

ON THE PLANE REPRESENTATION OF THE SPHERICAL SUB-TENDED ANGLE

by

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1. **The spherical sub-tended angle** is the locus of the surface points of a sphere so that the great circles connecting them to two given points of the sphere make a constant angle between them. This geometric locus which is an extension to the sphere of the plane sub-tended angle, has assumed some practical importance with the development of direction-finding by wireless. As a matter of fact, when a mobile wireless direction-finding station carried in a ship or plane, for instance, finds a fixed transmitting station of known position, that is to say, determines the azimuth in which this transmitting station is located, the observation furnishes a locus of the mobile station position which is nothing else but the spherical sub-tended angle of the measured azimuth described on the geographic pole and on the known station.

The spherical sub-tended angle has formed the subject of various analytical or geometrical studies, several methods have been propounded for its practical application, some users, in particular, have considered it desirable to have some diagrams or projections available on which the sub-tended angle would be an easily drawn curve. Although this solution is not free from criticism, it should be interesting to deal with the problem which it raises, which is precisely the purpose of this study. As far as we know, two methods of plane representation have already been suggested, one by Mr. Lecoq⁽¹⁾ a Professor of hydrography, who has conceived a new projection, the other by Civil Engineer Bourgonnier⁽²⁾ who on this occasion re-discovered Littrow's conformal projection.

We will at first establish the existence of a small circle of the sphere in punctual correspondence with the spherical sub-tended angle. The consideration of this auxiliary curve permits to deal in a simple manner with various problems relating to the spherical sub-tended angle amongst which that of its plane representation which is tantamount to that of a small circle. We will then show that the conformal spheric projection of exponent 2 and equatorial origin causes plane sub-tended angles of the same angle to correspond to spherical sub-tended angles described on two points of the equator.

Littrow's projection, which up to the last few years was considered as a curiosity without great interest and which one hardly knew how to classify

(1) *Le segment capable sphérique*, by G. LECOQ. (*Annales Hydrographiques* 1933 and *Hydrographic Review*, Vol. X, N° 2, p. 99.)

(2) *Note au sujet d'une projection et d'un diagramme pour l'étude et le tracé du segment capable sphérique*, by G. BOURGONNIER. (*Annales Hydrographiques*, 1933.)

among projections is the transverse aspect of the preceding projection; it is therefore a spherical projection for the same reason as Gauss's projection is a cylindrical projection; it enjoys, with respect to spherical sub-tended angles whose base is borne by the zero meridian, the same property as the conformal spherical projection of exponent 2 in regard to segments of equatorial basis. By shifting on the central meridian the origin of Littrow's projection, the grid of the meridians and parallels of the representation is altered, but not its properties, so that the use of the projection can be improved when the field of application is relatively distant from the equator; particularly when locating the origin on the 45th parallel or at the geographic pole, representations suitable for medium and high latitudes are obtained; we will give some particulars of these various systems so as to facilitate their possible utilisation. Still, there are a good many other methods of representation likely to be employed for drawing spherical sub-tended angles; in general, they retain neither the angles nor the relative expanse of surfaces and some of them have the disadvantage of being suitable only for segments described on a definite basis, so that, when applying these latter systems one diagram per transmitting station under consideration must be available; nevertheless these projections may be of interest in certain cases, consequently we will make a fairly brief study of four of them, one projection with orthogonal grid and one polyconic projection both deriving by homographic transformation from conformal spherical projections already considered with the same general application, then two projections suitable only for spherical sub-tended angles relating to two given points, Lecoq's polyconic projection and one projection with parallel and rectilinear meridians.

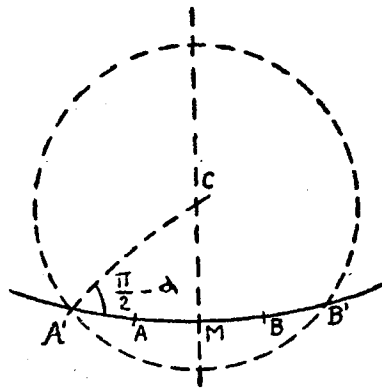


FIG. I

2. Circular correspondent of the spherical sub-tended angle

Let us take a spherical sub-tended angle of the angle α described on two points A and B (*Fig. 1*), let us denote by M the middle of the arc AB and by A' B' the symmetric of the point M in relation to the extremities of the base. Through one of the points A' or B' let us describe the great circle

making the angle $\frac{\pi}{2} - \alpha$ with A'B' and which cuts at C the great circle perpendicular to the base in its middle.

Let us now consider the small circle of pole C and spherical radius CA' which therefore passes through A' and B'. We maintain that there exists a punctual correspondance between this small circle and the sub-tended angle.

Let us assume a system of geographical coordinates admitting the great circle AB as equator and the great circle CM as zero meridian (Fig. 2). If Δ denotes the length of the arc AB, the longitudes of A, B, A', B' are respectively $-\frac{\Delta}{2}$, $\frac{\Delta}{2}$, $-\Delta$ and Δ . Let us consider the pole P and any point S of the sub-tended angle, having as geographical coordinates L (latitude) and M (longitude). We have, in the rectilateral triangles tan PSB and PSA, the relations :—

$$\tan \hat{\text{PSB}} = -\frac{\tan(M - \frac{\Delta}{2})}{\sin L}, \quad \tan \hat{\text{PSA}} = -\frac{\tan(M + \frac{\Delta}{2})}{\sin L}.$$

Let us now introduce the isometric latitude l defined by :—

$$l = \text{Log. tan} \left(\frac{\pi}{4} + \frac{L}{2} \right)$$

and calculate the expression of α , which is equal to PSB-PSA.

There emerges, after some simple reductions :—

$$\tan \alpha = \frac{\text{sh } 2 l \sin \Delta}{\text{ch } 2 l \cos \Delta - \cos 2 M}.$$

We can then write the equation of the sub-tended angle in the form :—

$$(1) \quad \cos 2 M = \text{ch } 2 l \cos \Delta - \text{sh } 2 l \sin \Delta \cot \alpha.$$

Let us now try to find the equation of the small circle the latitude of centre of which we shall denote by L_0 and the spherical radius CA' by r . In the spherical right-angled triangle CMA', we have the relations :—

$$\tan L_0 = \sin \Delta \cot \alpha, \quad \tan r = \frac{\tan \Delta}{\sin \alpha}, \quad \cos r = \cos \Delta \cos L_0.$$

The equation of the small circle :—

$$\cos r = \sin L \sin L_0 + \cos L \cos L_0 \cos M$$

is written with due regard of the expression of r and L_0 :—

$$\cos \Delta = \cos L \cos M + \sin L \sin \Delta \cot \alpha,$$

or else, by substituting for the geographical latitude the isometric latitude :—

$$(2) \quad \cos M = \text{ch } l \cos \Delta - \text{sh } l \sin \Delta \cot \alpha.$$

The comparison of equations (1) and (2) shows that we pass from a point of the sub-tended angle to a point of the small circle by doubling its isometric coordinates l and M .

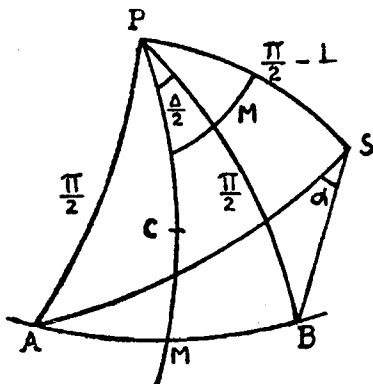


FIG. 2

In other words, if we denote by l' and M' the coordinates of the point of the circle corresponding to the point l , M of the sub-tended angle, we have between these coordinates the relations :—

$$(3) \quad l' = 2 l, \quad M' = 2 M.$$

Auxiliary sphere :

The correspondence between the sub-tended angle and the auxiliary circle is not dependent on the selection of the zero meridian. Consequently, we can associate with the given sphere an auxiliary sphere, so that to each spherical sub-tended angle described on two points of the equator of the first, will correspond a small circle of the second. The coordinates of the corresponding points of both surfaces must therefore check the relations (3).

REMARKS

a) It is desirable to bring together the method of generation of the auxiliary circle with that of the circumference, plane sub-tended angle of

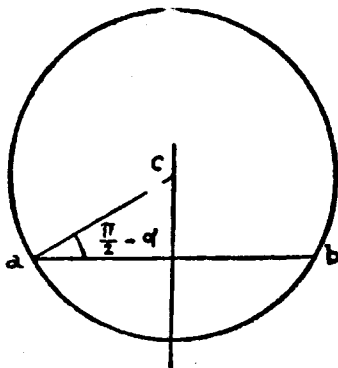


FIG. 3

the angle α , the centre of which is also obtained by drawing through an extremity of the base ab a straight line making the angle $\frac{\pi}{2} - \alpha$ with the base and taking the intersection c of this straight line with the perpendicular raised in the middle of the base (*Fig. 3*).

b) The corresponding points on the auxiliary circle and on the sub-tended angle are homothetic in relation 2 about the middle of the base and according to the loxodromic originating in this point. The "*loxodromic similarities*" whose preceding correspondence is a particular example can be utilized to give a geometrical interpretation of the "exponent" of the conformal projections, being the coefficient of the complex element $l + iM$ in the expression of $Y + iX$.

c) As Mercator projection is defined by the law :—

$$X = M, \quad Y = l,$$

the transformation, in this method of representation, of the spherical segment of equatorial base is deduced from the transformation of the auxiliary circle by a similarity of relation $\frac{1}{2}$ having for its centre the projection of the middle of the base. The spherical sub-tended angles are therefore projected according to the three types of curves which are distinguished in the classical theory of the curves of altitudes, which are images of the small circles of altitude drawn on the sphere.

It may be observed in this respect that navigators using generally circles of altitude as geometric loci of the ship's position, have practically never pointed out the need of a projection enabling them to draw easily an image of these loci, one can therefore wonder whether such a convenience is really warranted in the case of the spherical sub-tended angle.

3. Plane representation of the spherical sub-tended angle

All stereographic projections of the auxiliary sphere represent the auxiliary circles by circumferences. If, instead of drawing on the projection the grid l', M' of the parallels and meridians of the auxiliary sphere, one lays off the grid $l = \frac{l'}{2}, M = \frac{M'}{2}$ of the parallels and meridians of the given sphere, these circumferences will be the projections of the spherical sub-tended angles described on two points of the equator. The law of correspondence between the two spheres causing meridians to correspond to meridians and parallels to parallels, the grid will consist of circumferences like that of the stereographic projections. The projection thus defined is the conformal spherical projection of exponent 2 (coefficient of $\frac{l + iM}{2}$ equal to 2).

The conformal spherical projection of exponent 2 has therefore the property of representing by circumferences the spherical sub-tended angles described on two points of the equator.

This projection, however, will only solve the problem calling for attention if these circumferences are easy to draw. We know that the spherical segment sub-tending an angle makes at the extremities of the base with the basic great circle (*Fig. 4*) angles equal to α . This property is retained in a



FIG. 4

conformal representation so that, if such a projection transforms the sub-tended angle into a circumference and the base into a straight line, the circumference will be the segment sub-tending the angle α described on the projection of the base.

In the case of the conformal spherical projection, the base, being laid on to the equator, will have a rectilinear projection if the central point of the projection be located on this great circle. In consequence, the conformal spherical projection of exponent 2 and of equatorial central point represents the spherical sub-tended angles described on two points of the equator by plane segments sub-tending the same angle and constructed on the projections of the two points.

4. Conformal spherical projection of exponent 2 and of equatorial origin

Formulae. — This projection is defined by the relation :—

$$Y + iX = \text{th} (l + iM).$$

The right-angled coordinates have therefore the following expressions in terms of geographical coordinates :—

$$X = \frac{\sin 2M}{\text{ch } 2l + \cos 2M}, \quad Y = \frac{\text{sh } 2l}{\text{ch } 2l + \cos 2M},$$

or else as a function of geographical latitude :—

$$X = \frac{\sin M \cos M \cos^2 L}{1 - \sin^2 M \cos^2 L} = \frac{\sin M \cos M}{\tan^2 L + \cos M}, \quad Y = \frac{\sin L}{1 - \sin^2 M \cos^2 L} = \frac{\sin L (1 + \tan^2 L)}{\tan^2 L + \cos^2 M}.$$

If we denote by s and Z the length and the azimuth of transmission of the great circle arc joining the origin of the projection to the point of coordinates we have :—

$$X = \frac{\sin s \cos s \sin Z}{1 - \sin^2 s \sin^2 Z}, \quad Y = \frac{\sin s \cos Z}{1 - \sin^2 s \sin^2 Z}.$$

Further, the distance on the projection between the origin and the point XY has for its square :—

$$X^2 + Y^2 = \frac{\operatorname{ch} 2l - \cos 2M}{\operatorname{ch} 2l + \cos 2M} = \frac{1 - \cos^2 M \cos^2 L}{1 - \sin^2 M \cos^2 L} = \frac{\tan^2 L + \sin^2 M}{\tan^2 L + \cos^2 M} = \frac{\sin^2 s}{1 - \sin^2 s - \sin^2 L}$$

The inverse formulae are the equations of the transformations of the parallels and meridians :—

$$\operatorname{th} 2l = \frac{2Y}{1 + X^2 + Y^2}, \quad \tan 2M = \frac{2X}{1 - X^2 - Y^2}.$$

Grid. (Fig. 5). — The projection rejecting to infinity (critical points) the poles of the zero meridian points on the equator situated at a distance $\frac{\pi}{2}$ of the central point, it can only be used for representing the hemisphere having the origin as its pole and therefore limited to the meridians in longitude $\mp \frac{\pi}{2}$.

The grid of meridians and parallels is symmetrical about the coordinate axes. The poles are represented by points on the Y axis of the ± 1 ordinate. The segment of this axis comprised between the projections of the poles corresponds to the meridian of origin, the parts exterior to this segment represent the meridians in longitude $\pm \frac{\pi}{2}$. The equator is projected along the X axis.

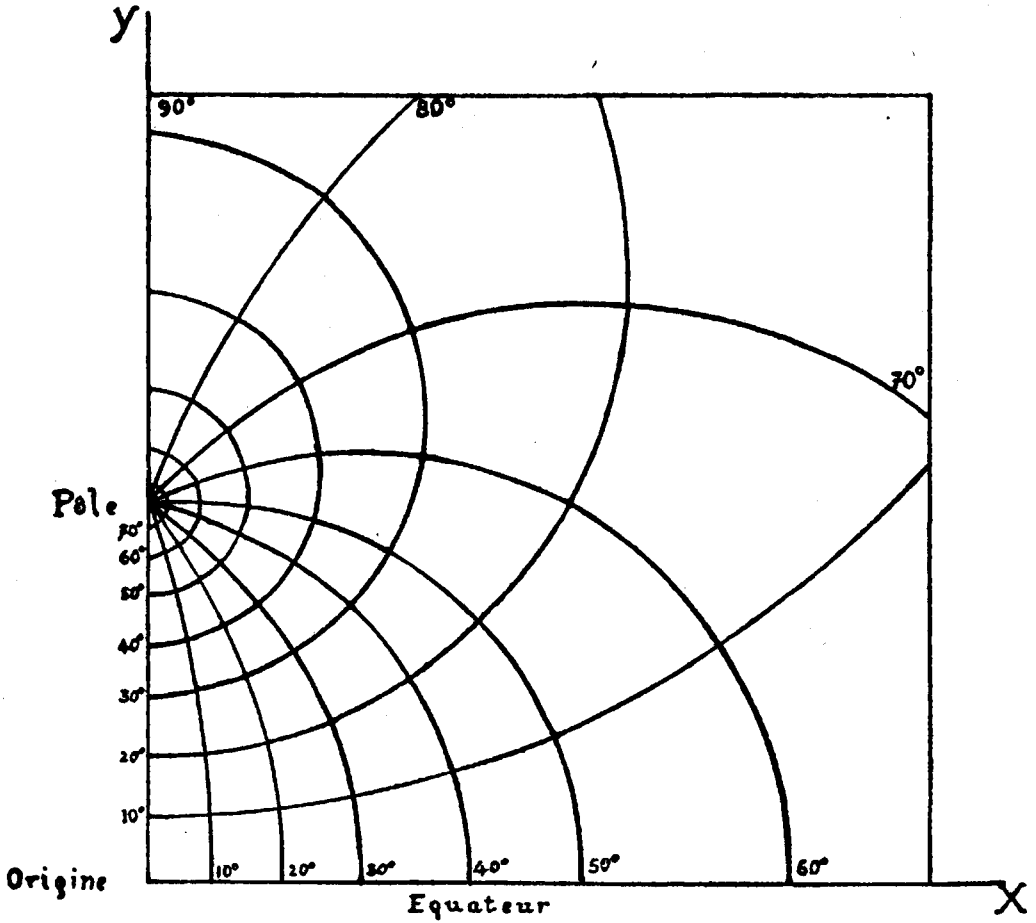
The parallels of the chart are circumferences whose centre is on the Y axis in ordinate $\frac{1}{\operatorname{th} 2l}$ or $\frac{1 + \sin^2 L}{2 \sin L}$ and whose radius is equal to $\frac{1}{\operatorname{sh} 2l}$ or $\frac{\cos^2 L}{2 \sin L}$. They cut the Y axis at the point of ordinate $Y = \operatorname{th} l$ for $M = 0$ and $Y = \frac{1}{\operatorname{th} L}$ for $M = \pm \frac{\pi}{2}$. The distances uv from the projection of the point LM to the projection of the poles being :—

$$\sqrt{X^2 + (Y \pm 1)^2} = e^{\pm l} \sqrt{\frac{2}{\operatorname{ch} 2l + \cos 2M}} = \sqrt{\frac{1 \pm \sin s \cos Z}{1 - \sin^2 s \sin^2 Z}}$$

the parallels of the chart can be considered as the locus of the points whose ratio of distances to projections of the poles is constant and equal to e^{2l} or $\tan^2 \left(\frac{\pi}{4} + \frac{L}{2} \right)$.

To the meridians correspond circumferences with their centre on the X axis at the abscisse $-\cot 2M$ and whose radius is equal to $\frac{1}{\sin 2M}$ they cut the X axis (equator) at the abscisse point $\tan M$. These circumferences are the angles sub-tending the angle $\pi - 2M$, constructed on the projections of the poles. The meridians in longitude $\pm \frac{\pi}{4}$ are projected in particular

FIG. 5



CONFORMAL SPHERIC PROJECTION

MERIDIANS				PARALLELS			
Longit.	Absc. of the centre	Radius	Intersect. with O X	Lat.	Ordon. of the centre	Radius	Intersect. with O Y
10°	— 137.4	146.2	+ 8.8	10°	148.3	139.6	8.7
20°	— 59.6	77.8	18.2	20	81.6	64.5	17.1
30	— 28.9	57.7	28.9	30	62.5	37.5	25.0
40	— 8.8	50.8	42.0	40	55.0	22.8	32.1
50	+ 8.8	50.8	59.6	50	51.8	13.5	38.3
60	+ 28.9	57.7	86.6	60	50.5	7.2	43.3
70	+ 59.6	77.8	137.4	70	50.1	3.1	47.0
80	+ 137.4	146.2		80	50.0	0.8	49.2
				Pôle	50.0	0.0	50.8

Limit the projection to the frame : X = 105,0, Y = 105,0.

along the circumference having the pole line as its diameter. Two right-angled meridians have as their projections two complementary arcs of a single circumference, joining at the poles.

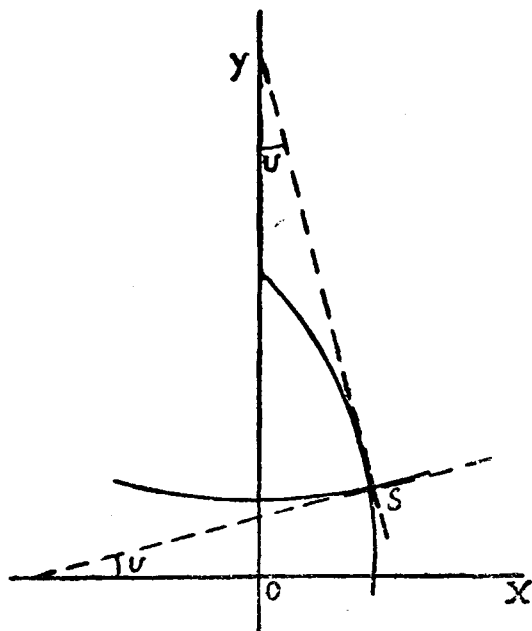


FIG. 6

If we denote by U (*Fig. 6*) the angle at the centre of the meridian and parallel of a point S , an angle which has the same value for both curves on account of the orthogonality of the grid, we have :—

$$\cos U = \operatorname{ch} 2l - Y \operatorname{sh} 2l,$$

and therefore :—

$$\tan \frac{U}{2} = \operatorname{th} l \tan M.$$

The bearing of the tangent to the parallel is $\frac{\pi}{2} - U$, that of the tangent to the meridian $\pi - U$. The parallels admit of the tangent parallel to OY and the meridians a tangent parallel to OX at the coordinate points :—

$$X = \frac{1}{\tan 2M}, \quad Y = \frac{1}{\operatorname{th} 2l},$$

whose locus is the equilateral hyperbola

$$Y^2 - X^2 = 1,$$

being a projection of the sphere curves corresponding to $U = \pm \frac{\pi}{2}$, which have therefore as their equation :—

$$\operatorname{th} l \operatorname{tg} M = \pm 1,$$

and which are the spherical sub-tended angles of a demi-right angle described on the pole and critical points.

The projections of the poles are single points at which the meridians cut one another under angles double those which they make between them on the globe.

Ratio of similarity. — The ratio of similarity $\frac{\text{plan}}{\text{sphere}}$ is given by the relation :—

$$K = \frac{2 \operatorname{ch} l}{\operatorname{ch} 2l + \cos 2M} = \frac{\cos L}{1 - \sin^2 M \cos^2 L} = \frac{\sqrt{1 - \sin^2 s \cos^2 L}}{1 - \sin^2 s \sin^2 L}.$$

We may also write in terms of the distances from the position to the projections of the poles

$$K = \operatorname{ch} l \operatorname{uv} = \sqrt{\frac{\operatorname{uv}}{1 - \sin^2 s \sin^2 L}} = \frac{\operatorname{uv}}{\sqrt{1 - \sin^2 s \cos^2 L}}.$$

Further, we have the two relations :—

$$K^2 + 2 Y^2 = \frac{2 \operatorname{ch} 2l (1 + \operatorname{ch} 2l)}{(\operatorname{ch} 2l + \cos 2M)^2}, \quad \sqrt{K^2 + Y^2} = \frac{1 + \operatorname{ch} 2l}{\operatorname{ch} 2l + \cos 2M}.$$

as, on the other hand

$$X^2 + Y^2 + 1 = \frac{2 \operatorname{ch} 2l}{\operatorname{ch} 2l + \cos 2M},$$

the following relation between K and the rectangular co-ordinates can be deduced :—

$$X^2 + Y^2 + 1 = \frac{K^2 + 2 Y^2}{\sqrt{K^2 + Y^2}},$$

which is the equation of the curves of equal linear distortion.

On the X axis (equator) the ratio of similarity is : $K = \frac{1}{\cos^2 M} = 1 + X^2$, on the Y axis between the projections of the poles (zero meridian) it is equal to $\cos L$, or $\sqrt{1 - Y^2}$; when the absolute value of Y is more than 1 (meridian in longitude $\pm \frac{\pi}{2}$) its expression is $\frac{\cos L}{\sin^2 L}$ or $Y \sqrt{Y^2 - 1}$.

The curves of linear equal distortion are symmetrical closed curves about the coordinate axes, when K is less than 1, they are each divided into two curves symmetrical to each other about OX (Fig. 7)⁽¹⁾. These curves cut the Y axis at the points whose ordinate is $\pm \sqrt{\frac{1 + \sqrt{1 + 4 K^2}}{2}}$ and also, but only when K is less than the unity at $\pm \sqrt{1 - K^2}$. When K is more

(1) See also Fig. 10, while interchanging the rectangular coordinate axes of the diagram.

than 1, they intersect the X axis at the abscissa points $\pm \sqrt{K-1}$. These curves are the projections of the sphere curves having for their equation:—

$$\sin M = \frac{1}{\cos L} \sqrt{1 - \frac{\cos L}{K}}$$

which cut the meridian of origin, when K is less than 1, at the points in latitude given by $\cos L = K$. When K is more than 1, they intersect the equator at the points in longitude M such as $\sec^2 M = K$. They intersect in any case the meridian in longitude $\pm \frac{\pi}{2}$, at the latitude given by $\frac{\cos L}{\sin^2 L} = K$.

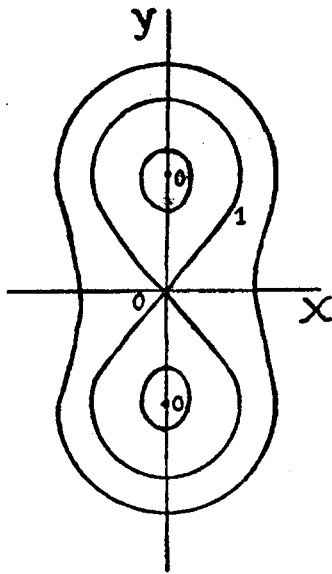


FIG. 7

Among the curves of equal linear distortion there is one for which the scale is preserved ($K = 1$). Its equation may be written:—

$$X^2 = (\sqrt{1 + Y^2} - 1) \left(\frac{1}{\sqrt{1 + Y^2}} - \sqrt{1 + Y^2} + 1 \right).$$

This curve is 8-shaped, symmetrical with respect to the coordinate axes and stretched along OY. It passes through the origin which is a point of inflexion and where its tangent has the bearing $\pm \arctan \frac{1}{\sqrt{2}}$, or $\pm 35^\circ 16'$; its points of intersection with the OY axis have for their ordinate $\pm \sqrt{\frac{1 + \sqrt{5}}{2}} = \pm 1.2720$, they correspond to the points of the sphere in longitude $\pm 90^\circ$ and latitude $\pm 51^\circ 50'$ ($\sin L = \cot L$).

Outside the curve of the preserved scale the ratio of similarity is more than the unity, it becomes infinite at the critical points; within the curve, it is less than 1 and becomes zero at the poles.

In the neighbourhood of the origin, K differs little from the unity and we have as a first approximation :—

$$K = 1 + X^2 - \frac{Y^2}{2};$$

the curves of linear equal distortion are merged with the hyperbola $X^2 - \frac{Y^2}{2} = \text{constant}$ and the projection is of the hyperbolic type.

When K is very small, the curves of equal linear distortion are obviously circumferences whose centre is a point very near to the projection of the poles, with which they merge when $K = \text{zero}$. Considering K as an infinitesimal of the first order, the intersections of the curves with OY have as ordinates $1 + \frac{K^2}{2} - \frac{5K^4}{8} + \dots$ and $1 - \frac{K^2}{2} - \frac{K^4}{8} + \dots$; being values very near to the unity; the equations of the curves can also be written neglecting the terms of the sixth order :—

$$X^2 + Y^2 + 1 = 2Y \left(1 + \frac{K^4}{8y^4} \right) = 2Y \left(1 + \frac{K^4}{8} \right);$$

they represent a circumference whose centre is on OY at the ordinate $1 + \frac{K^4}{8}$ and whose radius is equal to $\frac{K^2}{2}$.

On the contrary, when the ratio of similarity is very great, the curves of equal distortion are merged with circumferences whose centre is the origin of the coordinates and whose radius is $\sqrt{K - 1}$. Let us recollect, in this respect, that in the stereographic projection, the curves of equal distortion are strictly circumferences of radius $2\sqrt{K - 1}$ centred on the origin, at a same distance from the origin (when it is great) the ratio of similarity is therefore about four times smaller than in the considered conformal spherical projection.

Transformations of the spherical sub-tended angles.

In the method of representation which we are considering, the spherical subtended angles described on two points of the equator have as projections the segments subtending the same angle, described on the projections of the two points.

When one of the extremities of the base of a segment is a critical point which the projection rejects to infinity (poles of the meridian of origin $L = O$, $M = \pm \frac{\pi}{2}$), the spherical subtended angle is represented by a straight line passing through the projection of the other extremity of the base and making with the OX axis, being the projection of the equator an angle equal to that of the segment.

Conversely, every circumference of the projection intersecting the X axis is the image of a spherical subtended angle described on two points of the

equator. If we consider the circumference as a segment subtending an angle α , constructed on its two points of intersection with the X axis, the spherical segment subtends the same α angle.

Lastly, every straight line of the projection corresponds to a spherical subtended angle described on two points of the equator, one of which is in $\frac{\pi}{2}$ longitude. The angle of the segment is equal to the inclination of the straight line on the X axis.

Change in the meridian of origin.

Let us at first seek the point of intersection of the X axis with the straight line joining the projections of the two points on a single parallel of latitude L in the longitudes $M_1 M_2$. If $X_1 Y_1, X_2 Y_2$ denote the right-angled coordinates of the two points, we have as abscissa of the point of intersection :—

$$X = \frac{X_1 Y_2 - X_2 Y_1}{Y_2 - Y_1} = \frac{\sin 2 M_1 - \sin 2 M_2}{\cos 2 M_1 - \cos 2 M_2} = -\cot (M_2 + M_1).$$

Let us now consider two conformal spherical projections having as a central meridian the first:— the meridian of origin of the longitudes, the second :— the meridian in longitude M_0 and let us superimpose the coordinate axes of these projections (*Fig. 8*). Let M_1 and M_2 be the projections of a single point M of the geographic coordinates L and M; the points M_1 and M_2 are on the circumference representing the parallel in latitude L, which is common to both systems.

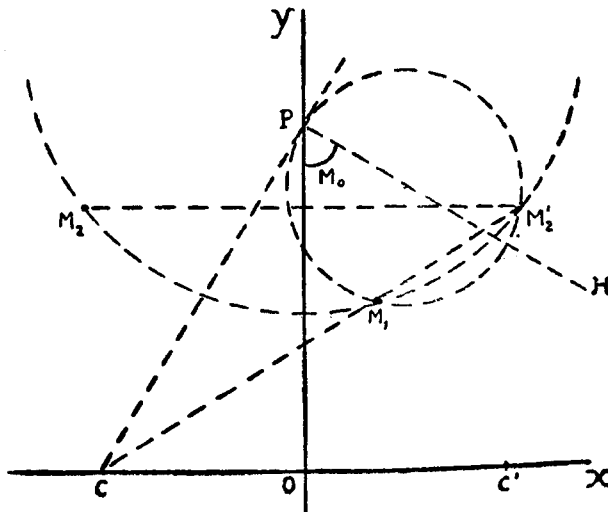


FIG. 8

Let us denote by M'_2 the symmetric of M_2 with respect to OY in the second projection this point represents the point of the sphere of the geo-

graphic coordinates L and M_0 — $(M-M_0)$ or $2M_0-M$; in the first it corresponds to the point L , M_0-M . According to the above demonstrated proposition, the point C where the straight line $M_1 M'_2$ meets the axis OK has therefore as abscissa $-\cot M_0$, it is the projection of the point of the equator in longitude $M_0 - \frac{\pi}{2}$, a critical point of the second system, it is also independent of the considered point M on the sphere.

On the other hand the product CM_1, CM'_2 is equal to the power of the point C with respect to the circumference which is a projection of the parallel; or :—

$$CM_1 \cdot CM'_2 = \cot^2 M_0 + \frac{1}{\text{th}^2 2l} - \frac{1}{\text{sh}^2 2l} = 1 + \cot^2 M_0 = \frac{1}{\sin^2 M_0} = \overline{CP}^2.$$

This product is therefore also independent of the point M . Consequently we pass from the first projection to the second through an inversion of the centre C and constant $\frac{1}{\sin^2 M_0}$, then through a symmetric with respect to OY .

We obtain M'_2 geometrically by taking the intersection of CM_1 with the circumference passing through M_1 and tangent to CP at P , a circumference whose centre is on the straight line PH which is perpendicular to CP at P and has consequently $\pi - M_0$ as a bearing. It should be noted that C is symmetrical with respect to OY of the point C' , a projection in the second system of the critical point of the first.

5. Transverse projections

In practice, in most cases, the angle of the spherical subtended angle is an azimuth, the pole is therefore one of the extremities of the base and this is orientated along a meridian. It is therefore necessary to use the transverse form of the conformal spherical projection, i.e. to adopt as a point of representation the pole of the meridian bearing the base of the segment.

For the representation of spherical subtended angles, the meridian of origin of the projection thus obtained enjoys the same properties as the equator of the spherical projection from which it is derived. As regards the ratio of similarity and linear distortions, the results are identical for both projections, when expressed in plane coordinates and due account is taken of the difference of orientation of the coordinate axes of both systems which is equal to a right angle.

On the other hand the grid of meridians and parallels is very different in both methods of representation and the transverse projection, in particular, no longer shows the simplicity of definition and drawing which characterises spherical projection.

FORMULAE

Let O be the central point of the part affected by the projection and L_0 the latitude of this point (*Fig. 9*). Let the coordinate bear the index 1 when referred to the point of the equator in longitude $\frac{\pi}{2}$ with respect to the

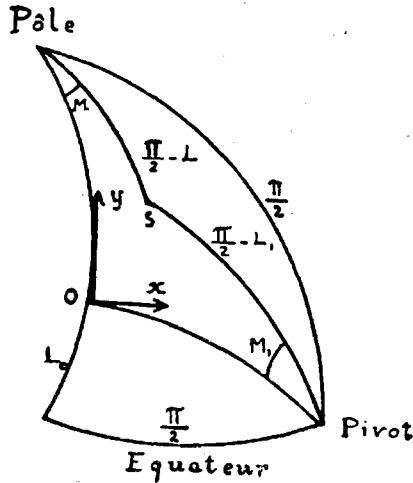


FIG. 9

meridian of the point O . The Y axis being orientated along this meridian and the X axis along the great circle perpendicular to this meridian at O , the formula of the conformal spherical projection is :—

$$X - iY = \text{th} (l_1 + iM_1).$$

Let us consider on the other hand the rectilateral triangle formed by the pole, the pivot and any point S of the coordinates L and M ; the change of coordinates is defined by the following relations :—

$$\sin L_1 = \cos L \sin M. \quad \text{or: } \text{th } l_1 = \frac{\sin M}{\text{ch } l} = \frac{1}{i} \frac{\text{sh } iM}{\text{ch } l}$$

$$\cot (L_0 - M_1) = \cot L \cos M, \quad \text{and } \tan M_1 = \frac{\cos M \text{ sh } l_0 - \text{sh } l}{\cos M + \text{sh } l_0 \text{ sh } l}$$

$$\text{th } iM_1 = \frac{1 \text{ sh } l - \text{sh } l_0 \text{ ch } iM}{i \text{ ch } iM + \text{sh } l_0 \text{ sh } l}$$

By substitution there emerges :—

$$Y + iX = \frac{\text{sh} (l + iM) - \text{sh } l_0}{1 + \text{sh } l_0 \text{ sh} (l + iM)}$$

From which are deduced for the right-angled coordinates :—

$$X = \frac{\text{ch}^2 l_0 \text{ ch } l \sin M}{(\text{sh}^2 l + \sin^2 M) \text{ sh}^2 l_0 + 2 \text{ sh } l_0 \text{ sh } l \cos M + 1} = \frac{\sin M \cos L}{1 - (\sin L \cos L_0 - \sin L_0 \cos L \cos M)^2}$$

$$Y = \frac{(1 - \text{sh } l_0)^2 \text{sh } l \cos M + \text{sh } l_0 (\text{sh}^2 l - \cos^2 M)}{(\text{sh}^2 l + \sin^2 M) \text{sh}^2 l_0 + 2 \text{sh } l_0 \text{sh } l \cos M + 1}$$

$$= \frac{1}{2} \frac{\cos 2 L_0 \sin 2 L \cos M + \sin 2 L_0 (\sin^2 L - \cos^2 L \cos^2 M)}{1 - (\sin L \cos L_0 - \sin L_0 \cos L \cos M)^2}$$

These expressions could also be obtained from the formulae in s and Z of the spherical projection, in which $-Y$ should be substituted for X , X for Y and $Z - \frac{\pi}{2}$ for Z ; which gives :—

$$X = \frac{\sin s \sin Z}{1 - \sin^2 s \cos^2 Z}, \quad Y = \frac{\sin s \cos s \cos Z}{1 - \sin^2 s \cos^2 Z}$$

We have, further :—

$$Y^2 + X^2 = (Y + iX)(Y - iX) = \frac{(\text{sh}^2 l + \sin^2 M) - 2 \text{sh } l_0 \text{sh } l \cos M + \text{sh}^2 l_0}{(\text{sh}^2 l_0 + \text{sh}^2 M) \text{sh}^2 l_0 + 2 \text{sh } l_0 \text{sh } l \cos M + 1}$$

$$= \frac{1 - (\sin L \sin L_0 + \cos L \cos L_0 \cos M)^2}{1 - (\sin L \cos L_0 - \sin L_0 \cos L \cos M)^2} = \frac{\sin^2 s}{1 - \sin^2 s \cos^2 Z}$$

The poles of the meridian of origin, being pivots of the spherical projection, are singular points where the angles are doubled; these points are projected on the Y axis at abscissae ± 1 . The distances uv from the point of the coordinate LM to these points are expressed by :—

$$\sqrt{Y^2 + (X \pm 1)^2} = \frac{\text{ch } l_0 (\text{ch } l \pm \sin M)}{\sqrt{(\text{sh}^2 l + \sin^2 M) \text{sh}^2 l_0 + 2 \text{sh } l_0 \text{sh } l \cos M + 1}}$$

$$= \frac{1 \pm \sin s \sin Z}{\sqrt{1 - \sin^2 s \cos^2 Z}}$$

Grid. — If we substitute $-L$ for L and $\pi - M$ for M , the expressions of the right-angled coordinates do not change. Consequently, any point of the projection represents two points of the globe, but there is no overlapping if we confine ourselves to the representation of one hemisphere; the projection of the great circle limiting this surface is thus a discontinuity line of the representation; when it is crossed over it is necessary to change the sign of the latitude of the parallels and replace the longitude of the meridians by its supplement.

The grid of meridians and parallels admits the axis OY as the axis of symmetry. The X axis, as regards the part comprised between the singular points is the projection of the great circle passing through the origin and the poles of the meridian of origin :— in this case we have :—

$$\text{sh } l = \text{sh } l_0 \cos M,$$

and therefore :—

$$X = \frac{\sin M}{\text{ch } l} = \sin M \cotg L.$$

The part outside the K axis represents the great circle whose origin is the pole; it corresponds to the relations :—

$$\text{sh } l = \frac{\cos M}{\text{sh } l_0}, \quad X = \frac{\text{ch } l}{\sin M} = \frac{1}{\sin M \cos L}.$$

The Y axis is the projection of the meridian of origin; its graduation in latitude follows the law :—

$$Y = \frac{\text{sh } l - \text{sh } l_0}{1 + \text{sh } l \text{ sh } l_0} = \tan (L - L_0).$$

It is also the projection of the meridian of longitude π , it is necessary in this case to change the sign of L. We have then :—

$$Y = -\tan (L + L_0)$$

The poles are projected on the Y axis at the point whose ordinate is :—

$\frac{1}{\text{sh } l_0} = \cot L_0$. The equator and meridians of longitude $\pm \frac{\pi}{2}$ being right-angled meridians of the spherical projection are represented by two complementary axes of a single circumference, being arcs joining at the singular points of the axis OX. This circumference which delimits the representation of one quarter of the surface of the globe has its centre on OY at the ordinate $2L_0$, its radius is $\frac{1}{\sin 2L_0}$ is cuts the OY axis at the points $\cot L_0$ and $-\tan L_0$, the first of which is the projection of the pole.

The projection rejects to infinity the point in latitude $L_0 - \frac{\pi}{2}$ of the meridian of origin and its antipode. The parallels of these two critical points have the same projection. For the first of these points for instance we have :—

$$\text{sh } l = -\frac{1}{\text{sh } l_0}, \quad \text{ch } l = \frac{1}{\text{th } l_0},$$

and its parallel has consequently for its parametric projection :—

$$X = \frac{1}{2 \text{ sh } l_0} \frac{\text{ch}^3 l_0 \cot \frac{M}{2}}{1 + \text{sh}^2 l_0 \cos^2 \frac{M}{2}}, \quad Y = \frac{1}{2 \text{ sh } l_0} \frac{1 + \text{sh}^2 l_0 \cos M}{1 + \text{sh}^2 l_0 \cos^2 \frac{M}{2}}.$$

This curve cuts the Y axis at the point whose ordinate is $\cot 2L_0$, which is the centre of the circumference bearing the projection of the equator. It admits as asymptote the straight line $Y = \frac{1}{2 \text{ sh } l_0} = \frac{\cot L_0}{2}$ which is half-way between the projection of the pole and the X axis.

The other parallels of the globe are represented by closed curves. Those that are comprised between the critical points surround the two singular points of the X axis which they cut outside the segment $+1, -1$ at the abscissa points :—

$$X = \pm \frac{1}{\sqrt{1 + \text{th}^2 l \text{ ch}^2 l}} = \pm \frac{1}{\sqrt{1 - \frac{\sin^2 L}{\cos^2 L_0}}}.$$

The parallels comprised between the poles and the critical points are projected according to closed curves surrounding the projection of the pole and leaving outside the singular points on the axis OX.

The parallels situated between the parallel of the origin and the symmetric parallel include only curves of the first of the two preceding categories si $L_0 < \frac{\pi}{4}$; in the opposite case, they include all those of the first category and a portion of those of the second. All the corresponding curves cut the X axis at 2 points located between the singular points and having as abscissa :—

$$X = \pm \sqrt{1 - \frac{\text{th}^2 l}{\text{th}^2 l_0}} = \pm \sqrt{1 - \frac{\sin^2 L}{\sin^2 L_0}}$$

The parallel of the origin which has as its parametric representation:—

$$X = \frac{\text{ch}^3 l_0 \sin M}{\text{ch}^4 l_0 - 4 \text{sh}^2 l_0 \sin^2 \frac{M}{2}}, \quad Y = \frac{2 \text{sh} l_0 \sin^2 \frac{M}{2} \left(\text{ch}^2 l_0 - 2 \sin^2 \frac{M}{2} \right)}{\text{ch}^4 l_0 - 4 \text{sh}^2 l_0 \sin^2 \frac{M}{2}}$$

offers no essential peculiarity.

The meridians are represented by 8-shaped closed curves, which are symmetrical with respect to OY, whose point of intersection is the projection of the pole; at this point the angles are preserved and the branches of the curve make with the axis OY angles equal to the longitude of the meridian and to its supplement. Further, each loop of the curve surround a singular point with abscissae :—

$$X = \frac{1}{\sqrt{1 + \frac{\text{cotg}^2 M}{\cos^2 L_0}}}, \quad X = \sqrt{1 + \frac{\text{cotg}^2 M}{\sin^2 L_0}}$$

Ratio of similarity.

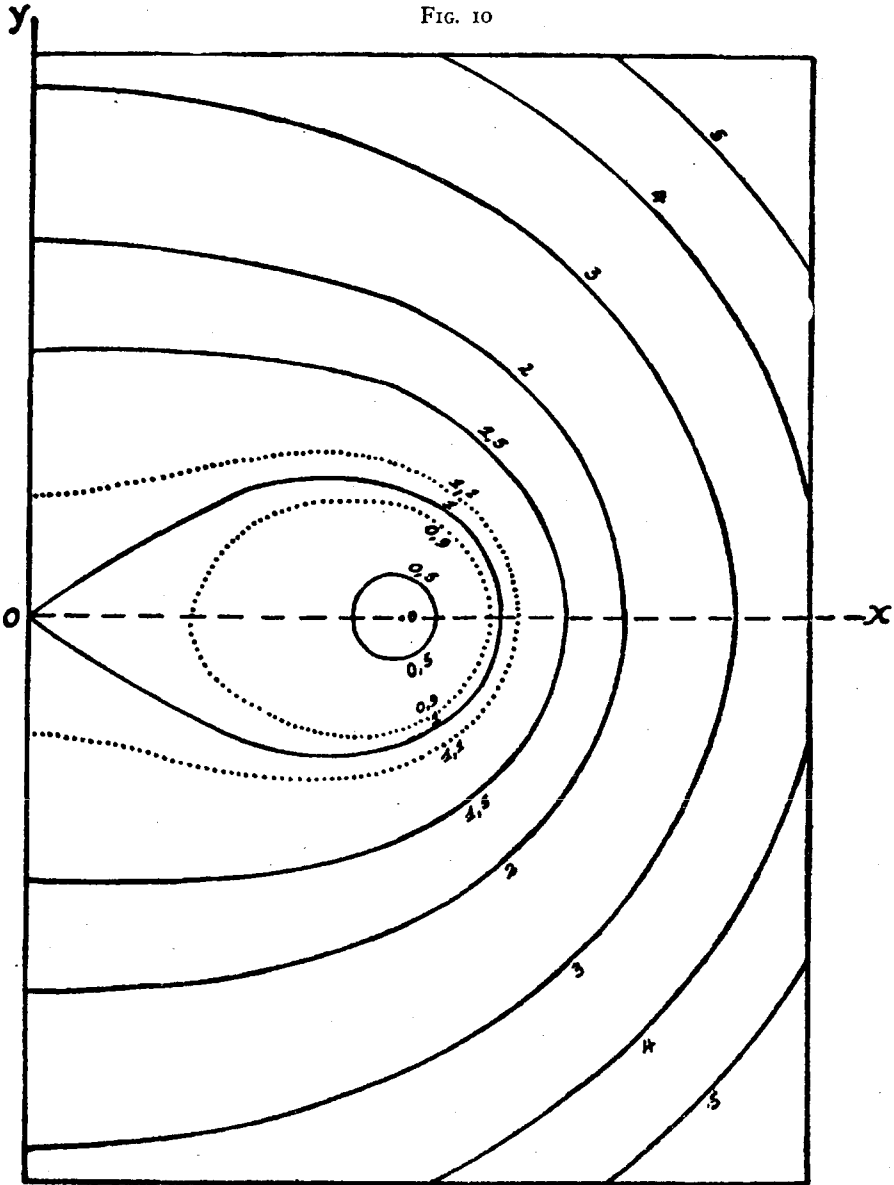
As a function of the geographic co-ordinates, the ratio of similarity $\frac{\text{plan}}{\text{sphere}}$ is expressed by :—

$$K = \frac{\text{ch}^2 l_0 \text{ch} l \sqrt{\text{ch}^2 l - \sin^2 M}}{(\text{sh}^2 l + \sin^2 M) \text{sh}^2 l_0 + 2 \text{sh} l \text{sh} l_0 \cos M + 1} \\ = \frac{\sqrt{1 - \sin^2 M \cos^2 L}}{1 - (\sin L \cos L_0 - \sin L_0 \cos L \cos M)^2}$$

We have also :—

$$K = \frac{\sqrt{1 - \sin^2 s \sin^2 Z}}{1 - \sin^2 s \cos^2 Z} = \sqrt{\frac{uv}{1 - \sin^2 s \cos^2 Z}} = \frac{uv}{\sqrt{1 - \sin^2 s \sin^2 Z}}$$

The ratio of similarity, equal to 1 at the origin is cancelled out at the singular points of the axis OX and become infinite at the critical points, at the pole, it is equal to $1 + \cot^2 L_0$.



CURVES OF EQUAL LINEAR DISTORTION

K = 0.5		K = 1		K = 1.5		K = 2		K = 3		K = 4		K = 5	
X	Y	X	Y	X	Y	X	Y	X	Y	X	Y	X	Y
43.3	0.0	0.0	0.0	0.0	35.4	0.0	50.0	0.0	70.7	50.0	76.9	80.0	73.8
45.0	4.0	5.0	3.5	10.0	35.3	10.0	49.7	10.0	70.4	60.0	72.1	95.0	59.5
47.5	5.4	10.0	6.9	20.0	35.2	20.0	48.9	20.0	69.3	70.0	66.0	105.0	45.5
50.0	5.6	15.0	10.0	30.0	34.6	30.0	47.4	30.0	67.4	80.0	57.9		
52.5	4.6	20.0	12.8	40.0	33.0	40.0	45.0	40.0	64.6	90.0	47.0		
54.9	0.0	25.0	15.2	50.0	29.9	50.0	41.3	50.0	60.8	105.0	14.5		
		30.0	17.0	60.0	24.0	60.0	35.7	60.0	55.6				
		35.0	18.2	70.0	10.4	70.0	26.7	70.0	48.5				
		40.0	18.7	72.2	0.0	75.0	19.4	80.0	38.5				
		45.0	18.5			80.0	1.4	90.0	21.5				
		50.0	17.4			80.1	0.0	94.4	0.0				
		55.0	15.0										
		60.0	10.5										
		63.6	0.0										

(Continued at the bottom of the next page.)

The relation between K and the right-angled coordinates is obtained by a permutation of X^2 and Y^2 , in the analogous formula relative to the conformal spherical projection. We have therefore :—

$$X^2 + Y^2 + 1 = \frac{K^2 + 2X^2}{\sqrt{K^2 + X^2}}$$

The grid of curves of equal linear distortion is deduced from that of the spherical projection by a rotation equal to a right angle. (*Fig. 10.*)

Relations between the various transverse projections.

When considering two transverse projections (*Fig. 11*) whose origins are located on a single meridian in latitudes L_1 and L_2 , we pass from one to the other through an inversion and a symmetric as when we effect a change of meridian of origin in the conformal spherical projection. The centre of inversion is the point C located on the axis OY at the ordinate $\cot(L_1 - L_2)$, a projection in the first system of the critical point of the second. The constant of inversion is $1 + \cot^2(L_1 - L_2)$. From a point M_1 of the first projection, we obtain a point M'_2 by taking the intersection of CML with the circumference passing through M_1 and tangent to CS at S , S being a singular point of the projection. The point M_2 which in the second system represents the same point of the sphere as M_1 in the first is symmetrical to M'_2 about OX .

Transformations of the spherical subtended angles.

In this system, the spherical subtended angles described on two points of the meridian of origin are represented by the segments subtending the same angle described on the projections of the two points.

If one of the points is a critical point, the projection of the subtended angle is a straight line passing through the projection of the other point and making with the axis OY an angle equal to the angle of the segment.

The diagram must be symmetrical about OX .
 The curves $K = 0,5$ and $1,5$ will be in dashes.
 Limit the diagram to a frame $X = 0$, $X = 105$, $Y = +75$, $Y = -75$.
 Add in dotted lines.

$K = 0.9$		$K = 1.1$	
X	Y	X	Y
21.8	0.0	0.0	15.8
25.0	7.5	5.0	16.1
30.0	11.7	10.0	16.9
40.0	15.5	20.0	19.2
50.0	14.9	30.0	21.2
55.0	12.5	40.0	21.8
60.0	7.1	50.0	20.0
61.8	0.0	55.0	17.6
		60.0	13.5
		62.5	10.1
		65.4	0.0

Various transverse projections.

These projections give a fairly satisfactory representation of the parts in the vicinity of the origin. But they involve important distortions when receding from this point and particularly along the central meridian. So that if a fairly large scale is desired at the origin, the projection cannot be expanded to a great extent in latitude. If on a Mercator's chart, we adopt for example 6 millimeters as the length of a degree of the equator, the 45th and 75th parallels are located respectively at 301 and 695 millimeters from the projection of the equator ; whilst on a transverse projection of equatorial origin with the same scale at the central point, these parallels intersect the Y axis at 344 and 1.283 millimeters from the origin.

It is therefore necessary, in practice, to restrict the field of application of these projections, but it seems sufficient for current needs to consider three projections with their origin at 0,45 and 90 degrees of latitude and respectively suitable for low, medium and high latitudes.

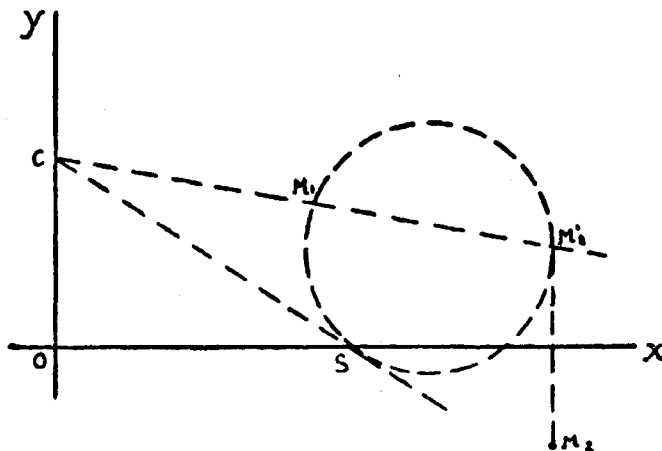


FIG. 11

6. Transverse projection of equatorial origin (Littrow's projection)

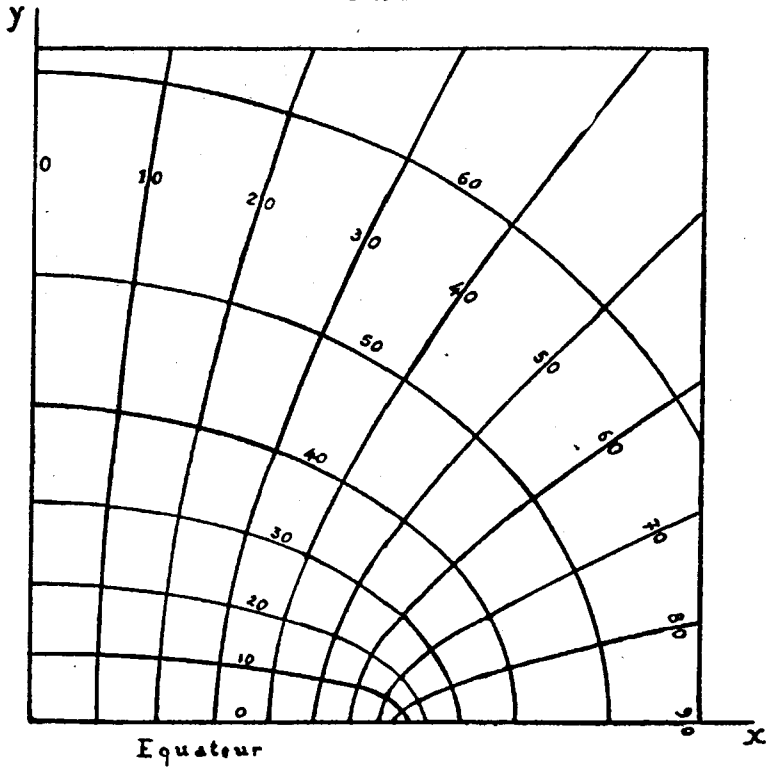
The transverse projection of equatorial origin is known under the name of *Littrow's projection*; for a long time, it was considered as "more curious than useful" ⁽¹⁾.

This projection for which $L_0 = 0$ complies with the formula :—

$$Y + i X = \text{sh} (l + i M),$$

(1) DRIENCOURT et LABORDE, *Traité des Projections des cartes géographiques*, fascicule I, page 224. — See also: *Hydrographic Review*, Vol. V, N° 2, p. 49; Vol. X, N° 2, p. 98; Vol. XIV, N° 1, p. 61.

FIG. 12



LITROW'S PROJECTION.
GRID OF MERIDIANS AND PARALLELS

LATITUDES

Longit.	0	10	20	30	40	50	60	70	80
0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	0.0	8.8	18.2	28.9	42.0	59.6	86.6		
10	8.7	8.8	9.2	10.0	11.4	13.5	17.4		
	0.0	8.7	17.9	28.4	41.3	58.7	85.3		
20	17.1	17.4	18.2	19.7	22.3	26.6	34.2		
	0.0	8.3	17.1	27.1	39.4	56.0	81.4		
30	25.0	25.4	26.6	28.9	32.6	38.9	50.0		
	0.0	7.6	15.8	25.0	36.3	51.6	75.0		
40	32.1	32.6	34.2	37.1	42.0	50.0	64.3		
	0.0	6.8	13.9	22.1	32.2	45.6	66.3		
50	38.3	38.9	40.8	44.2	50.0	59.6	76.6		
	0.0	5.7	11.7	18.6	27.0	38.3	55.7		
60	43.3	44.0	46.1	50.0	56.5	67.4	86.6		
	0.0	4.4	9.1	14.4	21.0	29.8	43.3		
70	47.0	47.7	50.0	54.3	61.3	73.1	94.0		
	0.0	3.0	6.2	9.9	14.3	20.4	29.6		
80	49.1	50.0	52.4	56.9	64.3	76.6			
	0.0	1.5	3.2	5.0	7.3	10.3			
90	50.0	50.8	53.2	57.7	65.3	77.8			
	0.0	0.0	0.0	0.0	0.0	0.0			

The first line gives the X, the second gives the values of Y.
X and Y are positive.
The equator, prolonged by the meridian of 90° is borne by the X axis.
The meridians and parallels intersect at right angles.
Limit the projection to the frame: X = 0, X = 90, Y = 0, Y = 90.

COMPLEMENTARY POINTS

Meridians	X	Y	X	Y
10	15.1	70.3	18.1	90.0
20	29.8	67.1	37.0	90.0
30	43.6	61.8	57.7	90.0
40	56.0	54.7	82.1	90.0
50	66.8	45.9	90.0	68.3
60	75.5	35.7	90.0	45.6
70	81.9	24.4	90.0	27.9
80	85.8	12.3	90.0	13.3

from which is deduced :—

$$X = \sin M \operatorname{ch} l = \sin M \sec L, \quad Y = \cos M \operatorname{sh} l = \cos M \tan L, \\ X^2 + Y^2 = \tan^2 L + \sin^2 M.$$

The grid of this projection (*Fig. 12*) is symmetrical about the axis OY. Further there is symmetry about OX, for the points on the same parallel with supplementary longitudes or for those of a single meridian with symmetrical latitudes. The poles are rejected to infinity. The segment of the X axis comprised between the abscissa points ± 1 corresponds to the equator, the remainder of the axis to the meridians in longitude $\pm \frac{\pi}{2}$. The meridians of longitude 0 and π are projected in the direction of the Y axis.

The parallels are represented by homofocal ellipses of equation

$$\frac{X^2}{\operatorname{ch}^2 l} + \frac{Y^2}{\operatorname{sh}^2 l} = 1$$

with foci at the singular points on the axis OX and whose axes have the half lengths $\operatorname{ch} l$ along oX and $\operatorname{sh} l$ along OY. The dimensions of these ellipses increase with latitude. The transformation of the meridians is a branch of the homofocal hyperbolae of equation

$$\frac{X^2}{\sin^2 M} - \frac{Y^2}{\cos^2 M} = 1.$$

These conics admit the same foci as the ellipses representing the parallels; the length of their half-axes is $\sin M$ and $\cos M$; their asymptotes pass through the origin of the coordinates and have as a bearing $\pm M$, the angle of the asymptotes, equal to $\pi - 2M$ diminishes therefore from π to 0 from the meridian of origin to the meridians of longitude $\pm \frac{\pi}{2}$.

The bearings v_p and v_m of the tangents at a point on a parallel and at a point on a meridian are given by :—

$$\tan v_p = \frac{1}{\operatorname{th} l \tan M} = \frac{1}{\sin L \tan M}, \quad \tan v_m = \operatorname{th} l \tan M = \sin L \tan M.$$

If, as in the case of the conformal spherical projection, we denote by U the auxiliary angle defined by

$$\tan \frac{U}{2} = \operatorname{th} l \tan M,$$

we have :—

$$v_m = \frac{U}{2}, \quad v_p = \frac{\pi}{2} + \frac{U}{2},$$

whilst in the conformal spherical projections :—

$$v_m = \pi - U, \quad v_p = \frac{\pi}{2} - U.$$

The ratio of similarly is expressed by

$$K = \operatorname{ch} l \sqrt{\operatorname{ch}^2 l - \sin^2 M} = \frac{1}{\cos L} \sqrt{\operatorname{tg}^2 L + \cos^2 M}.$$

The pole being a critical point, the spherical subtended angle relative to an azimuth bearing, taken on a point of the meridian of origin is represented by a bearing straight line α passing through the projection of this point. This property renders Littrow's projection particularly useful for drawing radiogoniometric bearings.

On the other hand, as the points of the sphere in the same longitude and symmetrical latitudes are projected symmetrically about OX the grid of this projection and that of any transverse projection are inverse to each other. If L_1 is the latitude of the origin of the second projection, the centre of inversion is the point on the Y axis whose ordinate is $-\cot L_1$ and the constant of inversion has the value $1 + \cot^2 L_1$.

Remark 1. — Of course, we can reach Littrow's projection in a more simple way, by endeavouring to find a plane representation through which a straight line shall correspond to a spherical subtended angle described on the pole and a point of the zero meridian.

If the latitude of this point be denoted by L_1 and the observed azimuth by α , the geometrical locus equation will be :—

$$\tan L \cos M - \frac{\sin M}{\cos L} \cot \alpha = \tan L_1.$$

The locus will be represented by the straight line

$$Y - X \cot \alpha = \tan L_1$$

and if we adopt a projection defined by :—

$$X = \frac{\sin M}{\cos L}, \quad Y = \cos M \tan L.$$

The straight line bearing is α and passes through the point whose coordinates are $X = 0$, $Y = \tan L_1$ being the projection of the point whose bearing has been taken.

This property has for its immediate consequence that a spherical subtended angle described on any two points of the zero meridian has for its transformation the segment subtending the same angle constructed on the projection of the base.

Remark 2. — Any homographic transformation of Littrow's projection furnishes a projection in which the spherical subtended angles described on the pole and a point of the central meridian are represented by straight lines passing through the projection of the latter point.

We reach one of these projections by endeavouring to find a nomographic

solution of the problem. The equation of the subtended angle relative to the point whose bearing has been taken in latitude L_1 and at the observed azimuth α

$$\tan L_1 \cos L + \cot \alpha \sin M - \sin L \cos M$$

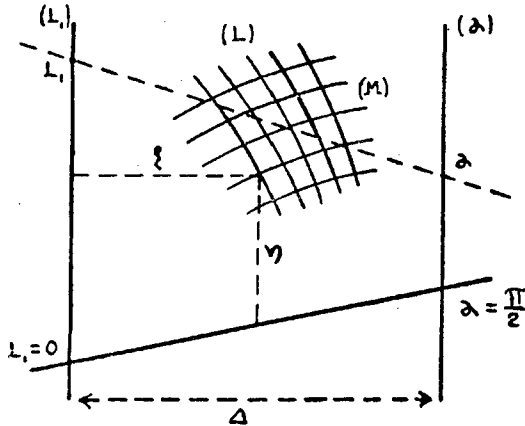


FIG. 13

can be represented by a nomogram with aligned points (*Fig. 13*) comprising two scales $\tan L_1$ and $\cot \alpha$, graduated respectively in L_1 , and α together with two grids of curves (L) and (M) described by:—

$$\xi = \Delta \frac{\sin M}{\cos L + \sin M}, \quad \eta = \frac{\sin L \cos M}{\cos L + \sin M}$$

Δ denoting the distance of the parallel scales, ξ the distance from the point L, M to the scale (L_1) , η its distance to the base of the nomogram (a straight line passing through the zeros of the parallel scales) measured in a direction parallel to the scales (L_1) and (α) .

If, through the given coordinate points L_1 and α , we draw a straight line, this meets the grids (L) (M) at the points whose coordinates satisfy the equation of the segment. The straight line is thus the projection of the subtended angle in a plane representation constituted by the two grids of curves, a representation which, as can easily be seen, is a homographic transformation of Littrow's projection.

Use of an auxiliary projection by inversion of the grid.

The ratio of similarity of Littrow's projection increases rapidly when receding from the central point, so that it is impossible to show regions of high latitude on the chart, if it be desired to preserve an adequate scale in the neighbourhood of the origin. It is, however, possible, in most cases, to obtain, in a simple manner, the projection of the circumpolar part of the spherical subtended angle, by means of the following expedient.

If the colatitude $\frac{\pi}{2} - L$ be denoted by λ , the equation of the subtended angle described on the pole and the point in latitude L_1 of the zero meridian is written :—

$$\cot \lambda_1 \sin \lambda + \cot \alpha \sin M - \cos \lambda \cos M = 0$$

It can be noted that if longitudes and colatitudes be interchanged on the one hand, and the angle λ_1 , and α on the other hand, the equation undergoes no change. Let us then consider the projection obtained by inverting the co-latitudes and longitudes of Littrow's projection, it has the same grid as the latter representation, but the parallel of latitude L becomes the meridian in longitude $\frac{\pi}{2} - L$, the meridian in longitude M becomes the parallel in latitude $\frac{\pi}{2} - M$. The new projection is therefore defined by :—

$$x = \frac{\cos L}{\sin M} = \frac{1}{X} \quad y = \cot M \sin L = \frac{Y}{X}$$

It is therefore a homographic transformation of Littrow's projection, the spherical subtended angle therein has also a rectilinear projection, the straight line of equation

$$x \tan L_1 - y + \cot \alpha = 0$$

with the bearing $\frac{\pi}{2} - L_1$, and which passes through the Y axis point whose ordinate is $\cot \alpha$. It suffices to change the figuring of Littrow's projection grid to obtain this auxiliary projection on which the drawing of the geometrical locus is as simple as on Littrow's projection.

In practice, the same grid can be used for both projections and the two transformations of the spherical subtended angle are the straight lines AC ($OA = \tan L_1$) of the bearing α and BC ($OB = \cot \alpha$), of the bearing $\frac{\pi}{2} - L_1$ (*Fig. 14*). The straight lines intersect at the point C of right-angled coordinates

$$X = x = 1$$

$$Y - y = \tan L_1 + \cot \alpha = OA + OB$$

This point represents the point on the spherical subtended angle with the same projection in both systems, therefore the one whose longitude is equal to its colatitude, its geographical coordinates L_c et $M_c = \frac{\pi}{2} - L_c$ are given by

$$\frac{\sin^2 L_c}{\cos L_c} = \frac{\cos^2 M_c}{\sin M_c} = \tan L_1 + \cot \alpha$$

The broken line ACB represents the spherical subtended angle between the point whose bearing has been taken and the pole; the segment AC corresponds, in Littrow's projection, to the arc running from the southern extremity

of the base to the point $L_c M_c$; the segment CB represents the arc comprised between the point $L_c M_c$ and the pole, in the auxiliary projection.

The procedure fails when the points ABC are close to the limits of the chart or outside these limits, i.e. if the point whose bearing has been taken

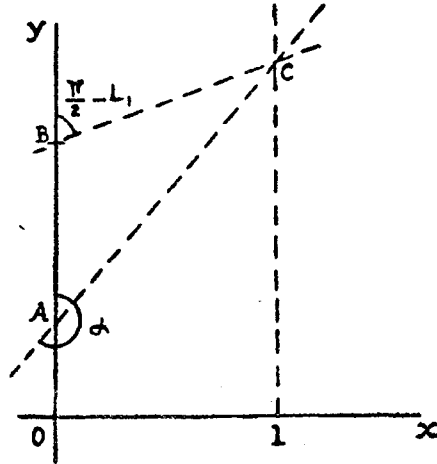


FIG. 14

is in a high latitude and if at the same time the azimuth of observation is close to 180 degrees, i.e. if the spherical subtended angle under consideration is very close to the polar region of the zero meridian.

We shall describe further on (paragraph 12) the auxiliary projection whose grid is rectangular, but which is neither conformal nor equivalent. (Fig. 15.)

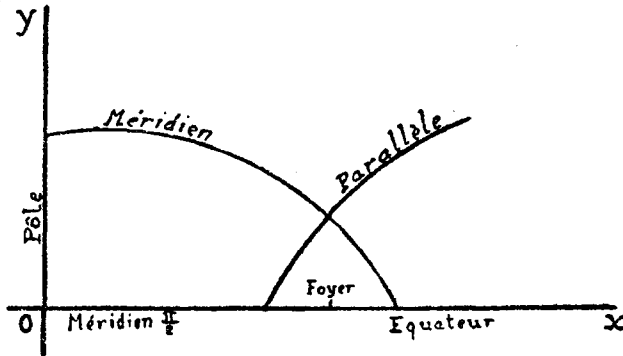


FIG. 15

7. Transverse projection of medium origin

If we take the origin of the transverse projection on the parallel in latitude $\frac{\pi}{4}$, we obtain a system which is advantageous for the plane representation of spherical subtended angles concerning medium latitudes.

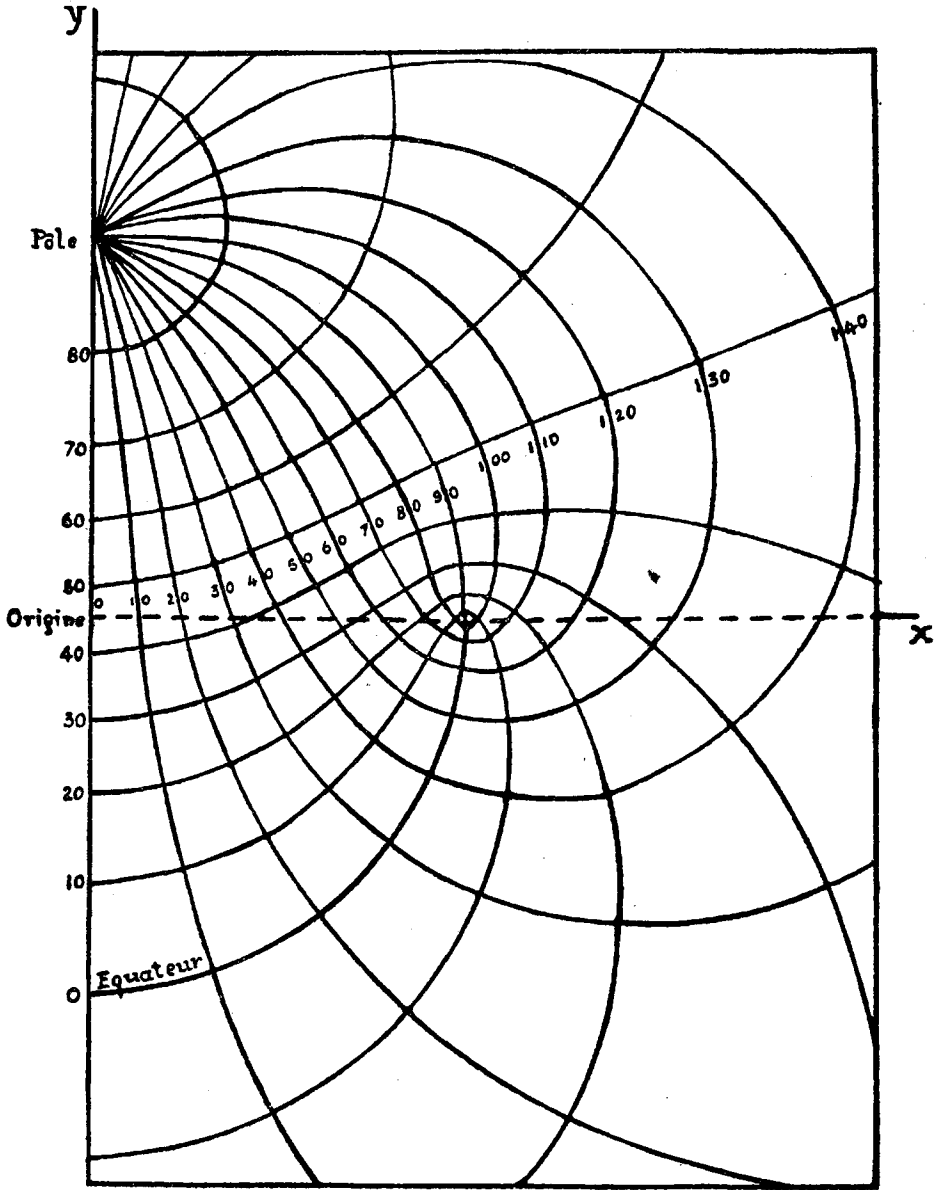


FIG. 16

As in this case $\text{sh } l_0 = 1$, comes

$$Y + i X = \frac{\text{sh}(l + i M) - 1}{\text{sh}(l + i M) + 1},$$

from which are deduced for the right-angled coordinates

$$X = \frac{2 \text{ch } l \sin M}{\text{ch}^2 l + \sin^2 M + 2 \text{sh } l \cos M} = \frac{2 \cos L \sin M}{1 + \cos^2 L \sin^2 M + \sin 2 L \cos M},$$

$$Y = \frac{\text{sh}^2 l - \cos^2 M}{\text{ch}^2 l + \sin^2 M + 2 \text{sh } l \cos M} = \frac{\sin^2 L - \cos^2 M \cos^2 L}{1 + \cos^2 L \sin^2 M + \sin 2 L \cos M}.$$

further

$$X^2 + Y^2 = \frac{\text{ch}^2 l + \sin^2 M - 2 \text{sh } l \cos M}{\text{ch}^2 l + \sin^2 M + 2 \text{sh } l \cos M}$$

$$= \frac{1 + \cos^2 L \sin^2 M - \sin 2 L \cos M}{1 + \cos^2 L \sin^2 M + \sin 2 L \cos M}.$$

The grid of the meridians and parallels (*Fig. 16*) is symmetrical about OY. The segment of the OX axis comprised between the abscissae $+1$ and -1 represents the great circle passing through the origin normally to the meridian of this point; the great circle having the origin as its pole is projected along the part of the X axis which is outside the singular points. The Y axis corresponds to the meridian of longitude 0 and π ; the projection of the pole is on this axis at the ordinate 1. The circumference of radius 1 having the origin as its centre represents the equator (demi-circumference located on the side of the negative Y's) and the meridians in longitude $\pm \frac{\pi}{2}$ (demi-circumference located on the side of the positive Y's); the point of the equator in longitude M and the point of the meridian $\frac{\pi}{2}$ in latitude $\frac{\pi}{2} - M$ have projections symmetrical about OX.

The projection rejects to infinity the points of geographical co-ordinates $-\frac{\pi}{4}$, 0 and $\frac{\pi}{4}$, π ; the parallels of these critical points have as their transformation the cubic:—

$$X = (1 - Y) \sqrt{\frac{2Y}{1 - 2Y}}$$

which passes through the origin and admits of the asymptote $Y = \frac{1}{2}$ (*Fig. 17*); this curve is also the projection of the parallel of the origin. ²

The parallels and meridians are projected according to the curves of equation:—

$$X^2 + (1 - Y)^2 = \frac{v + 1}{2v} \left[1 - 2Y + v \pm \sqrt{1 - v} \sqrt{(1 - 2Y)^2 - v} \right]$$

in which the parameter v must be taken as equal to $\cos 2L$ for the parallels, to $1 + 2 \cot^2 M$ for the meridians. The circumference $X^2 + Y^2 = 1$ which represents the equator and the meridians in longitude $\pm \frac{\pi}{2}$ corresponds to the borderline case between the two series of curves, for which $v = 1$.

The parallels of latitude less than $\frac{\pi}{4}$ are represented by closed curves surrounding the two singular points of the X axis which they intersect at the abscissa points $\sqrt{\cos 2L}$ and $\frac{1}{\sqrt{\cos 2L}}$; the projection of the other parallels consists of closed curves surrounding the projection of the pole. All these curves meet the Y axis at the ordinate points $\tan(L - \frac{\pi}{4})$.

The projections of the meridians are 8-shaped closed curves intersecting at the pole and whose every loop surrounds a singular point; their points of intersection with OX have as abscissae $\frac{1}{\sqrt{1 + 2 \cot^2 M}}$ and $\sqrt{1 + 2 \cot^2 M}$.

The bearing V of the tangent to a curve of the grid is given by :—

$$\tan V = \frac{1 - Y}{X} - \frac{1}{2X} \frac{v + 1}{v} \left[1 - (1 - 2Y) \sqrt{\frac{1 - v}{(1 - 2y)^2 - v}} \right].$$

These curves admit therefore of the tangents parallel to OX for $X = 0$ ($M = 0$ or π), which is obvious on account of the conformity of the projection, and also for $1 - 2Y = \pm \sqrt{v}$; the abscissa of the contact point is then given by

$$X^2 = \frac{v + 1}{4} \pm \frac{1}{2\sqrt{v}}.$$

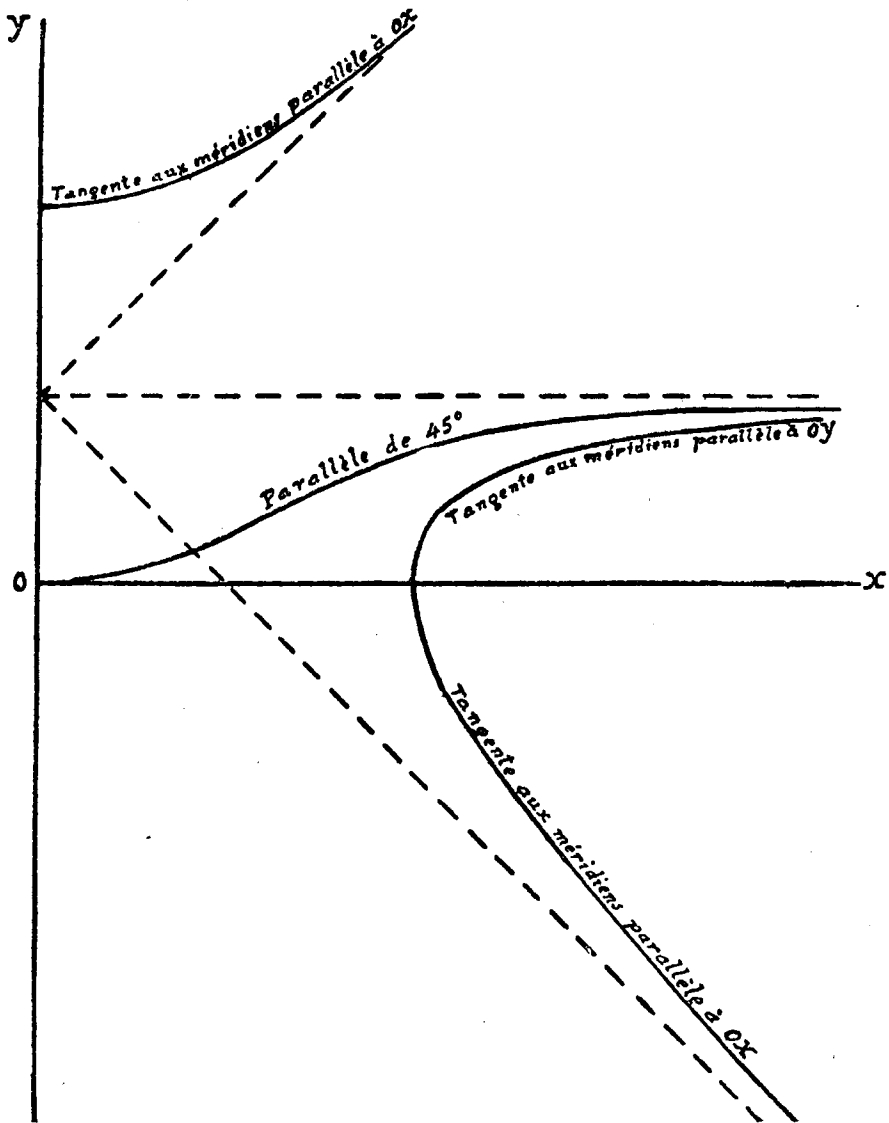
As regards the meridians, there are always two tangents parallel to OX whose ordinates are $Y = \frac{1}{2} (1 \pm \sqrt{1 + 2 \cot^2 M})$. For the parallels, the point of contact is real only when their latitude is less than $\frac{\pi}{4}$; the ordinate of the point of contact is

$$Y = \frac{1}{2} (1 - \sqrt{\cos 2L}).$$

The locus of the contact points of tangents parallel to OX is obtained by substituting $(1 - 2Y)^2$ for v , in the equation of the meridian and parallel transformations. A cubic with three hyperbolic branches the equation of which is :—

$$X^2 = \frac{(1 - Y)(1 - Y + 2Y^2)}{1 - 2Y}$$

FIG. 17



Parallel of 45°

X	Y	
0.0	0.0	
6.1	+	0.4
12.1	++	1.5
17.8	+++	3.1
23.0	++++	5.2
28.0	+++++	7.6
32.7	+++++	10.0
37.3	+++++	12.4
42.0	+++++	14.6
47.0	+++++	16.7
53.1	+++++	18.5
60.4	+++++	20.1
70.0	+++++	21.4
83.3	+++++	22.5
103.2	+++++	23.4

Asymptote

$Y = + 25.0$

Locus of the contact points of the tangents to the meridians.

X	Y	X	Y		
50.0	+	75.0	105.2	-	75.0
44.2		70.0	99.9		70.0
37.5		65.0	95.2		65.0
30.9		60.0	90.6		60.0
22.0		55.0	86.1		55.0
15.6		52.5	81.6		50.0
0.0		50.0	77.3		45.0
			73.0		40.0
			68.9		35.0
			65.0		30.0
			61.2		25.0
			57.8		20.0
			54.8		15.0
			52.4		10.0
			50.7	-	5.0
			50.0		0.0
			50.9	+	5.0
			54.2	+	10.0
			62.0		15.0
			83.1		20.0
			91.9		21.0
			105.1	+	22.0

which admits the asymptotes $Y = \frac{1}{2}$ (like the parallel of the origin) and $Y \pm X = \frac{1}{2}$ (Fig. 17).

This curve is the projection of the curves of the sphere

$$\tan^3 L + \tan L \cos^2 M + 2 \cos M = 0$$

for the arc relative to the points of contact of the parallels, and

$$\cos^3 M + \cos M \tan^2 L + 2 \tan L = 0$$

for the arc concerning the meridians.

As the projection is conformal, the points of contact of the tangents parallel to OY for a series of curves are those of the tangents parallel to OX of the other series; but the right angled co-ordinates of these points are not so easily expressed as a function of L or M as in the case of the tangents parallel to OX⁽¹⁾.

At the points of intersection with the X axis, the grid curves have bearings symmetrical about OY. At these points, we have as for the meridians:—

When X is greater than 1: $\tan V_m = \frac{1}{X} = \frac{1}{\sqrt{1 + 2 \cot^2 M}}$,

When X is less than 1: $\tan V_m = -X = -\frac{1}{\sqrt{1 + 2 \cot^2 M}}$.

Likewise for the parallels:—

for $X > 1$: $\tan V_p = -X = -\frac{1}{\sqrt{\cos 2L}}$,

for $X < 1$: $\tan V_p = \frac{1}{X} = \frac{1}{\sqrt{\cos 2L}}$.

(1) If we put $z = 1 - 2Y$, the condition $\tan V = 0$ is written:—

$$vz^3 - (2 + v)z^2 + (1 + 2v)z - 1 = 0,$$

equation of the third degree in z.

As the points of contact are on the cubic $X^2 = \frac{(1 - Y)(1 - Y + 2Y^2)}{1 - 2Y}$, X is obtained by this expression or by $X^2 = \frac{(1 + z)(2 - z + z^2)}{4z}$. In order to obtain the relation between X and v, z should be eliminated between the latter formula and the equation of the third degree in z.

We could also obtain the points of contact, by using the orthogonal curve whose tangent is then parallel to OX and for which we have, if we denote by v' the corresponding value of the parameter v: $z = \sqrt{v'}, X^2 = \frac{v' + 1}{4} + \frac{1}{2\sqrt{v'}}$. The parameter v' is obtained as a function of v by the equation of the third degree $t^3 + p^2 t + 2p = 0$, in which t represents $\pm \frac{1 - v'}{1 + v'}$ and $p = \mp \frac{1 - v}{1 + v}$.

If we consider the intersections with the parallel to OX' drawn through the projection of the pole ($Y = 1$), they are given by $X^2 = \frac{v^2 - 1}{v}$: at these points, we have also: $\tan^2 V = \frac{v + 1}{v(v - 1)}$.

We have therefore for the parallels $X^2 = -\frac{\sin^2 2L}{\cos 2L}$, $\text{tg } V_p = \frac{\text{cotg } L}{\sqrt{-\cos 2L}}$, and for the meridians $X^2 = \frac{4 \cos^2 M}{1 - \cos^4 M}$, $\text{tg } V_m = \frac{\text{tg } M}{\sqrt{1 + \cos^2 M}}$.

The ratio of similarity is described by the expression:—

$$K = \frac{2 \text{ch } l \sqrt{\text{ch}^2 l - \sin^2 M}}{\text{ch}^2 l + \sin^2 M + 2 \text{sh } l \cos M} = \frac{2 \sqrt{1 - \sin^2 M \cos^2 L}}{1 + \cos^2 L \sin^2 M + \sin 2L \cos M}$$

The grid of the projection can be obtained by inversion from that of Littrow's projection; the centre of inversion is the point of ordinate $+1$ of the Y axis and the constant of inversion is equal to 2.

8. Transverse projection of polar origin (Fig. 18) (1)

By placing the central point of the projection at the pole, we obtain a representation which is easily available for high latitudes.

For $L_0 = \frac{\pi}{2}$, $\text{sh } l_0$ is infinite and the projection formula is written:—

$$Y + iX = -\frac{1}{\text{sh}(l + iM)}$$

We see at once that its grid is inverse to that of Littrow's projection with respect to the origin of the coordinates, subject to the changing of the sign of the latitude of points obtained by inversion. The right angled coordinates are:—

$$X = \frac{\text{ch } l \sin M}{\text{ch}^2 l - \cos^2 M} = \frac{\cos L \sin M}{1 - \cos^2 L \cos^2 M},$$

$$Y = -\frac{\text{sh } l \cos M}{\text{ch}^2 l - \cos^2 M} = -\frac{\sin L \cos L \cos M}{1 - \cos^2 L \cos^2 M}.$$

Further,

$$X^2 + Y^2 = \frac{1}{\text{ch}^2 l - \cos^2 M} = \frac{1}{\text{tg}^2 L + \sin^2 M}.$$

The grid of the meridians and parallels is symmetrical about OY but, in addition, the points in the same longitude and in symmetrical latitudes are themselves symmetrical about OX , likewise those of the same parallel with additional longitudes.

(1) See: *Hydrographic Review*, Vol. IX, n° 1, p. 251.

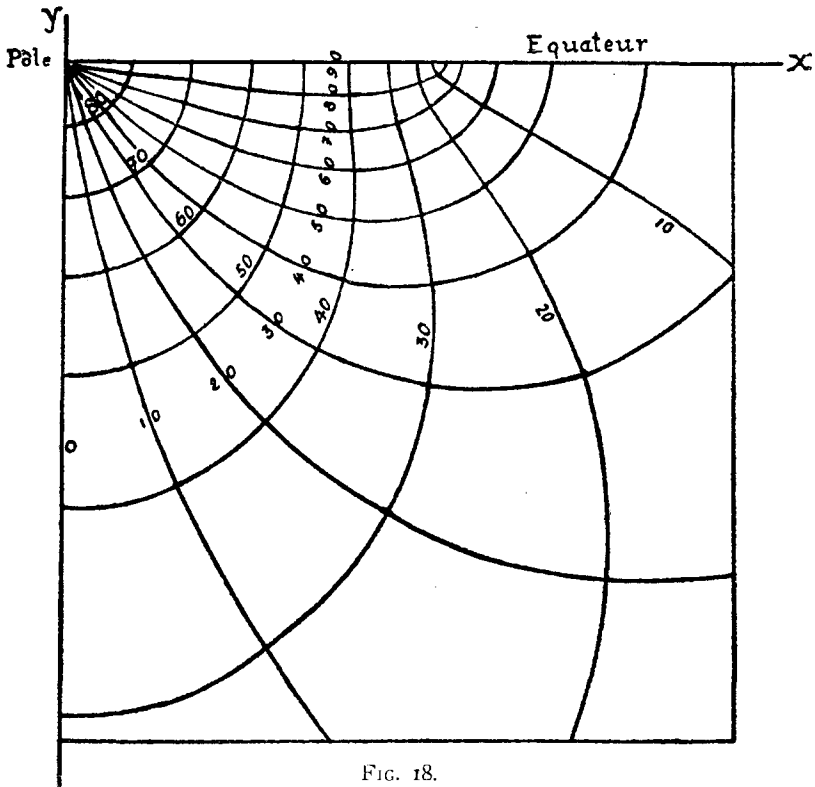


FIG. 18.

GRID OF THE MERIDIANS AND PARALLELS
LATITUDES

Longitudes	0	10	20	30	40	50	60	70	80
0				0.0	0.0	0.0	0.0	0.0	0.0
10			56.8	86.6	59.6	42.0	28.9	18.2	8.8
20			110.2	78.2	56.3	40.5	28.1	17.9	8.7
30	100.0	90.3	69.6	49.5	34.2	23.3	15.4	9.4	4.4
40	0.0	27.2	41.2	42.9	38.1	30.9	23.1	15.3	7.6
50	77.8	73.5	62.7	49.7	37.6	27.3	18.8	11.8	5.7
60	0.0	15.2	25.5	29.6	28.8	24.9	19.4	13.2	6.7
70	65.3	62.9	56.7	48.1	38.7	29.7	21.4	13.8	6.7
80	0.0	9.2	16.3	20.2	20.9	19.1	15.5	10.9	5.6
	57.7	56.3	52.2	46.2	38.9	31.0	23.1	15.3	7.6
	0.0	5.6	10.3	13.3	14.4	13.7	11.5	8.3	4.3
	53.2	52.2	49.2	44.6	38.6	31.7	24.2	16.3	8.2
	0.0	3.3	6.1	8.1	9.0	8.8	7.6	5.6	2.9
	50.8	50.0	47.5	43.6	38.4	32.1	24.8	16.9	8.6
	0.0	1.5	2.9	3.8	4.4	4.3	3.8	2.8	1.5
	50.0	49.2	47.0	43.3	38.3	32.1	25.0	17.1	8.7
	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

The first line gives the X's always positive the second the Y's always negative. The pole coincides with the origin.

The equator coincides with the X axis, it prolongs the meridian of 90°.

At the pole the tangent to the meridians makes with the downward Y axis an angle equal to longitude.

The meridians and parallels intersect at right angle.

Limit the projection to the frame X = 0, X = 90, Y = 0, Y = 90.

COMPLEMENTARY POINTS

Meridians	X	Y	Observations	Parallels	X	Y	Observations
10°	38.7	92.7		10°	81.8	20.1	
10	20.4	66.2		20	73.1	64.0	tang. // oy
20	93.8	66.7		20	72.3	53.0	
20	77.3	68.7	tang. // ox	20	69.0	88.1	
20	56.4	65.5		30	50.0	35.4	tang. // oy
20	34.4	54.2		30	37.3	69.6	
30	80.4	36.1		30	14.8	84.3	
30	59.0	43.2		40	38.9	16.2	tang. // oy
30	55.9	43.3	tang. // ox				
40	46.2	29.8	—				
50	41.1	21.0	—				
60	38.2	14.4	—				
70	36.5	9.1	—				
80	35.6	4.4	—				

The segment of the X axis comprised between the singular points represents the meridians in longitude $\pm \frac{\pi}{2}$, the remainder of the axis is the projection of the equator. The Y axis corresponds to the meridians in longitude 0 and π .

The meridians and parallels are represented by the curves of the fourth degree :—

$$\frac{X^2}{\sin^2 M} - \frac{Y^2}{\cos^2 M} = (X^2 + Y^2)^2, \quad \frac{X^2}{\operatorname{ch}^2 l} + \frac{Y^2}{\operatorname{sh}^2 l} = (X^2 + Y^2)^2.$$

The projections of the parallels are closed curves surrounding the pole and leaving outside the singular points on the axis OX. They intersect the Y axis at the points whose ordinate is $\pm \cot L$ and the X axis at the abscissae $\pm \cos L$.

The meridians are 8-shaped curves, admitting of the origin as a centre of symmetry and including a singular point in each loop; their intersection with OX have for abscissae $\pm \frac{1}{\sin M}$.

The bearings V_m and V_p of the tangents to the projections of the meridians and parallels are expressed by :—

$$\begin{aligned} \tan V_m = -\cot V_p &= -\operatorname{th} l \tan M \frac{\operatorname{ch}^2 l + \cos^2 M}{\operatorname{sh}^2 l - \sin^2 M} \\ &= -\sin L \tan M \frac{1 + \cos^3 L \cos^2 M}{\sin^2 L - \cos^2 L \sin^2 M}. \end{aligned}$$

If we denote again by U the angle at the centre of the conformal spherical projection such as

$$\tan \frac{U}{2} = \operatorname{th} l \tan M,$$

if we denote by θ the bearing of the point X, Y in relation to the origin of the coordinates,

$$\tan \theta = \frac{X}{Y} = -\frac{\tan M}{\operatorname{th} l},$$

it is easy to establish that

$$\tan V_m = \tan \left(\frac{U}{2} + 2\theta \right),$$

whence

$$V_m = \frac{U}{2} + 2\theta, \quad V_p = \frac{\pi}{2} + \frac{U}{2} + 2\theta.$$

The bearing V_m is cancelled out at the same time as $\operatorname{th} l$ and $\tan M$; which is obvious since the projection is conformal. On the other hand V_m becomes equal to $\frac{\pi}{2}$ and V_p null for

$$\tan L = \pm \sin M,$$

being the equation of the great circles of the sphere perpendicular to the meridians in longitude $\pm \frac{\pi}{2}$ at the points in latitude $\pm \frac{\pi}{4}$. The projection of these great circles is the locus of the points on the meridians where the tangent is parallel to OX and consequently of the points on the parallels where the tangent has a bearing equal to zero. The coordinates of the contact points are :—

$$\text{for the meridians :— } X = \pm \frac{1}{2} \sqrt{2 + \cot^2 M}, \quad Y = \pm \frac{1}{2} \cot M;$$

$$\text{for the parallels :— } X = \pm \frac{1}{2 \sin L}, \quad Y = \pm \frac{\sqrt{\cos 2 L}}{2 \sin L} = \pm \frac{1}{2} \sqrt{\cot^2 L - 1}.$$

There are therefore contact points for parallels only in latitudes less than $\frac{\pi}{4}$.

The locus of the contact points is the equilateral hyperbola :—

$$X^2 - Y^2 = \frac{1}{2}$$

whose asymptotes are the bissectrices of the coordinates axes.

The parallels admitting of tangents parallel to OY at the preceding points and at their intersection with the X axis, offer an inflection between these points. In order to obtain its position, it is necessary to seek the value of M which cancels $\frac{d V_p}{d M}$ with a change of sign. Now,

$$\frac{d V_p}{d M} = - \frac{\cos^2 V_p}{4 \operatorname{th} l \sin^2 M} \frac{(\operatorname{ch} 2 l + \cos 2 M)^2 - 4 \cos^2 2 M}{(\operatorname{ch}^2 l + \cos^2 M)^2},$$

as $\operatorname{ch} 2 l$ is greater than 1 this is a solution only for :—

$$\operatorname{ch} 2 l + 3 \cos 2 M = 0,$$

which is an equation defining on the sphere the segment subtending a right angle constructed on the points of the equator in longitude $\pm 54^\circ 44'$

($\cos 2 M = -\frac{1}{3}$). The projection of this curve is the locus of the points of inflection. The right-angled coordinates of these points, as a function of the latitude of the parallel, are expressed by :—

$$X = \frac{\operatorname{ch} l}{2 \operatorname{ch} 2 l} \sqrt{1 + \operatorname{ch}^2 l} = \frac{\sqrt{1 + \cos^2 L}}{2(1 + \sin^2 L)}, \quad Y = \frac{\operatorname{sh} l}{2 \operatorname{ch} 2 l} \sqrt{1 - \operatorname{sh}^2 l} = \frac{\sin L \sqrt{\cos^2 L}}{2(1 + \sin^2 L)}.$$

At these points, the bearing of the tangent to the parallel is expressed by :—

$$\tan V_p = - \frac{1}{\operatorname{th} l} \left(\frac{3 - \operatorname{ch} 2 l}{3 + \operatorname{ch} 2 l} \right)^{\frac{3}{2}} = - \frac{1}{\sin L} \left(\frac{1 - \tan^2 L}{2 + \tan^2 L} \right)^{\frac{3}{2}}.$$

The equation of the locus of the points of inflection is :—

$$4(Y^2 + X^2)^2 + 8(Y^2 - X^2) + 3 = 0.$$

It is constituted by two closed curves symmetrical to each other about OY and admitting of OX as axis of symmetry. We have for the bearing V of the tangent to the curve :—

$$\tan V = \frac{Y}{X} \frac{1 + X^2 + Y^2}{1 - X^2 - Y^2}$$

The tangent is parallel to OY at the points of intersection with the X axis, which have for abscissae $\pm \frac{1}{\sqrt{2}}$ ($L = 45^\circ$ $M = 90^\circ$) and $\pm \sqrt{\frac{3}{2}}$ ($L = 0$, $M = 54^\circ 44'$) these latter points are the points of inflection of the projection (rectilinear) of the equator. The locus admits a tangent parallel to OX at the points defined by $X^2 + Y^2 = 1$ and whose coordinates are consequently : $X = \pm \frac{\sqrt{15}}{4}$, $Y = \pm \frac{1}{4}$ the direction of the tangent to the projection of the parallel y is given by : $\tan V_p = \pm \frac{1}{3} \sqrt{\frac{5}{3}}$, i.e. $V_p = 23^\circ 17'$ or $156^\circ 43'$.

These points are on the meridians of longitude 60 or 120° in latitude $26^\circ 34'$ ($\tan L = \frac{1}{2}$).

The ratio of similarity of the projection of polar origin is :—

$$K = \text{ch } l \frac{\sqrt{\text{ch}^2 l - \sin^2 M}}{\text{ch}^2 l - \cos^2 M} = \frac{\sqrt{1 - \sin^2 M \cos^2 L}}{1 - \cos^2 M \cos^2 L}$$

9. Construction of transverse projections

In order to obtain, in the transverse projection whose origin is in latitude L_0 , the projection of the point whose geographical coordinates are LM, we can, instead of calculating the right-angled coordinates, go through the conformal spherical projection and construct the orthogonal circumferences representing in this system, the parallel and meridian of the point under consideration, whose coordinates about the pivot are the arcs L_1 and M_1 defined by :—

$$\sin L_1 = \cos L \sin M, \quad \cot (L_0 - M_1) = \cot L \cos M.$$

Further, the spherical projection complies with the formula :—

$$X - i Y = \text{th} (l_1 + i M_1)$$

which takes into account the orientation of the axes about the geographical meridian.

The parallel of latitude L_1 , has its centre on OX at the abscissa :—

$$\xi = \frac{1 + \sin^2 L_1}{2 \sin L_1} = \frac{1 + \cos^2 L \sin^2 M}{2 \cos L \sin M}$$

and its radius is expressed by :—

$$\rho_p = \frac{\cos^2 L_1}{2 \sin L_1} = \frac{1 - \cos^2 L \sin^2 M}{2 \cos L \sin M}$$

If we denote by φ the auxiliary angle defined by :—

$$\tan \varphi = \cos L \sin M,$$

these quantities are written :—

$$\xi = \frac{1}{\sin 2 \varphi}, \quad \rho_p = \cot 2 \varphi.$$

As regards the meridian of longitude M_1 its centre is on OY at the ordinate $\eta = \cot 2 M_1$ and its radius is expressed by $\rho_m = \frac{1}{\sin 2 M_1}$ the arc M_1 being considered as an auxiliary variable calculated as a function of L and M. As the meridians of the auxiliary spherical projection pass through the singular points of the X axis, it is sufficient to calculate the ordinate of their centre, the value of the radius is only useful for checking purposes.

If no great accuracy be desired, the co-ordinates of the circumference centres as well as the radii can be obtained by means of two nomograms with aligned points. The first, relative to parallels, is valid for all transverse projections, because it is not dependent on L_o ; it includes two parallel scales of L and M graduated according to the laws $\log \cos L$ and $\log \sin M$; half-way between the two, is plotted a scale $1/2 \log \sin L$ doubly graduated as for ξ and ρ_p , which allows a direct reading of these quantities. The second nomogram, established in a similar way, is constituted by two L and M parallel scales of $\log \cot L$ and $\log \cos M$, and by a middle scale of $1/2 \log \cot (L_o - M_1)$ bearing a double graduation for η and ρ_m .

In order to obtain the point $L_1 M$, we could also avail ourselves of the inversion ratio existing between Littrow's projection and the transverse projection under consideration. In particular, the straight lines of Littrow's projection $X = \text{constant}$, $Y = \text{constant}$, or else the straight line $\frac{X}{Y} = \text{constant}$ and the circumference $X^2 + Y^2 = \text{constant}$, furnish by inversion with respect to the point $X = O$, $Y = -\cot L_o$, some orthogonal circumferences, two by two, and which are loci of the point $L_1 M$, but the definition of these loci is less simple then in the case of the conformal spherical projection.

When, however, we wish to obtain the transverse projection of medium origin ($L_o = \frac{\pi}{4}$) some simplifications become apparent. The straight line passing through the origin of Littrow's projection with the bearing θ ,

$$\frac{X}{Y} = \frac{\tan M}{\sin L} = \tan \theta,$$

has for its inverse a circumference whose centre is on OX at the abscissa $-\cot \theta$ and whose radius is equal to $\frac{1}{\sin \theta}$. As this locus passes through the points whose ordinate is ± 1 on the Y axis, the calculation of the radius by means of the auxiliary angle θ is only useful for checking purposes. On

the other hand, the circumference of Litrow's projection, whose centre is the origin,

$$X^2 + Y^2 = \tan^2 L + \sin^2 M = \tan^2 \left(\frac{\pi}{4} - \psi \right),$$

when denoting by ψ an other auxiliary variable, is transformed by inversion into a circumference whose centre is on OY at the ordinate $\frac{1}{\sin 2\psi}$ and whose radius has the value $\cot 2\psi$. These two grids of orthogonal circumferences constitute a conformal spherical projection of exponent 2 relative to a system of isometric coordinates L_2 and M_2 defined by $\tan l_2 = \tan \psi$ and $M_2 = \frac{\theta}{2}$ or else $e^{-2l_2} = \frac{\sin \sigma}{\cos L}$, $\cos 2M_2 = \frac{\tan L}{\tan \sigma}$ when denoting by σ the spherical distance from the point L_1M to the equatorial point of the zero meridian.

10. Use of transverse projections

Transverse projections have the property of making a simple figure correspond to spherical subtended angles only when the base of these is borne by the zero meridian. They are therefore not intended for the compilation of geographical maps, their object is limited to the graphic determination of spherical subtended angles. They will therefore be used only in the schematic form of a grid of meridians and parallels in which the base of the spherical subtended angle will always coincide with the zero meridian and where the subtended angle projection will be constructed graphically. The geographical coordinates of a few points of the locus in the prescribed area will then be taken, it will thus be possible to plot it point by point on the navigation chart; in this operation, due account must be taken of the proper longitude of the segment base, whose value shall be added algebraically to the longitudes taken. As these representations are conformal, use may be made, in the plotting, of the value of the azimuths of the locus at some of its points, these angles being then preserved on projection.

These projections offering also some important linear and fairly rapidly variable distortions, it will be desirable to establish a fairly close grid in order to minimize the interpolations in taking up the positions, these interpolations will practically always be effected on inspection. It is however useless to try for great accuracy, the angle of the segment, being obtained in the present state of radiogoniometric technique, only with a fair measure of uncertainty. It seems sufficient, for the time being to draw the grid curves from degree to degree or every two degrees.

From a practical stand point, Littrow's projection seems a priori the more advantageous, since the radiogoniometric subtended angle is represented by a straight line; but it has the disadvantage of not being easily extensible to high latitudes, especially when it is desired to preserve a fairly large scale in the vicinity of the equator. We did show that the difficulty can be obviated

by reversing the meridians and parallels of the grid (it will be remembered that the projection thus obtained is no longer conformal), which makes it possible to represent by a second straight line the polar portion of the spherical subtended angle; still, apart from the fact that this procedure requires additional attention on the part of the user, it may fail when the subtended angle is comprised entirely in an area far away from the equator.

This case which is fairly rare in sea navigation is likely to become more and more frequent in air navigation for which circumpolar regions afford no great difficulty and constitute a useful way through in intercontinental relations.

It might then be thought advisable to add to Littrow's projection a transverse projection of polar origin, in which the spherical subtended angle is represented by a plane segment subtending the same angle. The user should be trained to use either of the methods of representation of the segment.

Thus, it seems to us more rational to waive the special advantage of Littrow's projection (rectilinear projection of the spherical subtended angle) and to have recourse to the transverse projection of medium origin which might probably enable us to deal with most cases in practice. A projection covering for example 130 degrees of the zero meridian, is more useful when its central point is in latitude 45 degrees than at the equator; instead of running from 65° S. to 65° N., it extends from 20° S. to 20° beyond the pole and consequently it enables us to solve, in addition to all problems relative to mediums latitudes, nearly all those affecting either the polar regions or the equatorial regions. The grid of the projection of medium origin not being symmetrical about the X axis, calculations required for its establishment take evidently longer than for projections of equatorial or polar origin; the drawing of the plane subtended angle takes also longer than that of a straight line; but this defect seems to be made good by the fact that this method of resolution is the only one for all cases, which must in all probability do away with misgivings and errors on the part of the operator.

11. Other Methods of representation of spherical subtended angles

Apart from the preceding conformal representations through which a plane subtended angle corresponds to spherical subtended angles, described on two points of the equator or two points of the zero meridian, there exist other methods of representation, aphyllactic, this time, which are capable of being utilized for drawing spherical subtended angles.

These systems are of two kinds, some deriving from the conformal projections which we have studied, are applicable to spherical subtended angles whose base is subject only to the condition of being borne by a definite great circle; taking this great circle as a zero meridian, the diagram thus obtained can be utilized as that of Littrow's projection; on the contrary, as regards other systems, the projection grid depends on the position of the base

extremities and consequently can only be used for the drawing of the segments described on this base; in this case, there must be as many diagrams available as there are transmitting stations under consideration.

Professor Lecoq's projection belongs to this second category and causes spherical subtended angles described on a definite base to correspond to plane segments subtending the same angle.

The most useful representations, for practical purposes, are those in which the geometrical locus is projected according to a straight line. Now, as regards all projections, whether conformal or not, in which the spherical subtended angle has for its transformation a plane segment subtending the same angle, it is possible, through a suitable choice of the point or by inversion, to reject to infinity the projection of an extremity of the base; the effect of which is to transform the projection of the spherical segment into a straight line passing through the new projection of the other extremity and inclined on the base projection at an angle equal to the segment angle; the new system gives evidently but the projection of a more or less extended arc of the segment in the vicinity of the base extremity.

In the case of Lecoq's projection, we may proceed either with one or with the other extremity, according to the arc which it is desired to preserve. In the case of a conformal projection (transverse) as, in practice, the pole is an extremity common to all bases, it is its projection which should be at infinity, if it is desired that the representation may still be used with any meridian base; we then have Littrow's projection; if, on the contrary, we chose to establish a special representation for each base, we may proceed as with Lecoq's projection.

From the systems in which the spherical subtended angle has a rectilinear projection, it is possible, through homographic transformation, to deduce others possessing the same property and complying with other practical requirements; we have already met with an instance of this transformation in connection with Littrow's projection (auxiliary projection through inversion of the grid). Those of these systems which are only suitable for a definite base may be the subject of another transformation which affects in no way the representation of the segments, but which is likely to improve the projection grid in the useful area; if the point of concurrence of the straight lines representing the spherical segments be taken as the origin of the right-angled coordinates and these coordinates be multiplied by any same function of the geographical coordinates, the straight lines corresponding to the segments, are not affected, but the grid of the meridians and parallels is altered; the function is chosen so as to give a special property to the representation, generally some facility in grid construction; we will give a simple instance of this transformation by radial amplification.

All the projections intended for drawing spherical subtended angles are used as shown in the case of transverse projections; for those of a general

application, there can be no question of introducing geographical drawings, since in each case, the base of the subtended angle must have as its projection the central meridian of the chart; the systems suitable only for a definite base might include a cartographic sketch, but its use would only be of minor importance. Normally, these various projections will only be employed in the shape of a relatively close grid of meridians and parallels on which the transformation of the geometrical locus shall be drawn and on which shall then be plotted the geographical coordinates of a few points of this curve in the useful area so as to plot it point by point on the navigation chart.

On the other hand, these representations involve important alterations in the lengths, angles and surfaces; the determination of these distortions is only of minor interest because these projections must not include geographical drawings and also because the problem which they solve leaves generally no choice between several solutions corresponding to different alterations.

12. Inverted Littrow's projection

We have seen (paragraph 6) that by inverting longitudes and co-latitudes in Littrow's projection grid, we obtained a new system which, in homographic relation with that of Littrow, retained as a consequence, the rectilinear shape of the radiogoniometric subtended angle transformations and could be utilized for drawing the projection of the circumpolar part of these geometric loci.

This auxiliary projection is defined by the relations :—

$$X = \frac{\cos L}{\sin M}, \quad Y = \cot M \sin L.$$

Its grid is rectangular like that of Littrow's projection; the Y axis is a singular line representing the pole; there are on the X axis, at the abscissae ± 1 two singular points corresponding to the poles of the zero meridian, a great circle which the projection rejects to infinity; the meridian of longitude $\pm \frac{\pi}{2}$ is projected according to the X axis comprised between these two points; the remainder of the axis corresponds to the equator.

The meridians are represented by homofocal ellipses of equation :—

$$X^2 \sin^2 M + Y^2 \operatorname{tg}^2 M = 1,$$

whose foci are the singular points of the X axis.

To the parallels correspond the homofocal hyperbolas :—

$$\frac{X^2}{\cos^2 L} - \frac{Y^2}{\sin^2 L} = 1,$$

with the same foci as the preceding ellipses and whose asymptotes passing through the origin of the coordinates and having as bearing $\pm \left(\frac{\pi}{2} - L\right)$ make the angle $2L$ between them.

The major axis of Tissot's indicatrix ellipse is orientated along the parallel and has for half-length $\frac{\sqrt{\sin^2 L + \cotan^2 M}}{\sin M \cos L}$; that of the minor axis is $\sqrt{\sin^2 L + \cotan^2 M}$; the ratio of the corresponding surfaces is therefore expressed by $\frac{\sin^2 L + \cotan^2 M}{\sin M \cos L}$, lastly the maximum ω of the angular demi-distortion is given by :—

$$\tan\left(\frac{\pi}{4} + \frac{\omega}{2}\right) = \sqrt{\sin M \cos L}.$$

In this system, any spherical subtended angle described on the pole and a point of the zero meridian, is projected according to a straight line passing through the point of the Y axis corresponding to a longitude equal to the segment angle and making with this axis an angle equal to the base length, estimated in parts of the radius.

The homographic relation between this projection and that of Littrow's is written :—

$$X = \frac{1}{X_L}, \quad Y = \frac{Y_L}{X_L}$$

whence is deduced :—

$$\frac{Y}{X+1} = \frac{Y_L}{X_L+1}.$$

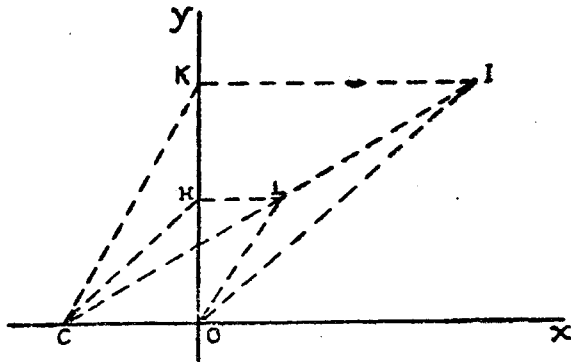


FIG. 19

The points of the same right-angled coordinates of both systems are such that the colatitude of one is equal to the longitude of the other; the straight line $X = 1$ representing the locus of the points of the sphere whose longitude is equal to colatitude, is common to both systems.

The points L and I (*Fig. 19*) representing a single point of the sphere in both projections, are aligned on the point C of the X axis, with abscissa -1 ; further, the straight line joining one of these points to the origin of the coordinates is parallel to that joining the point C to the projection of the other point on the Y axis.

13. Polyconic projection

Littrow's projection is the transverse aspect of the conformal spherical projection of exponent 2 and equatorial origin; the foregoing inverted projection is the transverse aspect of a polyconic projection which we are going to deduce from the conformal spherical projection through the same homographic transformation which connects Littrow's projection with the inverted projection, while, of course, taking into account the difference of orientation obtaining between the axes of the direct and transverse system coordinates. This homographic relation is therefore written :—

$$X = \frac{X_s}{Y_s}, \quad Y = \frac{1}{Y_s};$$

whence we further deduce

$$\frac{X}{Y + 1} = \frac{X_s}{Y_s + 1}.$$

The geometric correspondence between the points of the spherical projection and polyconic representing a single point of the sphere is therefore also given by Fig. 19, but while causing the coordinates axes to turn by $-\frac{\pi}{2}$.

The right-angled coordinates of the polyconic projection are expressed as follows, in terms of the geographical co-ordinates :—

$$X = \frac{\sin 2 M}{\text{sh } 2 l} = \frac{\sin M \cos M \cos^2 L}{\sin L}, \quad Y = \frac{\text{ch } 2 l + \cos 2 M}{\text{sh } 2 l} = \frac{1 - \cos^2 L \sin^2 M}{\sin L}.$$

The parallels are represented by the circumferences of equation :—

$$\text{th } 2 l = \frac{2 Y}{1 + X^2 + Y^2},$$

whose centre is on OY at the ordinate $\frac{1}{\tan 2 l}$ and whose radius is equal to $\frac{1}{\text{sh } 2 l}$; these circumferences also represent the parallels in the conformal spherical projection, the effect of the homographic transformation is simply to shift the points of the projection along their parallel.

To the meridians correspond the equilateral hyperbolas of equation :—

$$\tan 2 M = \frac{2 X Y}{Y^2 - X^2 - 1},$$

which pass through the points of the Y axis whose ordinate is ± 1 , being the projections of the poles, which admit as asymptotes the straight lines of the bearing M and $\frac{\pi}{2} + M$, coming from the origin.

The parallels being projected along circumferences whose centres are in a straight line, the representation is polyconic and characterized by the following expressions of the quantities r , s and U (Fig. 20):

$$r = \frac{1}{\text{th } 2l} = \frac{\cos^2 L}{2 \sin L}, \quad s = \frac{1}{\text{th}^2 l} = \frac{1 + \sin^2 L}{2 \sin L}, \quad u = \pi - 2M.$$

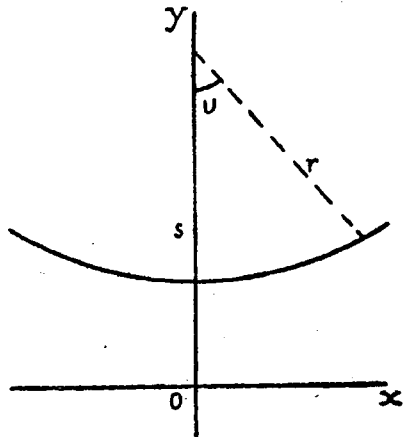


FIG. 20

In the conformal spherical projection of exponent 2, r and s have identical expressions but U is defined by the relation:—

$$\tan \frac{U}{2} = \text{th } l \tan M = \sin L \tan M.$$

The points of both systems which have the same right-angled coordinates represent therefore two points of the sphere in the same latitude L and whose longitudes M and M_1 are linked by the relation:—

$$\cot M \cot M_1 = \sin L.$$

The straight line $Y = 1$ which represents the sphere curve of equation $\cot^2 M = \sin L$ is common to both systems.

The polyconic projection under consideration rejects to infinity the projection of the equator, the segment comprised between the projections of the poles corresponds to the meridians of longitude $\pm \frac{\pi}{2}$, the remainder of the Y axis represents the meridians of longitude 0 and π ; lastly the X axis is a singular line which is the projection of the zero meridian poles.

A spherical segment subtending an angle α and whose base has for extremities one of these poles and another point of the equator in longitude M_0 is projected along a straight line bearing M_0 passing through the point on the axis OX whose abscissa is $-\cot \alpha$. This property, so far, is of no practical interest and the projection must be considered at present as a mere curiosity.

The alterations corresponding to this method of representation are important; the ratios between lengths along the meridian and the parallel are expressed respectively by

$$\cot L \sqrt{1 - \sin^2 M \cos^2 L + \cos^2 M \cot^2 L} \quad \text{and} \quad \cot L$$

The right-angle formed by a meridian and a parallel is subject to the alteration I given by :—

$$\tan I = - \frac{\sin M \cos M}{\text{tg}^2 L + \cos^2 M} = - \frac{X}{Y},$$

and therefore equal to the supplement of the straight line bearing joining the point under consideration to the origin of the coordinates. The axes of the indicatrix ellipse have for half-lengths

$$\frac{\cot L \sqrt{1 - \cos^2 L \sin^2 M}}{\sin L} \quad \text{and} \quad \cot L \sqrt{1 - \cos^2 L \sin^2 M};$$

the ratio between surfaces is expressed by $\frac{\cos^2 L}{\sin^3 L} (1 - \cos^2 L \sin^2 M)$, that is $Y \cot^2 L$; lastly the maximum ω of half the angular alteration at any point depends only on latitude and is given by :—

$$\sin \omega = - \tan^2 \left(\frac{\pi}{4} - \frac{L}{2} \right).$$

14. Lecoq's projection⁽¹⁾

Lecoq's projection is based on the following projective property of the spherical triangle. If we consider a spherical triangle ABC, drawn on a sphere with centre O (*Fig. 21*) and its apexes are projected in a b c on the sheer plan perpendicular to OB, in a direction parallel to the bissectrix OM of the side BOC, the angle a of the plane triangle a b c is equal to the angle A of the spherical triangle.

Consequently, if the point A describes a spherical subtended angle on the base BC, the projection of this curve is the segment subtending the same angle described on the projection plane, taking as a base the projection b c of BC.

With this method of representation, the parallels and meridians are projected along ellipses whose elements are easily obtainable. In particular, if the extremity B of the base is at the geographical pole, the projection plane is the equator's plane and the point M is the middle of the base; the parallels are then projected in true size along circumferences of radius $\sin \lambda$ (λ colatitude) whose centres are aligned on the straight line representing the meridian bearing the base and are at a distance $2 \sin^2 \frac{\lambda}{2} \tan \frac{\lambda_0}{2}$ from the projection of the polar extremity of the base (denoting by λ_0 the colatitude

(1) See: *Hydrographic Review*, Vol. X, N° 2, p. 101.

of the other extremity, that is, the length of the base). By dividing these circumferences into equal parts, we can next easily draw point by point the meridian half-ellipses. The projection of the base, serving as a base for the plane subtending angles representing the spherical segments, has a length of

$$2 \tan \frac{\lambda_0}{2}.$$

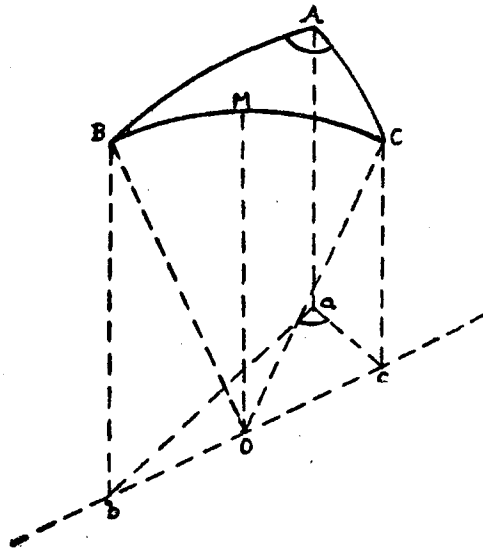
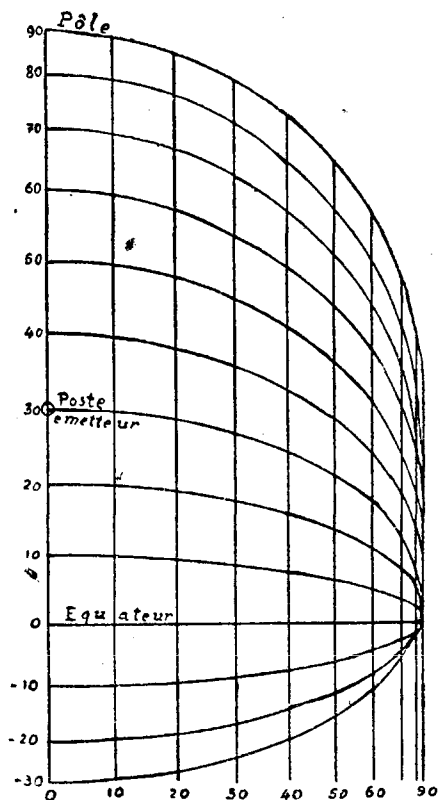


FIG. 21

Lecoq's projection which is polyconic but is neither conformal nor equivalent, is evidently only suitable for subtended angles described on the base under consideration, but this drawback is partly compensated for by an easily established grid; this representation like that of Littrow is used in the form of a diagram of meridians and parallels on which is drawn the subtended angle of the azimuth furnished by radiogoniometry, the useful part of the locus is then plotted on the navigation chart by means of geographical coordinates.

In order to simplify the drawing of the geometrical locus, Professor Lecoq proposed to replace the projection by its inverse figure in relation to the pole projection. This transformation, which rejects the projection of the pole to infinity, causes a spherical subtended angle to correspond to a straight line making with the projection of the base an angle equal to the observed azimuth; but the grid of this new representation is less advantageous; if the circular form of the parallels is preserved, their graduation in longitude can no longer be effected in equal parts and requires to be calculated beforehand or to be constructed graphically; further, the circumpolar part of the subtended angle can no longer be represented.

FIG. 22



PROJECTION BY RADIAL AMPLIFICATION
 (Transmitting station in latitude 30°)
GRID OF THE MERIDIANS AND PARALLELS
 LONGITUDES

	0	10	20	30	40	50	60	70	80	90
X	0.0	8.7	17.1	25.0	32.1	38.3	43.3	47.0	49.2	50.0
Latitudes	Values of Y									
90	50.0	49.2	47.0	43.3	38.3	32.1	25.0	17.1	8.7	0.0
80	44.2	43.5	41.3	37.6	32.7	26.6	19.6	11.8	3.5	— 5.0
70	37.1	36.4	34.3	30.8	26.1	20.3	13.6	6.2	— 1.7	— 9.9
60	28.9	28.2	26.3	23.1	18.7	13.4	7.2	0.3	— 6.9	— 14.4
50	19.7	19.2	17.4	14.6	10.8	6.1	0.6	— 5.5	— 11.9	— 18.6
40	10.0	9.5	8.1	5.7	2.5	— 1.5	— 6.0	— 11.1	— 16.5	— 22.1
30	0.0	— 0.4	— 1.5	— 3.3	— 5.8	— 8.9	— 12.5	— 16.4	— 20.7	— 25.0
20	— 10.0	— 10.3	— 11.1	— 12.3	— 14.0	— 16.1	— 18.6	— 21.3	— 24.2	— 27.1
10	— 19.7	— 19.9	— 20.3	— 20.9	— 21.8	— 22.8	— 24.1	— 25.5	— 26.9	— 28.4
0	— 28.9	— 28.9	— 28.9	— 28.9	— 28.9	— 28.9	— 28.9	— 28.9	— 28.9	— 28.9
— 10	— 37.1	— 37.0	— 36.5	— 35.9	— 35.1	— 34.0	— 32.8	— 31.4	— 29.9	— 28.4
— 20	— 44.2	— 44.0	— 43.2	— 41.9	— 40.2	— 38.1	— 35.7	— 33.0	— 30.1	— 27.1
— 30	— 50.0	— 49.6	— 48.5	— 46.7	— 44.2	— 41.1	— 37.5	— 33.6	— 29.3	— 25.0

The parallel of 90° representing the pole is a circumference with a 50 mm. radius and 0.0 centre.

15. Projections by radial amplification

From Littrow's projection, for instance, defined by

$$X = \frac{\sin M}{\cos L}, \quad Y = \tan L \cos M,$$

can be deduced a projection peculiar to a base comprised between the pole and the point in latitude L_1 , by multiplying by a function $\varphi (L_1M)$ the right-angled coordinates referred to the southern extremity of the base, that is

$$X = \frac{\sin M}{\cos L}, \quad Y = \tan L \cos M - \tan L_1.$$

If the function $\cos L$ be taken for φ we get

$$X = \sin M \quad Y = \sin L \cos M - \tan L_1 \cos L.$$

In this system (*Fig. 22*) the meridians are represented by straight lines parallel to the Y axis, the parallels by ellipses of eccentricity equal to $\cos L$, so that the transformation of the pole is a circumference whose centre is the origin of the coordinates. The spherical subtended angles are projected along the radii of the circumference. The projection thus obtained is quite acceptable and could be put to practical use; it has the advantage of giving a complete representation of the geometrical locus, from one to the other extremity of the base.

By adopting φ the function $\frac{\cos L}{\cos M}$, we obtain :—

$$X = \tan M, \quad Y = \sin L - \tan L_1 \frac{\cos L}{\cos M}.$$

The meridians are still represented by parallels to OY, but hyperbolas correspond to the geographical parallels; the projection of the pole is the parallel to OX whose ordinate is $Y = 1$.

We come directly to the foregoing projections, like to Littrow's projection, through the simple consideration of the spherical subtended angle equation.

$$\sin L \cos M - \sin M \cot \alpha - \tan L_1 \cos L = 0.$$

Such would no longer be the case if radial amplification were employed on a different projection for instance, on Lecoq's projection inversed in relation to the pole which also causes converging straight lines to correspond to the spherical subtended angles.

