MISURA DELLE DISTANZE CON IL METODO DELLA "PICCOLA BASE" NEL RELIEVO A SCALE A GRANDE DENOMINATORE.

(Measurements of distances by the "Short-Base" method in surveys on scales with a large denominator (Small-Scale Surveys).

by

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Regular survey work is in progress for the compilation of the basic map of the Empire on a 1 : 400.000 scale.

Due to the scale, the survey of the country is of an essentially sketchy character; but, having nevertheless to do with an homogeneous and regular map, it is desirable to put into practice all the resources which, without involving an appreciable increase of time, confer upon the map the greatest possible accuracy.

The Administration of the Military Geographical Institute issued, a short time ago, the "Direttive tecniche per la formazione della carta 1 : 400.000 dell'Impero" (Technical guiding principles for the construction of the 1 : 400.000 map of the Empire). These guiding principles, the fruit of a broad experience gained in various surveys of extensive African zones, prescribes, among other things, that the geodetic frame be constituted by points with a mutual spacing of from 30 to 40 km., determined by provisional astronomical observations and with developments of partial triangulations.

The survey is carried on by overlapping traverses of polygonal networks developed in the direction of the geodetic points : the sides of the figures being measured by telemeters, kilometre-counters, podometers etc. The expeditious use, in the initial stage of aerophotograms, is also provided for. For the measurement of distances, the use also of the "short-base" method is recommended.

This method, the subject of the present article, was not employed in the past, not because if was unknown, but because, due to the lack of precision of the instruments then available (theodolite to within 10" to 20"), good results could not be assured. In surveying for the new 1 : 400.000 map, on the contrary, Zeiss II theodolites are used, in which the approximation in the reading of the circles is 1".

These instruments, which are among the best in existence, serve both for the terrestrial and provisional astronomical determinations, and due to their fine qualities of accuracy, lightness, celerity and simplicity of use, are all that can be desired for this kind of work. The availability of instruments of such a high degree of accuracy, gives this method of measurement of distances, by the short base method, a possibility of attaining the optimum of practical utilization. Adequate experience is lacking, at least in Italy, in the realm of running surveys; but it may be anticipated that this method will confirm in practice the advantages foreseen for it.

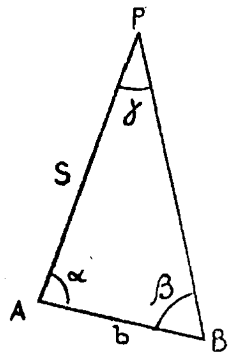
In theory, the problem is of great simplicity.

It is desired from a point A, to determine the distance S, by measuring the horizontal distance b (short base) between point A and point B chosen in such a manner that b is almost normal to S, and the angles α and β being mesured.

We have :

$$\gamma = 180 - (\alpha + \beta)$$

$$S = \frac{b \sin \beta}{\sin \gamma}.$$

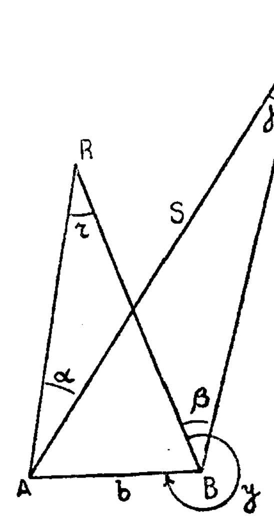


The practical solution of the problem thus expressed is fraught, however, with some difficulties. It is actually obvious that, to an ever so small variation of the angle γ; there corresponds an appreciable variation of the distance S. It is therefore necessary that the value of γ be ascertained with great accuracy.

But γ is deduced from the angles α and β; consequently, it is essential that these two angles be very accurately measured. For obtaining such accuracy, it is essential that, whilst carrying out the measurements, the theodolite be very well centred on the ends of the base-line, and that the subtense bars, which are substituted alternatively to the theodolite, be just as carefully set up. The strict observance of this requirement is essential, because it is of importance that the angle γ, as will be seen further on, be measured with an approximation of a few seconds. For obtaining such a result, good initial marks (base sites) must be had for the measurement of the directions which, from the ends of the base, run to P.

In the case dealt with here, the initial marks for the measurement of these directions are alternately A and B; obviously, anything but good, owing to their short distance apart.

Assuming, in fact, that the base *b* be 100 m. long, a displacement of one millimetre in the position of this initial, in a direction normal to the base, is sufficient to produce an error of 2". Such an error is already notable, especially if it be considered that the initials in the same circumstances are two; whilst achieving in the field an approximation of one millimetre in the substitution of the theodolite for the subtense bar, and vice versa, is no easy matter. The problem, as it has been propounded, might be better applied to practice by disposing of two theodolites set up at the ends of the short base, or else of one theodolite and two equal instrument tripods with the clamping screw of the theodolite in a fixed position, or of any other suitable gear. But these solutions are not adequate, as they are too expensive and unwieldy. In the case of two theodolites, on the other hand, frequent adjustment of the telescope sight is necessary, which it is important to avoid, although with modern theodolites, the telescopes of which are of a fixed length, this operation does not produce appreciable displacements of the optical axis. In the other cases, although the operations may have been conducted with the utmost care, it might always be doubted whether the various manipulations have been executed with the requisite accuracy. To obviate these drawbacks, the following solution is proposed :



A point R, which constitutes the initial mark of reference for the various directions, both from A and from B, is chosen.

The point R must constitute an optimum origin. It must therefore be at a distance from the base not inferior to 2 km. and present an optimum vertical line of collimation. From the point R, the angle r is measured with the maximum possible accuracy. (In the event of the distance AR being known, r is easily calculated). From A and B, the angles α, β and y are measured.

Considering the figure, it is easily deduced that :

$$\gamma = \alpha + r - \beta \qquad S = \frac{b \sin(\beta - y)}{\sin \gamma}$$

In this manner, γ is independent of the reciprocal direction AB; but it is a function of the angles α and β (r is a constant which as has been said, is measured with the maximum possible accuracy), which, in the present case, may be measured with a good approximation even if the theodolite is not very accurately centred on the points A and B. Assuming the origin R to be at a distance of 4 km. from the base, a centring error of 1 cm. in setting up the theodolite, there would correspond a maximum error of 0.5" in the measurement of the angles α and β. An error which decreases in proportion to the increase in the distance of R, and with the decrease of the eccentricity.

Depending on the relative position of the various points which compose the figure, 8 different cases may present themselves, giving rise to changes in signs in the two formulae of solution.

But, considering that, in our case, we are interested only in the absolute value of γ and of S, the 8 different cases are solved by the following formulae :

$$S = \frac{b \sin(\pm \beta \mp y)}{\sin \gamma}$$

$$(1) \qquad \gamma = \pm(\alpha + r) \mp \beta$$

$$(2) \qquad \gamma = \pm(\alpha - r) \mp \beta$$

In so far as S is concerned, the formula is always the same, and it is immaterial to take β from y or vice versa. For the determination of γ; formula (1) should be used if the point B is to the right when viewing the point R from A; formula (2) if the point B is to the left. Here again it is immaterial to take $(\alpha + r)$ or $(\alpha - r)$ from β or vice versa. In all cases, therefore, it will be appropriate to take the smaller angle from the larger, considering, of course $(\alpha + r)$ or $(\alpha - r)$ as a single angle. We must distinguish the end A of the base (to which point the distances it is intended to measure, are referred) from the other : we shall call it the main point.

A special case of this method deserves particular attention, namely that when the reference angle r assumes the value zero, that is, when the point of reference is situated in the bearing AB or BA. This, particular *x* case can be very easily realised in practice. In actual fact, from the main point, which will be the known point (either because on it there will be located the provisional astronomical station, or because it will coincide with a vertex of the fundamental geodetic system), it will always be possible to pick out, at a convenient distance, in the determined angular sector of observation, a point in the field, which lends itself to selection as origin of reference for the various directions to be observed; along this direction it will be possible to measure the base b.

In this particular case, which, most probably, will always be realisable in practice, the formulae of solution, whatever the relative position of the points considered, are reduced to the following :

$$S = \frac{b \sin \beta}{\sin \gamma}$$

$$\gamma = \pm \alpha \mp \beta$$

The value of γ will thus be given by the difference between α and β, taking the smaller angle from the larger.

It is now of interest to establish, in the practical domain, what possibilities are offered by this method, in relation to the goal aimed at, i.e. the determination, by polar co-ordinates, of a certain number of points about the provisional astronomical station, for the purpose of the survey of the 1 : 400.000 map of the Empire.

It is first necessary to settle the accuracy it is intended to obtain. The graphical error in plotting the points in the map is, on the average, of 0.2 mm., which, at the scale of 1 : 400.000, is equivalent to 80 m. If all of the points were determined with errors not exceeding 80 m., we would obtain, considering the scale, a rigorously geometrical map.

As it is question of a map constructed from running surveys, and considering that the fundamental points themselves, determined by astronomical observations, may be affected with appreciably larger errors, even if for no other reason than the deflection of the vertical, it is safe to assume, especially for the points of an inferior order in the series of determinations, that an accuracy of even half, i.e. of about 150 m., is acceptable.

Having established this maximum error, we may examine up to what limit of distance the method expounded is liable to yield satisfactory results, having regard to the instruments employed.

The length of the short base will vary from 100 to 200 m., according to the nature of the ground about the fundamental point and according to the range of the distances it is desired to measure.

With the modern means of direct and indirect measurement of distances (the parallactic method with the subtense bar of fixed length, either horizontal or vertical, offers the possibility of precise and easy measurements); measurements may be obtained with an even greater appoximation than one two-thousandths of the distance itself.

Differentiating the formula of solution :

$$S = \frac{b \sin (\beta - y)}{\sin \gamma}$$

with respect to "*b*", we have :

$$d\,S = \frac{\sin (\beta - y)}{\sin \gamma} d\,b$$

and substituting $\frac{S}{b}$ for the ratio of the sines :

$$\frac{d\,S}{S} = \frac{d\,b}{b}$$

The relative error of the base and of the distance is the same.

For this reason, the error with which the measurement of the distance is affected on account of the error introduced in the measurement of the base is, in relation to the assumption made, of $\frac{1}{2000}$, which, at a distance of 40 km., corresponds to an error of 20 metres.

Differentiating with respect to $(\beta - y)$:

$$dS = \frac{b \cos (\beta - y)}{\sin \gamma} \quad d\,(\beta - y);$$

and since :

$$\frac{b}{\sin \gamma} = \frac{S}{\sin (\beta - y)}$$

by substituting, we have :

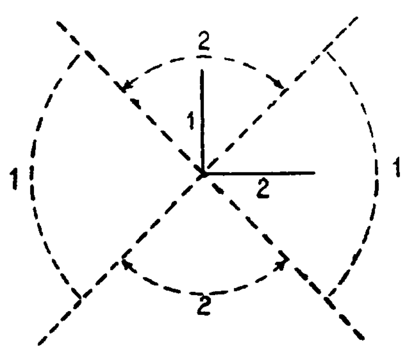
$$dS = \frac{S \cos (\beta - y)}{\sin (\beta - y)} \quad d\,(\beta - y)$$

or

$$(1) \quad \frac{d\,S}{S} = \operatorname{cotg} (\beta - y)\; d\,(\beta - y)''\, \text{arc}\; 1''.$$

It was said, at the outset, that the short base must be roughly perpendicular to the relative direction of the point of which it is desired to measure the distance. It is not possible to maintain this condition, because it would then be necessary to measure a short base for each small sector, and the method would thereby lose a great part of its practicability and expedition.

For determining distances on a round of angles, it suffices to measure two bases at right angles to one another.



In the sketch, the sectors of influence of the two bases are indicated, and from this it follows that the serviceable base for the calculation of the distance, i.e. the component of the base measured in the direction normal to the distance (*b'*) is, in the most unfavorable case, nearly one and a half times shorter than the measured base itself.

The two normal bases represent, however, a minimum, and, in practice, it will be expedient to measure three bases at about 120°, which will give the possibility of obtaining accuracy and of making checks. With three bases thus arranged, it is possible in fact for many points, to calculate the distance with respect to two different bases.

From these considerations it appears that the difference ($\beta - y$) will be contained between a maximum of about 90° and a minimum of about 45°. Consequently the cotg ($\beta - y$) will be able to reach a maximum value 1. By substituting this value in (1) we have :

$$\frac{d\,S}{S} = d\ (\beta - y)''\ \text{arc}\ 1''.$$

By substituting further for "arc 1" its approximate value 1 : 200 000, and considering that error in the measurement of ($\beta - y$), for the reason already stated, may amount to 20", we have :

$$\frac{d\,S}{S} = \frac{20}{200.000} = \frac{1}{10.000}$$

A negligible error in view of the goal aimed at.

Differentiating with respect to γ we have :

$$d\,S = \frac{-\,b \sin(\beta - y) \cos\gamma}{\sin^2\gamma}\, d\,\gamma$$

or

$$\frac{d\,S}{S} = -\ \text{cotg}\ \gamma\ d\,\gamma''\ \text{arc}\ 1''.$$

Here, it is necessary to settle the approximation with which γ can be determined.

The theodolite employed for the works it a Zeiss II, which, as already stated, allows an accuracy of 1" in the reading of the circles. The practical approximation of this instrument may be assumed to be 2". If we decide to measure each direction to the Point P four times with conjugated observations (8 single observations), then the mean relative error at each of the angles α, β, r will be $\frac{2}{\sqrt{8}}$; consequently, the mean error of γ which is derived from their algebraic summation, will be $\sqrt{3}\ \frac{2}{\sqrt{8}} = 1''.5$, which, owing to the small number of observations, it is advisable to consider as 2".

We have already seen that the relative error at ($\beta - y$) is negligible, and that due to the measurement of the short base may be assumed $\frac{1}{2000}$.

Supposing, now, that it is proposed to determine a point at a distance of 30 km. with an approximation of ± 80 m., and therefore with a relative error of $\frac{80}{30.000} = \frac{1}{375}$, the maximum relative error with which γ may be affected, in order this allowance be not exceeded, will be :

$$\frac{1}{375} - \frac{1}{2000} = \frac{1}{460}$$

From the expression

$$\frac{d\,S}{S} = -\ \text{cotg}\ \gamma\ d\,\gamma''\ \text{arc}\ 1''$$

omitting the sign of the 2nd member, which in our case has no significance, we deduce :

$$\cot g\,\gamma = \frac{d\,S}{S} \cdot \frac{1}{d\,\gamma''\ \text{arc}\ 1''}$$

Substituting the figures found, and giving the arc 1" the value 1 : 200 000, we get :

$$\cot g\,\gamma = \frac{1}{460} \cdot \frac{200.000}{2''} = 217; \qquad \gamma = 0°16'.\ \text{approximately.}$$

To this magnitude of γ there corresponds, at the distance of 30 km., a useful base *b'* of 139 m., and allocating to (β — *y*) the minimum value 45°, a measured base of 196 m. It follows, from what has been said up to now, that to obtain determinations with an approximation corresponding to the graphical error of the map, it suffices to measure short bases of 200 m., even for points at 30 km. distance, and disposed in the worst manner with respect to the orientation of the base. It follows, too, that, having allocated to γ a determined mean error, the relative error of the calculated distance is a function of γ alone.

In this manner, by the magnitude of γ, we can settle *a priori* the centesimal error with which the measured distance will be affected. For facilitating this calculation, the following table is given in which a mean error of 2" has been assigned to γ.

ERROR OF THE DISTANCES FOR γ TO WITHIN 2".

γ	$1000 . \frac{\Delta S}{S}$	Value of S for *b'* = 1	γ	$1000 . \frac{\Delta S}{S}$	Value of S for *b'* = 1
10'	3.33	344	50'	0.67	69
15'	2.22	229	1°	0.56	57
20'	1.67	172	1° 30'	0.37	38
25'	1.34	138	2°	0.28	29
30'	1.11	115	2° 30'	0.22	23
35'	0.95	98	3°	0.18	19
40'	0.83	86	4°	0.14	14
45'	0.74	76	5°	0.11	11

To arrive at these conclusions, we have placed ourselves in the most unfavourable conditions; as a matter of fact, when measuring in the field three bases set at 120°, it will no longer be essential to have values of (β — *y*) which attain 45°. Geometrically the value of (β — *y*) ought not to fall below 60°.

With the 200-metre base oriented in the best manner with respect to the direction of the point to be determined, and preserving the approximation of 1 : 500, it is possible to calculate distances up to 40 km.

It should be noted also that the relative error being a function of cotg γ the accuracy increases rapidly with γ. Thus, for points situated at 15 km., and under the most unfavourable circumstances with respect to the orientation of the base the relative error is approximately $\frac{1}{1000}$, i.e. a linear error of about 15 m. Under more favourable conditions the linear error is reduced to 11 m. We have assumed, also, that, in the measurement of the short bases, an error of $\frac{1}{2000}$ may have been involved; whereas, in practice, with the modern methods of measurement, it is possible, with comparative ease to reach an approximation even greater than $\frac{1}{4000}$, an error which, in relation to our aims, may be considered negligible. It may thus be asserted that the accuracy in calculation of the distances depends on the accuracy with which γ may be determined.

With the measurement of three bases in the field, or even of only two, it is possible to calculate, for any points, the distance twice independently; the table of the centesimal errors permits, besides, to make between these two values, which are a check on each other, an average estimation, which confers upon the definite value of the calculated distance enhanced accuracy and the assurance of not having gone astray.

Until now, however, we have considered the mean errors, which errors have a great chance of being exceeded. Assuming, practically, that the maximum error to be feared is three times the mean, we may have, for the limiting distance considered, errors up to 250 m., corresponding to somewhat over half a graphical millimetre.

But it is necessary to note that these errors would be involved at the maximum distances only. In fact, for distances of 20 km., the maximum error to be feared is approximately 80 m. under the most unfavourable conditions of orientation of the base.

It may therefore be concluded that, with bases of 200 m. a mean error of 2" in the measurement of γ, and an orientation of the base (β — y) not inferior to 45°, it is possible to calculate distances up to 20 km. with the almost absolute certainty of not having introduced errors superior to the graphical error.

Whilst it is possible to calculate distances up to 40 km. under favourable conditions of orientation of the base, with a probability of over 50% of not having exceeded the graphical error. and with the improbable possibility of introducing errors up to 0.5 graphical millimetres, it is possible, on the other hand, in many cases, to calculate the same distance from two different bases, thus obtaining, alongside greater accuracy, a check on the operations performed.

With the fundamental point determined by provisional astronomical observations, the technical guiding principles prescribe the determination of the direction of the graphical North.

In this manner, from the fundamental point, with the measurement of azimuth and of the distance, it is possible to plot on the map, with polar co-ordinates, as many point as desired about the fundamental point itself.

Along the direction of the traverse, it is advisable, when possible, to determine points at progressive distances, which allow the plotting of all, or of part, of the vertices of the polygonal chain (figures) which, as prescribed, link together the fundamental points.

On these vertices, if convenient, measurements by the short-base method may be effected; it will be possible to define the original direction of the polar co-ordinates from the fundamental point which, obviously, should be visible from these vertices. It will thus be possible to obtain, for certain more important points, double determinations from various stations, and, consequently, a reliable and controlled, in addition to accurate, framing of the survey. It will, besides, be made possible to extend the polygonal chain, without interruption, towards the succeeding fundamental point.

But it is necessary to recall that this method, if it is to yield the presumed results, must be carried out with great care; particular attention must, above all, be paid to the measurement of the angle γ.

It is therefore necessary to select those features of the country which offer a well-defined line of collimation. In the exposition made, it has been taken for granted that it is possible to obtain the measurement of γ to within 2" of mean error. This is a result easily attainable; but it is necessary to work with discernment and precision.

It is always advisable, and most likely will always be possible in a country as varied as that of the Empire, to find, in the mountainous part, initial points offering a good collimation, and which allow the definition of a direction along which to measure a base; otherwise it will be necessary to procure the reference point R.

Here it is necessary to be careful to set up a signal which lends itself to an easy and dependable collimation, and which is situated at a distance such as to render inappreciable the error due to the setting up of the theodolite and to the substitution of same by the subtense bar, and vice versa. It is therefore necessary that the angle r be measured with the utmost accuracy, because the approximation with which this angle is measured constitutes a systematic error which it is not possible to evaluate.

Requisite experience is lacking at present; but, insofar as permissable, it may be hoped that, owing to the great precision of the theodolites employed, the practical results will substantiate the predictions.

At any rate, the proposed method must not be understood as a rule, but merely as an auxiliary for which practical experience will decide as to its merits.

But the possibilities of this method have not ended here. In the technical guiding principles for the construction of the 1 : 400 000 map, it is said that it is possible, and necessary, to utilize photograms taken from the air; and furthermore that the aerial photographs will be utilized a second time for regular surveys by means of complementary successive and appropriate operations carried out on the ground.

In the direct survey of the 1 : 400 000 map it might be run counter, and with evident reason, to the exigencies of the regular aerophotogrammetric survey, whilst at the same time creating the best conditions for a more rational and profitable utilization of the aerial photograms to effect the survey of the 1 : 400 000 map.

In this connection a few remarks are indicated. In the 1 : 100 000 aerial survey by the new Santoni method, which provides for the integral utilization of an aerial triangulation, such as has been evolved and realized by Major Santoni himself with his new ingenious solar periscope, the preparation on the ground is reduced to a minimum. It is therefore a question of fixing along the axes of the strips, executed with the "Santoni four-lens colonial aerophotogrammetrical machine with a large angular field", a series of points up to 100 km. from one another, and determined by provisional astronomical observations. Close by each of these points, it is necessary to measure, in suitable directions, short bases about 1.5 km. long. Although not indispensable, it would be nevertheless useful to determine along the direction of the strips, and at a distance of 10 to 15 km. from the fundamental point, a point with a particularly accurate figure. The purpose of this point, at the beginning of each section of aerial triangulation, is that of causing to be evaluated, in first approximation, the amount of the "resultant systematic" error, which, introduced from the commencement of the calculations, will permit an appreciable reduction in the differences in closure of the various sections of aerial triangulation.

The geodetic frame for the survey of the 1 : 400 000 map is thus more than adequate for the regular aerophotogrammetric survey to 1 : 100 000 by the Santoni method. And it is such, not only by the density of the fundamental points, but by its accuracy also. In fact, the provisional astronomical stations prescribed for the survey of the fundamental map of the Empire are of an order of accuracy liable to satisfy the requirements of the aerophotogrammetric survey on a 1 : 100 000 scale.

Among the points determined by polar co-ordinates from the fundamental point, it is easy to select those which lend themselves best to altimetric checking.

For the measurement of the photogrammetrical bases, the short-base method affords the possibility of optimum determinations. In fact, for calculating distances of the order of 1 to 2 km., the angle γ assumes values comprised between about 3° and 5°, with measured bases of 100 and 200 m. respectively. From the synopsis of the relative errors, it follows that, for such a value of γ, the approximation of the relative distance is between 0.2 and 0.1 per thousand; an accuracy which, in reality, will be lower, because, for distances thus small, it is more difficult to measure γ within 2" of approximation; but, as in this particular case, the predominant error is obvously that introduced in the measurement of the short base, it may be asserted that the relative error corresponding to distances of this order of magnitude, is practically equal to that with which the short base is affected. We have to do, thus, with an order of accuracy more than sufficient for the proposed purpose.

It would seem, from what has been expounded, that the geodetic frame of the 1 : 400 000 survey, sagaciously executed, might constitute also the complete preparation on the ground for the 1 : 100 000 aerophotogrammetric survey. However, in practice, the solution of this problem is not so simple and easy. The reason is obvious : it is indispensable, therefore, that the points determined on the ground be reliably identifiable, too, on the photographs.

For this purpose, the Military Geographical Institute applies, as a rule, the method of determining on the ground those natural signals which are readily identifiable both on the photographs and on the ground. Hence, the photographic exposure precedes the preparation on the ground. But this rule does not preclude that, when it is convenient, another method be adopted.

I am of opinion that it is not appropriate, during the survey work of the 1 : 400 000 map, to make photographable the signals on the ground, for the following reasons :

(1) excessive overloading and, consequently, slowness in the survey of the fundamental map;

(2) difficulty in the construction of large-size planimetric signals, in order that they stand out clearly on photographs taken from a height of flight relative to the survey;

(3) difficulty of giving these signals the necessary qualities of stability and durability;

(4) because the survey of the fundamental map and that of the aerophotogrammetric map are fully independent of each other, it is not possible to foresee which will be the axes and the orientations of the "strips", and, consequently, it is not possible to establish what will be, on the ground, the location of the points meant to constitute the preparation on the ground. However, judicious selection of the fundamental points and of the frame points determined by polar co-ordinates, for the purpose of the 1 : 400 000 survey, might make a good part of this work utilizable for the aerophotogrammetric survey. In the technical guiding principles it is, in fact, said: "the astronomical stations shall be effected at points on the ground well-defined both naturally and artificially (wells, tumuli, rocks, signals, etc.) and, preferably, in dominating positions". Of these stations, there should be made an accurate monography, supplemented by photographs, if possible stereoscopic".

In these provisions there seems implicitly understood that points thus selected are to serve for aerophotogrammetric surveys; anyhow, whenever possible, it is certainly very useful to locate these points on natural or artificial objects of the country which it is presumed will stand out clearly on the aerial photographs.

As concerns the determinations carried out with polar co-ordinates by the short-base method, the matter is still easier, because, obviously, these determinations refer to natural signals of the country, which, while changing the sighting point, it is safe to hope are retraceable, at least partly, in the photographs from the air.

In any case, such precautions are necessary if it is proposed to make a rational use of the aerophotograms to satisfy the 1 : 400 000 survey.