e^{-2i\theta}\right)^2\right]
\end{aligned}$$

$$+\frac{7Z^6}{288N_o^6}\operatorname{tg}L_o\left[8\operatorname{tg}^4L_o+\frac{N_o}{\rho_o}\operatorname{tg}^2L_o\left(1-9\,n\,e^{-2i\theta}-40\,\upsilon\,e'^2\cos^2L_o\right)\right.$$
$$\left.+\frac{N_o^2}{\rho_o^2}\left(1+n\,e^{-2i\theta}\right)\left(2+ne^{-2i\theta}-4\,\upsilon\,e'^2\cos^2L_o\right)\right]$$
$$+\ \dots\ \dots\dots$$

Let us put :

$$\frac{Z}{N_o}=q\,e^{i\varphi},\qquad \operatorname{tg}\varphi=\frac{Y}{X},\qquad q=\frac{X}{N_o\cos\varphi}=\frac{Y}{N_o\sin\varphi}$$

the above formula will provide for $l-l_0$ and G.

In the same way, reciprocal of formula (6) will provide :

$$\tau=\frac{Z}{N_o}-\frac{Z^2}{2N_o^2}\operatorname{cotg}L'_o+\frac{Z^3}{12N_o^3}\left(3\operatorname{cotg}^2L'_o+e'^2\cos^2L_o+n\frac{N_o}{\rho_o}e^{-2i\theta}\right)$$
$$-\frac{Z^4}{24N_o^4}\operatorname{cotg}L'_o\left[3\operatorname{cotg}^2L'_o+2e'^2\sin^2L'_o\left(8+3\sin L_o-4\,\upsilon\frac{N_o}{\rho_o}\sin L_o\right)+2n\frac{N_o}{\rho_o}e^{-2i\theta}\right]$$
$$+\frac{Z^5}{48N_o^5}\left\{6\operatorname{cotg}^4L'_o-3\operatorname{cotg}^2L'_o\left[4e'^2\sin^2L'_o\left(\sin^2L'_o-4\,\upsilon\frac{N_o}{\rho_o}\sin L_o\right)\right.\right.$$
$$\left.\left.+n\frac{N_o}{\rho_o}e^{-2i\theta}\right]+\left(e'^2\cos^2L_o+n\frac{N_o}{\rho_o}e^{-2i\theta}\right)^2\right\}$$
$$-\frac{Z^6}{288N_o^6}\operatorname{cotg}L'_o\left(\begin{aligned}&81\operatorname{cotg}^4L'_o-18\operatorname{cotg}^2L'_o\left[e'^2\sin^2L'_o\left(10-3\sin L_o\right.\right.\\&\qquad\left.\left.-28\,\upsilon\frac{N_o}{\rho_o}\sin L_o\right)-5\,n\frac{N_o}{\rho_o}e^{-2i\theta}\right]\\&+7\left(e'^2\cos^2L_o+n\frac{N_o}{\rho_o}e^{-2i\theta}\right)\left[2e'^2\left(1-2\sin L_o\right.\right.\\&\qquad\left.\left.+4\,\upsilon\frac{N_o}{\rho_o}\sin L_o\right)\sin^2L'_o+n\frac{N_o}{\rho_o}e^{-2i\theta}\right]\end{aligned}\right)+\dots\dots$$

The real part P and the imaginary part $i\,Q$ will be calculated by the same method as above, from which is deduced :

$$\operatorname{tg}G=\frac{Q}{\lambda_0\dfrac{D_0}{a}-P},\qquad D=\frac{a\,Q}{\lambda_0\sin G}=\frac{\lambda_0D_0-a\,P}{\lambda_0\cos G}$$

IV.

The meridional part l used in formulae in which the complex coordinate ζ appears should be expressed in radians and not in minutes of arcs as usually given by Tables. It is easy to transform the tabulated values in to radians by multiplying by $\frac{\pi}{10800}$.

However, tables of meridional parts being in terms of round minutes, the interpolation implies the use of second differences. A higher accuracy is obtained as follows : The required value of l is, (M being the modulus of Neperian logarithms):

$$l = \frac{1}{M} \log \operatorname{tg} \left(45 + \frac{L}{2}\right) - \left(e^2 \sin L + \frac{e^4}{3} \sin^3 L + \frac{e^6}{5} \sin^5 L + \frac{e^8}{7} \sin^7 L + \dots\right)$$

The term $\frac{1}{M} \log \operatorname{tg} (45 + \frac{L}{2})$ is easily calculated with great exactness by means of logarithm tables to 8 or 10 places of decimals. On the other hand l_e and l_s being the values of the meridional parts as provided, for the ellipsoid and for the sphere, by tables I and VI of International Hydrographic Bureau Special Publication N° 21 is obtained :

$$e^2 \sin L + \frac{e^4}{3} \sin^3 L + \frac{e^6}{5} \sin^5 L + \frac{e^8}{7} \sin^7 L + \dots = \frac{\pi}{10800} (l_s - l_e),$$

the interpolation of $l_s - l_e$ is easily made without using second differences because its variation is rather slow (2).

It has been remarked that in higher latitudes and even, theoretically, as soon as the latitude of the selected origin is higher than 36° 52' 12", it is preferable to use the complex variable τ instead of ζ ; The later τ is easily deduced from the values of radii of parallels of the polar stereographic projection (formula 7) the expression of which is :

$$\log \frac{D}{a} = \log \operatorname{cotg} (45 + \frac{L}{2}) + M (e^2 \sin L + \frac{e^4}{3} \sin^3 L + \frac{e^6}{5} \sin^5 L + \frac{e^8}{7} \sin^7 L + \dots)$$
$$+ \log 2 - \frac{1 - e}{2} \log (1 - e) - \frac{1 + e}{2} \log (1 + e).$$

The first term is derived from the logarithm table; it is the only one varying rather rapidly in terms of the latitude. The second one involving factor M, equals $\frac{\pi M}{10800} (l_s - l_e)$ and is readily calculated by means of Special Publication N° 21. The other terms are constant and have the following values for the international ellipsoid :

$$\log 2 - \frac{1 - e}{2} \log (1 - e) - \frac{1 + e}{2} \log (1 + e) = 0.29956, 85463, 579.$$

(2) $\frac{\pi}{10800} = 0.00029, 08882, 08666$ $\qquad \log \frac{\pi}{10800} = 6.46372, 61172, 07184$

$M = 0.43429, 44819, 03251, 82765$ $\qquad \log M = 9.63778, 43113, 00536, 78912$

$\frac{1}{M} = 2.30258, 50929, 94045, 68402$

$\frac{\pi M}{10800} = 0.00012, 63311, 43874$ $\qquad \log \frac{\pi M}{10800} = 6.10151, 04285, 07721$

The following is calculated in the same way :

$$\log \lambda_0 = \lg 2 \sin^2 (45 + \frac{L_0}{2}) - M (e^2 \sin L_0 + \frac{e^4}{3} \sin^3 L_0 + \frac{e^6}{5} \sin^5 L_0 + \frac{e^8}{7}$$

$$\sin^7 L_0 + \ldots) - \left[\lg 2 - \frac{1-e}{2} \lg (1-e) - \frac{1+e}{2} \lg (1+e) \right]$$

Let us apply this process to the example given already in the Hydrographic Review Vol. VII N° 1 et Vol. XVI N° 2. The latitude of origin is high enough to use by preference the expansion according to the powers of the complex coordinate τ.

The results (notwithstanding errors in calculation) differ by several centimeters from the former; remark however has been made that the three projections used in these three calculations, although all three rigorously conformal, are not identical.

V.

We have seen that all the projections in which the scale is equal to unity and is a minimum at the origin may be expressed by the expansion (3) as a function of the powers of the Mercator complex coordinate: an expression in which the first two terms always remain the same. It might be convenient in certain cases which we shall try to specify, to limit the expression to these two terms and to employ the projection defined by:

$$\frac{Z}{N_0 \cos L_0} = \zeta - \frac{\zeta^2}{2} \sin L_0.$$

We thus obtain an expression which is very easy to calculate; but since the term of the third order of the scale increases very rapidly with the distance from the origin, it is necessary to take this term into consideration and employ the projection only at a limited distance from the origin. This scale will become the following :

$$M = 1 + \frac{X^2}{2 N_0^2} \left(\frac{N_0}{\rho_0} - tg^2 L_0 \right) + \frac{Y^2}{2 N_0^2} tg^2 L_0 + \frac{X (X^2 - 3 Y^2) tg L_0}{6 N_0^3}$$

$$\left(\frac{N_0}{\rho_0} - 5 \; tg^2 L_0 \right) - \frac{e'^2}{3} \frac{X^3 \sin 2 L_0}{N_0^2 \rho_0} + \ldots$$

We can no longer make use of the indicatrix curve for adapting it to the shape of the terrain. The shape of this curve is imposed by the position of the origin. But, in order that the scale may be a minimum at the origin, it is still necessary that the conic should be an ellipse, that is :

$$\frac{N_0}{\rho_0} \geqslant tg^2 L_0 ;$$

from which :

$$\cos^2 L_0 (2 + e'^2 \cos^2 L_0) \geqslant 1 ;$$

$$\frac{X}{N_0} = d\cos\delta + Ad^2\cos 2\delta + Bd^3\cos 3\delta + Cd^3\cos(3\delta-2\theta) + Ed^4\cos 4\delta + Fd^4\cos(4\delta-2\theta)$$

$$\frac{Y}{N_0} = d\sin\delta + Ad^2\sin 2\delta + Bd^3\sin 3\delta + Cd^3\sin(3\delta-2\theta) + Ed^4\sin 4\delta + Fd^4\sin(4\delta-2\theta)$$

$$A = \frac{1}{2}\mathrm{cotg}\,L'_0 \;,\quad B = \frac{1}{4}\left(\mathrm{cotg}^2 L'_0 - \frac{e'^2}{3}\cos^2 L_0\right) \;,\quad C = -\frac{1}{12}\frac{N_0}{\rho_0} \;,\quad E = \frac{1}{8}\mathrm{cotg}\,L'_0\left[\mathrm{cotg}^2 L'_0 - \frac{2}{3}e'^2(3-2\sin L_0)\sin^2 L'_0\right] \;,\quad F = -\frac{1}{8}\frac{N_0}{\rho_0}\mathrm{cotg}\,L'_0$$

$$L_0 = 41°\text{-}40' \;,\quad L'_0 = 65°\text{-}50' .$$

$\log \mathrm{cotg}\, L_0$	9,6519.7419
$\log 2$	0,3010.3000
$\log A$	9,3509.4419
$\log \frac{N_0}{\rho_0}$	0,0016.3722
$\log 12$	1,0791.8125
$\log C$ (-)	8,9224.5597

$\log \mathrm{cotg}^2 L'_0$	9,3039.4838
$\mathrm{cotg}^2 L'_0$	0,2013.4849
$\frac{e'^2\cos^2 L_0}{3}$	0,0012.5898
$4B$	0,2000.8951
B	0,0500.2238
$\log B$	8,6991.6433

$\log \cos^2 L_0$	9,7466.7036
$\log e'^2$	7,8304.7127
$\log e'^2\cos^2 L_0$	7,5771.4163
$e'^2\cos^2 L_0$	0,0037.7695

$\sin L_0$	0,6647.9586
$2\sin L_0$	1,3295.9173
$3-2\sin L_0$	1,6704.0827
$\log(3-2\sin L_0)$	0,2228.2264
$\log 2$	0,3010.3000
$\log \sin^2 L'_0$	9,9203.3099
$\log e'^2$	7,8304.7127
$\log 2e'^2(3-2\sin L_0)\sin^2 L'_0$	8,2746.5490
$2e'^2(3-2\sin L_0)\sin^2 L'_0$	0,0188.2153
$1/3$	0,0062.7384
$\mathrm{cotg}^2 L'_0$	0,2013.4849
$8E\,\mathrm{tg}\,L'_0$	0,1950.7465

$\log 8E\,\mathrm{tg}\,L'_0$	9,2902.0083
$\log \mathrm{cotg}\, L'_0$	9,6519.7419
$\log 8E$	8,9421.7502
$\log 8$	0,9030.8999
$\log E$	8,0390.8503

$\log \frac{N_0}{\rho_0}$	0,0016.3722
$\log \mathrm{cotg}\, L'_0$	9,6519.7419
	9,6536.1141
$\log 8$	0,9030.8999
$\log F$ (-)	8,7505.2142

L=43°-50'-29",68 L'=66°-55'-14",84
G=-4°-59'-21",73

	X	Y
1er terme	+0,0401.98408	-0,0617.07175
2e terme	-0,0004.91765	-0,0011.13060
3e terme	-0,0000.19721	-0,0000.03210
4e terme	-0,0000.07086	+0,0000.32651
5e terme	-0,0000.00217	+0,0000.00238
6e terme	+0,0000.01164	+0,0000.01178
Σ	+0,0396.80783	-0,0627.89378
$\log \Sigma$	8,5985.8023	8,7978.8618
$\log N_0$	6,8053.5706	6,8053.5706
	5,4039.3729	5,6032;4324

X' = +253476,262 Y' = -401091,300
-12,674 +20,055
X = +253463,588 Y = -401071,245

$\log \frac{D}{a}$	9,9311.1209		9,9311.1209
$\log \cos G$	9,9983.5127	$\lg \sin G$	8,9393.7401
$\log \frac{D}{a}\cos G$	9,9294.6336		8,8704.8610
$\frac{D}{a}\cos G$	0,8500.8697		8,6843.5929
$\frac{D_0}{a}$	0,8984.3283	$\lg \mathrm{tg}\,\delta$ (-)	0,1861.2681
$\frac{D_0}{a} - \frac{D}{a}\cos G$	0,0483.4586		

δ = - 56°-55'-05",68 θ = 133-30
2δ = - 113 -50 -11,36 2θ = 267-0
3δ = - 170 - 45-17,04 $3\delta-2\theta$ = - 77°-45'-17",04
4δ = - 227 -40 -22,72 $4\delta-2\theta$ = - 134°-40'-22",72

$\log d^3$	6,6014.4200	6,6014.4200
$\log \cos(3\delta-2\theta)$	9,3265.3440	(-) 9,9900.0607
$\log C$ (-)	8,9224.5597	(-) 8,9224.5597
$\log 4^e$ terme (-)	4,8504.3237	(+) 5,5139.0404

$\log d^4$	5,4685.8930	5,4685.8930
$\log \cos(4\delta-2\theta)$ (-)	9,8469.9202	(-) 9,8519.4970
$\log F$	(-) 8,7505.2142	(-) 8,7505.2142
$\log 6^e$ terme (+)	4,0661.0274	(+) 4,0710.6042

$\log \lambda_0$	9;9198.4956		9,9198.4956
$\log \frac{D_0 - D\cos G}{a}$	8,6843.5929		8,8704.8610
$\log 1^{er}$ terme	8,6042.0885	$\lg 1^{er}$ term	8,7903.3566
$\log \cos\delta$	9,7370.6152	$\lg \sin\delta$	9,9231.8833
$\log d$ =	8,8671.4733	$\log d$ =	8,8671.4733
$\log d^2$	7,7342.9465		7,7342.9465
$\log \cos 2\delta$ (-)	9,6065.1884	$\lg \sin$ (-)	9,9612.7986
$\log A$	9,3509.4419		9,3509.4419
$\log 2^e$ terme (-)	6,6917.5769	(-)	7,0465.1870

$\log d^3$	6,6014.4200		6,6014.4200
$\log \cos 3\delta$ (-)	9,9943.2139	$\lg \sin$	9,2059.1053
$\log B$	8,6991.6433		8,6991.6433
$\log$ 3e terme (-)	5,2949.2772	(-)	4,5065.1686

$\log d^4$	5,4685.8930		5,4685.8930
$\log \cos 4\delta$ (-)	9,8282.4813	$\lg \sin$ (+)	9,8688.2870
$\log E$	8,0390.8503		8,0390.8503
$\log$ 5e terme (-)	3,3359.2247	(+)	3,3765.0303

or approximately :

$$\cos L_0 \geqslant \frac{\sqrt{2}}{2}\left(1 - \frac{e'^2}{4}\right)$$

$$L_0 \leqslant 45^\circ\ 6'$$

If L_0 equals zero, we have the Mercator projection, of which the indicatrix is formed by straight lines parallel to the axis of Y. As L_0 increases, the indicatrix becomes an ellipse of which the major axis is oriented in accordance with the Y-axis; then for about $L_0 = 35^\circ 20'$ the indicatrix becomes a circle and the projection falls into the stereographic class. When L_0 continues to increase, the indicatrix ellipse will have its major axis turned towards the X-axis and when L_0 attains a value of $45^\circ 6'$ the indicatrix merges with the X axis and the projection is then of the Gauss type.

Let us note further that the term of the third order of the scale disappears when the origin L_0 reaches a value near 24°.

The meridians and the parallels in projection are parabolas having as axis the X - axis.

We have, in effect :

$$\frac{X}{N_0 \cos L_0} = v - \frac{v^2}{2} \sin L_0 + \frac{G^2}{2} \sin L_0 ,$$

$$\frac{Y}{N_0 \cos L_0} = G\,(1 - v \sin L_0).$$

The parallels have as equation :

$$Y^2 - 2\,N_0 \operatorname{cotg} L_0\,(1 - v \sin L_0)^2\,X + 2\,N_0^2 \operatorname{cotg} L_0 \cos L_0$$

$$v\,(1 - v \sin L_0)^2\,(1 - \frac{v}{2} \sin L_0) = 0$$

These parallels have their concavity turned towards the elevated pole so long as v does not reach a value of $\frac{1}{\sin L_0}$. At this moment the parabola merges with a part of the X axis. This limiting value of v is infinite if the origin is at the equator; and it diminishes gradually to about $\sqrt{2}\left(1 - \frac{e'^2}{4}\right)$ if L_0 reaches the value of about $45^\circ 6'$, which we have assigned as its limit. Even in this case we shall have as a limiting value for L a number greater than 78°. We see then that without reaching these limits, at which the distortion becomes inacceptable, the projection may still be employed over wide limits.

The meridians are represented by parabolas normal to those of the preceding paragraph. Their convexity is turned towards the elevated pole. Their focal length is equal to $\frac{G^2 N_0 \sin 2 L_0}{4}$.

Their equation is :

$$Y^2 + G^2 N_0 X \sin 2 L_0 - G^2 N_0^2 (1 + G^2 \sin^2 L_0) \cos^2 L_0 = 0$$

One of the branches of the parabola represents the meridian corresponding to the positive value of G, the other branch to the negative value. All the parabolas of the two systems have their foci on the X - axis at the same point.

$$X = \frac{N_0}{2} \operatorname{cotg} L_0$$

VI.

Let us see also how we may utilize as a working projection the conformal projection whose expression is limited to the first two terms of the expansion (5).

$$\frac{Z}{N_0} = \tau + \frac{\tau^2}{2} \operatorname{cotg} \left(45 + \frac{L_0}{2}\right).$$

For this it is necessary to adopt the values

$$A = 2 (3 \sin L_0 - 2)$$

$$B = \frac{6}{\sin L_0} - 7$$

The scale will be :

$$M = 1 + \frac{X^2}{2 N_0^2}\left(\frac{N_0}{\rho_0} - \frac{2 - \sin L_0}{1 + \sin L_0}\right) + \frac{Y^2}{2 N_0^2} \frac{2 - \sin L_0}{1 + \sin L_0} + \frac{X (X^2 - 3 Y^2)}{N_0^3}$$
$$\operatorname{cotg}^3 \left(45 + \frac{L_0}{2}\right) - e'^2 \frac{X (X^2 + Y^2)}{4 N_0^3} \sin 2 L_0 - e'^4 \frac{X^3 \sin 2 L_0 \cos^2 L_0}{3 N_0^3} + \ldots$$

In order that the origin may correspond to a scale minimum it is necessary that the coefficient of X^2 should be positive; this occurs when L_0 is greater than about 30°.

We may therefore retain the rule which we have given in chapter II : utilize the development in ζ when L_0 is smaller than 36° 53' 12" and the development in τ when it is greater than this value. (3)

---

(3) More exactly, if a and b are the axis of the indicatrix on OX and OY, and if L_0 lies between 30° to about 42°, it is necessary to take the development in ζ if we desire that a/b should be less than 2, and the development in τ if a/b is to be greater than 2.

When L_0 is comprised between about 42° to 45°, we take the development in τ if a/b is to be less than 2, and in ζ if it is to be greater.

(The two developments give the same value $a/b = 1.98$ when $\sin L_0 = 2/3$, that is when $L_0 = 41° 48' 30"$ about).

The meridians will all pass through the pole. For this point τ is equal to cos L_0. Its coordinates are therefore :

$$X = N_0 \frac{3 - \sin L_0}{2} \cos L_0 , \qquad Y = 0$$

If we shift the origin there provisionally by putting :

$$X = X' + N_0 \frac{3 - \sin L_0}{2} \cos L_0 , \qquad Y = Y'.$$

The coordinates of a point of the projection are given by :

$$(9) \quad \begin{aligned} \frac{X'}{N_0 \cos L_0} &= \frac{D^2}{D_0^2} \frac{1 - \sin L_0}{2} \cos 2 G - \frac{D}{D_0} (2 - \sin L_0) \cos G, \\ \frac{Y'}{N_0 \cos L_0} &= - \frac{D^2}{D_0^2} \frac{1 - \sin L_0}{2} \sin 2 G + \frac{D}{D_0} (2 - \sin L_0) \sin G. \end{aligned}$$

By adding the equations member by member after having multiplied the first by sin G and the second by cos G; we obtain

$$\frac{X' \sin G + Y' \cos G}{N_0 \cos L_0} = - \frac{D^2}{D_0^2} \frac{1 - \sin L_0}{2} \sin G.$$

Performing the same operation after having multiplied the first by sin $2 G$ and the second by cos $2 G$: we have:—

$$\frac{X' \sin 2 G + Y' \cos 2 G}{N_0 \cos L_0} = - \frac{D}{D_0} (2 - \sin L_0) \sin G.$$

Eliminating D/D_0 between the two equations, we have :

$$\left(\frac{X' \sin 2 G + Y' \cos 2 G}{N_0}\right)^2 + 2 \frac{X' \sin G + Y' \cos G}{N_0} (2 - \sin L_0)^2 \operatorname{tg} \left(45 + \frac{L_0}{2}\right) \sin G = 0.$$

This equation of the meridians represents the parabolas which all pass through the elevated pole, where they make an angle $- G$ with the axis of X'. The axes of these parabolas make an angle of $- 2 G$ with the axis of X'. They all have the same focus located on the axis of X' at a point of ordinate,

$$X' = - \frac{N_0}{2} (2 - \sin L_0)^2 \operatorname{tg} \left(45 + \frac{L_0}{2}\right),$$

$$X = - \frac{N_0}{2} \operatorname{tg} \left(45 + \frac{L_0}{2}\right),$$

and their directrices are all tangent to a circle having the pole as its center and $\frac{N_0}{2} (2 - \sin L_0)^2 \operatorname{tg} \left(45 + \frac{L_0}{2}\right)$ as radius.

By multiplying the 2 members of the first of equations (9) by cos 2 *G* and those of the second by sin 2 *G*, and subtracting member by member we obtain :—

$$(10)\quad \frac{X'\cos 2G - Y'\sin 2G}{N_0\cos L_0} = \frac{D^2}{D_0^2}\frac{1-\sin L_0}{2} - \frac{D}{D_0}(2-\sin L_0)\cos G.$$

For the rest, the equations (9) may be written :

$$\frac{D}{D_0}(2-\sin L_0)\cos G = \frac{D^2}{D_0^2}\frac{1-\sin L_0}{2}\cos 2G - \frac{X'}{N_0\cos L_0},$$

$$\frac{D}{D_0}(2-\sin L_0)\sin G = \frac{D^2}{D_0^2}\frac{1-\sin L_0}{2}\sin 2G + \frac{Y'}{N_0\cos L_0}.$$

Raising the two members to the square and adding member by member we have :

$$(11)\quad \frac{D^2}{D_0^2}(2-\sin L_0)^2 = \frac{D^4}{D_0^4}\left(\frac{1-\sin L_0}{2}\right)^2 + \frac{X'^2+Y'^2}{N_0^2\cos^2 L_0}$$
$$- 2\frac{D^2}{D_0^2}\frac{1-\sin L_0}{2}\,\frac{X'\cos 2G - Y'\sin 2G}{N_0\cos L_0}$$

From equations (10) and (11) we deduce :

$$\frac{D}{D_0}\cos G = \frac{2-\sin L_0}{1-\sin L_0} + \frac{D^2}{4D_0^2}\frac{1-\sin L_0}{2-\sin L_0} - \frac{X'^2+Y'^2}{N_0^2\cos^2 L^0}\,\frac{1}{(1-\sin L_0)(2-\sin L_0)}\,\frac{D_0^2}{D^2},$$

$$\frac{D^2}{D_0^2}\cos 2G = 2\left(\frac{2-\sin L_0}{1-\sin L_0}\right)^2 + \frac{D^4}{8D_0^4}\left(\frac{1-\sin L_0}{2-\sin L_0}\right)^2 + 2\left(\frac{X'^2+Y'^2}{N_0^2\cos^2 L_0}\right)^2$$
$$\frac{1}{(1-\sin L_0)^2(2-\sin L_0)^2}\,\frac{D_0^4}{D^4} - \frac{X'^2+Y'^2}{N_0^2\cos^2 L_0}\left[\frac{4\frac{D_0^2}{D^2}}{(1-\sin L_0)^2} + \frac{1}{(2-\sin L_0)^2}\right].$$

Substituting these values in the first of equations (9) we have the equation of the parallels :

$$\left\{\frac{X'^2+Y'^2}{N_0^2}\,\frac{D_0^2}{D^2(2-\sin L_0)} - \frac{\cos^2 L_0}{4(2-\sin L_0)}\left[2(2-\sin L_0)^2 + \frac{D^2}{D_0^2}(1-\sin L_0)^2\right]\right\}^2$$
$$= \frac{X'}{N_0}\cos^3 L_0(1-\sin L_0) + \frac{\cos^4 L_0}{4}\left[(2-\sin L_0)^2 + 2\frac{D^2}{D_0^2}(1-\sin L_0)^2\right]$$

This equation permits us, for any given value of *X'* to find readily the value of *Y'* and thus to plot the parallels. But it would be simpler to utilize the equations; (9) and to give to *G* and *D* successively the round values corresponding to the apexes of the grid of meridians and parallels which one wishes to trace.