NOTE ON THE CALCULATION OF THE POSITION AT SEA

by Ingénieur hydrographe en chef

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1) The determination of the line of position by the Marcq St Hilaire method involves the solution of the spherical triangle defined by the three consecutive elements: the assumed co-latitude of the vessel; 90° — L, the computed hour angle of the observed heavenly body; the polar distance of this body. The unknown elements are the third side 90° — h, where h represents the computed altitude and the angle Z, the azimuth of the observation.

The trigonometric calculation is simple and does not, perhaps, take any more time than the successive operations in fixing the position at sea, from the astronomical observation to the plotting of the line of position on the chart. However, there is always the fear that some material error may have crept in, which is possibly excusable in the difficult conditions under which the watch officers have to work, but which is, nevertheless, fraught with grave consequences. For this reason, numerous investigators have long sought to simplify the solution of the celestial triangle by the use of methods, tables, diagrams or various other apparatus.

2) Now, the spherical triangles which are defined by three consecutive elements (one side comprised between two angles or an angle contained between two sides) possesses very *special property:* they can, in fact, only be solved by equations of the general form:

(1) $\tan u = \cos v \tan w,$

in which, u, v, w, designate the elements (or their complements) of the given triangle or of its resolution into two right spherical triangles. Further, the elements to be determined in the course of the calculation, auxiliary variable or unknowns strictly speaking, always occur in the formulae as tangents, which fact, aside from the advantage deriving from the determination of the angles by this trigonometric function, always permits us to make the unknown elements correspond with the variable u in equation (1) so that the performance of the calculation remains exactly the same in the sequence of operations.

3) The application of this property to the solution of the problem of fixing the position at sea appears susceptible of furnishing an advantageous method for the calculation for the Marcq St Hilaire position line.

In fact, if we make use of logarithms, the calculation necessitates the use of a single table of logarithms of trigonometric functions and involves additions[1]. Usually it is carried out with five decimal places, but if we are satisfied with the computed altitude with an approximation of one minute of arc, four decimal places will suffice, since the logarithms of the tangent of arc vary by at least 2 1/2 units at the fourth place for a variation of 1' of arc.

But the principal interest of this method of calculation centers in the simplicity and uniformity of the applied formula, which lends itself to an extremely easy mechanical solution and thus permits the construction of "a machine for calculating the position," which is reliable, economical and not unwieldy.

4) The method is in reality, not new. In his *Traité d'hydrographie* published in 1882, the hydrographic engineer Germain, a propos of the solution of the spherical triangles and without regard to any particular practical application, mentions the advantages of the determination of the angles by means of their tangents and gives the group of formulae for the corresponding solution.

More recently Captain Radler de Aquino, of the Brazilian Navy, who for thirty years has been studying the various methods of bringing about a simplification in the calculations for position finding at sea and who prepared the tables with a view to their application, recognized in 1933 the advantage of the solution by tangents. He described it in the *Revista Maritima Brasileira* (supplement to the number of Sept-Oct 1933) and in the *United States Naval Institute Proceedings* (May 1937) and he has published since 1934 the tables of logarithms appropriate to its application.

Since, to the best of our knowledge, the sole french text pertaining to this method is a brief summary of the articles of Commander de Aquino which appeared in the *Hydrographic Review* (November 1937) it does not appear that it would be superfluous to develop them anew, and before showing what advantage may be gained by the construction of a machine to calculate the position, to show what advantages it already possesses in making the calculation with the aid of tables.

5) Let us consider the celestial triangle having as vertices the pole P, the star observed A and the Zenith Z of the assumed position.

(1) The difficulties experienced by certain persons in the use of the negative characteristics disappear when we resort to the artifice employed in various logarithm tables which consists in adding implicity 10 units to the logarithms of functions having a value less than unity, which is the same as replacing the characteristics —1, —2, —3, ... by 9, 8, 7, ...

The additions and subtractions are performed as though the characteristics were exact, it suffices in the greater number of operations to omit the figure giving the ten of the characteristics; for example :

$$7{,}2342 + 1{,}5195 = 8{,}7537\ ;$$
$$8{,}5222 - 9{,}4183 = 9{,}1039$$

Also; if log a = 8.3184 colog a = 1.6816.

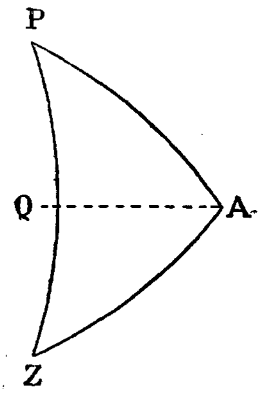


FIG. 1.

The latitudes and declinations are taken at their absolute values and are designated not by sign but by name, as North or South; the azimuths Z are computed from 0 to 360° from North by way of East.

Let us call Q the point of intersection of the meridian PZ with the great circle passing through A perpendicular to this meridian and designate by 90° — x and 90° — y the arcs of the circle PQ and QA. A consideration of the right spherical triangles PQA and AQZ leads to the following equations : —

$$
(2) \qquad \begin{aligned}
\cot x &= \cos P \cot D \\
\cot y &= \cos x \operatorname{tg} P \\
\cot z &= \sin (x \pm L) \tan y \\
\tan h &= \cos z \cot (x \pm L).
\end{aligned}
$$

The auxiliary arcs x, y and z are taken in the first quadrant. When the hour angle is acute, we take x — L (or L — x, if x is less than L) when the latitude and declination are of the same name; x + L when they are of the contrary name, and inversely when the hour angle is obtuse. This rule has been summarized in the following table N° I.

TABLE N° I

Latitude and declination of *same name*	P acute from 0 to 6 h or from 18 to 24 h	x — L or L — x	P obtuse from 6 to 18 h	Latitude and declination of *contrary names.*
	P obtuse from 6 to 18 h	x + L	P acute from 0 to 6 h or from 18 to 24 h	

To pass from z to Z, we apply the table II, which gives the expressions for Z as a function of z, according to the names of the latitude and the declination and the values of the hour angle.

TABLE N° II

	Hour angle in the case of *North Latitudes*			
	0 to 6 h	6 to 12 h	12 to 18 h	18 to 24 h
Declination North	if x > L 360° — z if x < L 180° + z	360° — z	z	if x > L z if x < L 180° — z
Declination South	180° + z	if x > L 180° + z if x < L 360° — z	if x > L 180° — z if x < L z	180° — z
	12 h to 6 h	6 to 0 h	24 to 18 h	18 to 12 h
	Hour angle in case of *South Latitudes*			

These formulae (2) call for the following explanations : —

a) The azimuth is obtained before the altitude by three applications only of the equation (1) such that the calculation by machine or the form for calculation may be utilized very advantageously when computing the astronomic azimuth, for instance, to obtain the compass error. We may, in that case, resort to the logarithms of three decimal places, the result being obtained to one tenth of a degree.

b) The unknown auxiliary *y* is given by the cotangent (second equation group 2); and it is subsequently utilized in its tangent function (third equation group 2) When making the calculation by logarithms this particular fact reduces the number of required entries in the table by one, because it is not essential to know the value of y.

6) *Logarithmic Calculation*

The calculation of h and Z by logarithms with the aid of the equations (2) necessitates only *six entries* in a single table of trigonometric functions (three given quantities, 2 unknowns and one auxiliary variable) and the inscription in addition of the 3 given quantities, of *17* quantities thus obtained, 10 from the table, 4 by addition, 1 by subtraction and 2 by addition or subtraction.

This furnishes directly the azimuth and the altitude of the assumed position without the necessity of recourse to an auxiliary position.

1 st example (5 decimals)

L	44°20',5 N								
D	43°18',5 N	l cot	0,02566						
P	$7^h\ 39^m\ 42^s$	l cos	9,62472	l tg	0.33282				
x	65°54',7	l cot	9,65038	l cos	9.61081				
y				l cot	9.94363	l tg	0,05637		
x + L	110°15',2					l sin	9.97228	l cot	9.56701
z	43°06',7	Z	316°53',3			l cot	0,02865	l cos	9.86334
h	15°04',6							l tg	9,43035

2 nd example (4 decimals)

L	44°20',5 N								
D	24°50' N	l cot	0.3346						
P	$3^h 29^m 10^s$	l cos	9.7865	l tg	0.1117				
x	37°06',7	l cot	0.1211	l cos	9.9017				
y				l cot	0.0134	l tg	9.9866		
L — x	7°13',8					l sin	9.0999	l cot	0.8967
z	83°02',5	Z	263°02',5			l cot	9.0865	l cos	9.0833
h	43°41',0							l tg	9.9800

This method may be more difficult to memorise than that of the calculation of h and Z by the fundamental formula and the analogy of the sine, but the use of the last two formulae necessitates eight entries in two tables (3 in the table of logarithms of numbers, 5 in the table of logarithmic functions) as well as writing down 18 quantities of which 13 are obtained from the tables and 5 obtained by addition or subtraction. Further, h and Z are given as sine functions, which is not without its disadvantage when their values are in the vicinity of 90°.

Note : I. Considering the triangle QZA, we have
$\sin h = \cos (x \pm L) \sin y.$

An error of d (x ± L) involves in h an error given by

$$\cos h \, dh = - \sin (x \pm L) \sin y. \quad d (x \pm L)$$

from which $dh = - \cos z. \, d (x \pm L)$.

The error in altitude is therefore always less than the error committed in x ± L.

In particular, if we do not seek for h an approximation closer than one minute, we may always round off x ± L to the nearest minute to avoid interpolations; the error which will result in the altitude will then be a maximum of one half minute.

When the observed azimuth differs by less than 11° 32' from the azimuths of 90° or 270°, cos z is less than 1/5th and the rounded value of x ± L to the nearest minute involves for h an error less than 1/10th of a minute which is therefore negligible.

Finally, in every case, one can always limit the approximation of x ± L to one tenth of a minute, by taking care however to utilize exactly the same value of this arc in the last two equations of group (2);

II — The angles x and z are obtained from their cotangents and intervene subsequently in the cosine function.

When those angles are close to o, the tabular difference of log cotan. is large, but that of log cos. is small and the interpolation necessary for the determination of log cos. is made by inspection.

On the other hand, when x and z are close to 90°, the tabular differences of log cotan. and log cos. are both large, but sensibly equal, so that log cotan x (or z) and log cos x differ by the same amount in the logarithms furnished by the table for the rounded off value closest to the angle, which permits a ready calculation of log cos x. If the table gives the log cosec. we may equally obtain log cos x by subtracting from log cotan x the log cosec x, which is very small, positive and which, varying so slightly, can be interpolated by inspection.

Examples : —

a) log cotan x = 1.25503

The tables gives: for 3° 10', log cot = 1.25708 log cos = 9.99934
for 3° 11', log cot = 1.25479 log cos = 9.99933

whence : x = 3°10'.9 log cos x = 9.99933.

b) log cot x = 8.99964

we find in the table: for 84° 17', log cot = 9.00046 log cos = 8.99830
for 84° 18', log cot = 8.99919 log cos = 8.99704

We proceed then : —

log cos x = log cos 84°18' + [log cot x — log cot 84°18'] = 8.99749
x = 84° 17',6.

Or better, if the table gives also: for 84° 17', log coses. = 0.00217
for 84° 18', log cosec. = 0.00215

log cosec. x = 0.00216 and log cos x. = log cot x — log cosec x = 8.99749.

7) *Calculating Machines.*

Formula (1) tg u = cos v tg w. may be solved with the aid of a slide rule composed of two logarithmic scales, scale (T) graduated in log tangents and scale (C) graduated in log cosines. One brings into coincidence the graduation v on scale (C) with the w on scale (T) and reads off u on scale (T) opposite the graduation o on scale (C).

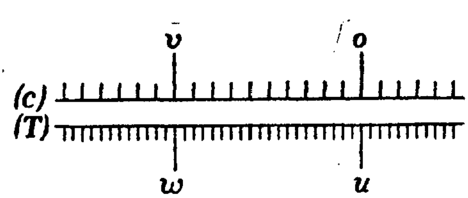


FIG. 2.

But these scales must be very long if we are to obtain the accurary requisite for nautical calculations. By adopting a length of 10 meters to represent one unit of logarithms, an interval of 1' on the scale of tangents corresponds to a minimum of 2.5 millimeters, which permits a least reading by inspection of 1/10th to 1/5th of a minute.

With this modulus, the length of scale (T) is then 76.75 meters between 0°0.5' and 89°59.5' : the scale (C) is one half as long.

In practice however, large displacements of the scales are not called for very frequently, scale (T) measures in fact.

8.78	meters	between	20°	and	70°
15.07	meters	between	10°	and	80°
21.16	meters	between	5°	and	85°

As to scale (C) it measures : —

4.66	meters	between	0°	and	70°
7.60	meters	between	0°	and	80°
10.60	meters	between	0°	and	85°

Therefore, it is only with angles in the vicinity of 0 and 90° that it becomes necessary to give the scales a displacement of more than about 15 metres.

It is evident that one cannot consider the employment of rigid scales of such length, but it is easy to devise a procedure utilising the graduated scales wound on drums or engraved on helices about a cylinder.

One may also arrange to read the scales, either on the graduated film or on counters operated mechanically or electrically, but with the same length of scale, the counters yield in somewhat less accurate results because they do not permit of interpolations.

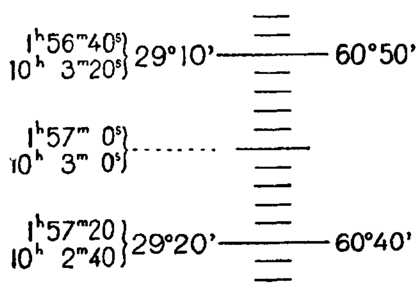
Certain requirements of maritime or aerial navigation may lead, from a practical standpoint, to the adoption of this or that calculating machine which may be developed, but, in this investigation, we shall consider solely the problem of fixing the position with the greatest possible accuracy.

In that case one of the best solutions with graduated scales appears to be that invented by Commander Le Sort for the construction of a calculating machine which is also designed to calculate the position at sea and which is manufactured by the firm of Carpentier[1]. The graduations are etched on the celluloid films, which are perforated along the edges like the cinema film, conveyed without slipping by spin wheels with gear teeth and wound on spools at the ends. Each film passes beneath a window fitted with a glace

(1) See *Revue Maritime*, 1928, 1st quarter, page 223.
See Hydrographic Review, Vol. VII N° 2 Nov., 1930, page 114.

plate, on the lower side of which is engraved a fine line to serve as a reference mark. The films are moved with the aid of a crank for rapid displacements and with a button for the fine setting.

The machines which we are considering comprise two graduated scales, (T) and (C). The graduations w of the film (T) and v of the film (C) having been placed on the corresponding indices, the spur wheels for the movement of the film are locked together and the two films are turned, either with the aid of the crank or else automatically under the influence of a retracting spring, until the film has unwound and its further movement arrested by a stop, when the graduation *o* of film (C) is under the index. It suffices then to read off u on the film (T) under the other index. The two films are then released from each other and the apparatus is ready to function anew.



In order to permit the execution of the four successive operations corresponding to the formulae (2) each film should carry three series of graduations, one in hours from $0^h\ 0^m\ 0^s$ to $11^h\ 59^m\ 58^s$ [1] and two complementary graduations in arcs from 0° 0.5' to 89° 59.5'.

If the current dimensions of the films will not lend themselves to all of these inscriptions — one scale and three rows of figures — we may juxtapose for each scale two films which are carried simultaneously on the wheel containing three sets of teeth; one film may carry then two rows of figures, the other the graduations and one row of figures. [2]

Finally, the spur wheels may be geared down and provided with pointers which move over a graduated dial, in order to show the operator the approximate position of the moving films and thus to save appreciable time in the setting of the films.

In order to avoid all errors in the copying down of the results or in the reading, the tabulation given below is kept in plain view of the operator showing him which row of figures are to be utilized for the calculation and permitting him to follow their movement if necessary.

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(1) When the hour angles are comprised between 12 and 24 h the operator first subtracts 12 hours.

(2) One may also arrange on the film of ordinary dimensions one graduation and the two complementary graduations in arc; thereupon there is arranged a third film absolutely independent of the other two, comprising a double graduation in arc and in hours, which gives by direct reading the arc of the first quadrant corresponding to the hour angle under consideration.

Latitude & déclinaison de *mêmes noms*	P de 0 à 6^h ou de 18 à 24^h	x — L ou L — x	P de 6 à 18^h	Latitude & déclinaison de *noms contraires*
Same name	P de 6 à 18^h	x + L	P de 0 à 6^h ou de 18 à 24^h	*contrary names.*

This tabulation may be replaced by four small opaque celluloid sheets each carrying the indications relative to one operation and are successively dropped into place in front of the windows with the aid of a hinged joint.

Electrical devices, ingeniously contrived, may bring to the operator considerable convenient aids in the detailed use, but even though the circuits may be simple and well protected, the use of electricity introduces the risk of improper functioning which should not be run, since the apparatus must offer the maximum possible reliability.

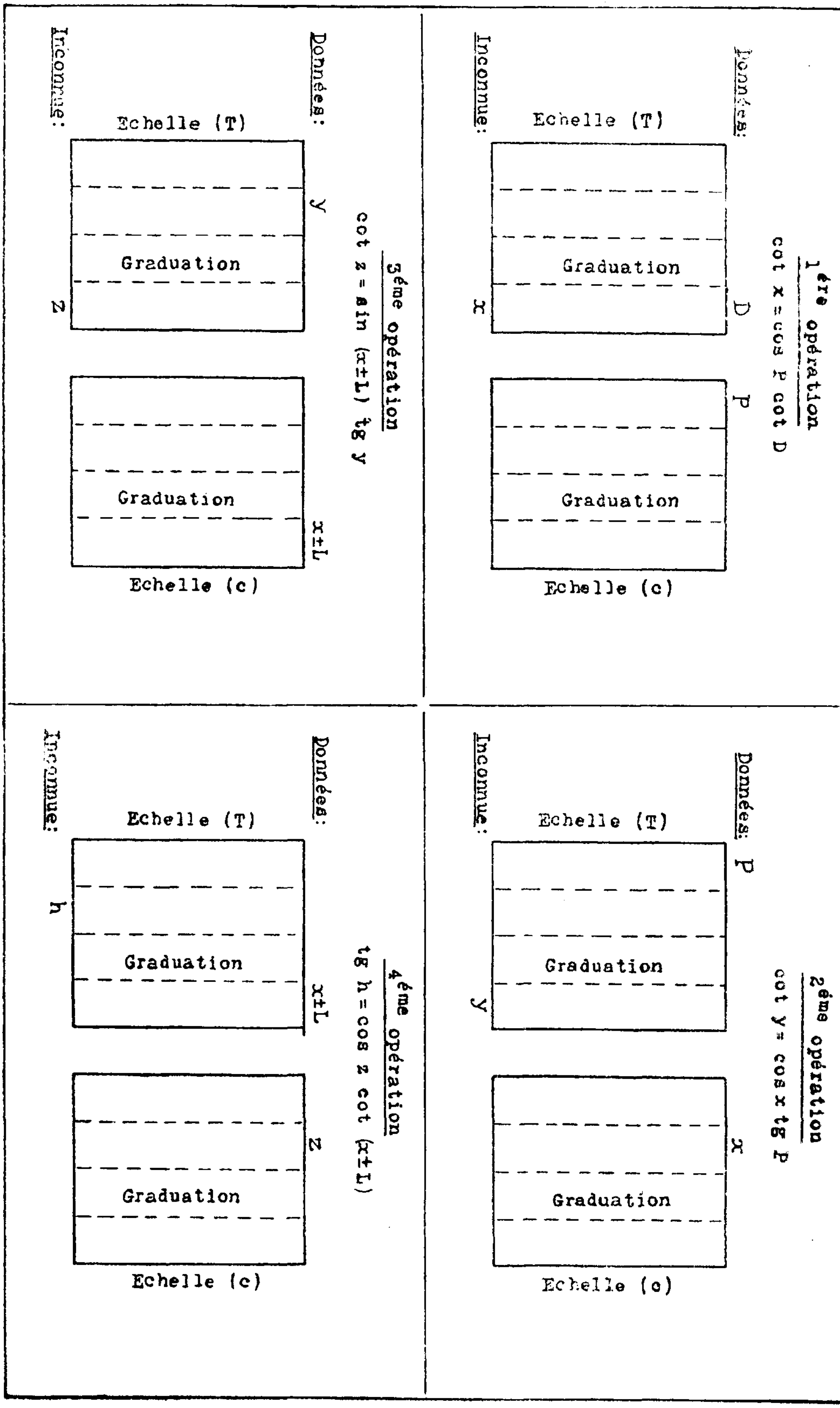
For the calculation of the position at sea the use of this machine necessitates eight inscriptions of the given elements and four readings of the unknown, the last two readings furnishing the elements determining the Marcq St Hilaire position line; azimuth and computed altitude.

The machine of Commander Le Sort, which comprises eight graduated films, four differentials and four counters necessitates 7 inscriptions and three readings to furnish the single computed altitude, the azimuth being obtained rapidly thereafter with the aid of an auxiliary calculating scale. The principle interest of the Le Sort machine is that it preserves the inscription of all the elements of the calculation until it is terminated. But since the price, its unwieldiness and the risk of break-down of a machine increases with the complexity of its parts, the solution which we recommend seams more advantageous from several points of view.

8) The method which we have explained and the apparatus which lends itself to its application also permits the calculation of nautical equations, other than those for the position line; for instance, the orthodromy, or the determination of the astronomical azimuth by the hour angle, which, as we have noted above, constitutes a part of the calculation for the position.

The formulae for the solution of the various problems are readily established. We thereupon determine the correspondences between the elements of the problem and the symbols employed in the method of calculation for the position or the machine. It is necessary however, in the application of these formulae to invite attention to the fact that in the calculation for the position, *h* is always an angle in the first quadrant while in some other problems the corresponding element may take any values whatever.

For the calculation of the orthodromic, for instance, we designate by P the extension in longitude and by D the latitude of the point of arrival. Z will then represent the azimuth of the point of departure and 90° — h the


1ère opération
cot x = cos P cot D
Données: D P
Inconnue: x
Echelle (T)
Graduation
Graduation
Echelle (c)
2ème opération
cot y = cos x tg P
Données: P x
Inconnue: y
Echelle (T)
Graduation
Graduation
Echelle (c)
3ème opération
cot z = sin (x±L) tg y
Données: y x±L
Inconnue: z
Echelle (T)
Graduation
Graduation
Echelle (c)
4ème opération
tg h = cos z cot (x±L)
Données: x±L z
Inconnue: h
Echelle (T)
Graduation
Graduation
Echelle (c)

orthodromic distance. We then obtain the difference in longitude l between the meridian of departure and that of the vertex, the latitude v of this point and the distance d to make good to reach it, by the equations

$$\begin{aligned} \cot l &= \sin L \tan Z \\ \tan d &= \cos Z \cot L \\ \cot v &= \sin d \tan Z \end{aligned}$$

or by other equivalent equations of the same form.