METHOD OF CALCULATION FOR THE ADJUSTMENT OF OBSERVATIONS RESULTS

by

INGÉNIEUR HYDROGRAPHE GÉNÉRAL P. DE VANSSAY DE BLAVOUS,
Director.

When n observations have been made and have given results : $l_1, l_2 \ldots l_n$, which depend linearly on quantities $x_1, x_2 \ldots x_p$ it is proposed to apply to the results corrections : $v_1, v_2 \ldots v_n$ so as to obtain, for x quantities, values which may satisfy all equations, although p is less then n, by bringing to a minimum the sum of the v correction squares.

This problem is particularly that of the adjustment of a geodetic net, for the purpose of obtaining, from observation values, such a corrected system that the figure should be geometrically consistent. These conditions of figures give rise to *equations of condition,* which developed for corrections may by differentiation, be limited to the first order of smallness and rendered linear.

Mr. Bertil HALLERT of Stockholm, dealing again with this question in pamphlet 11 of the *Zeitschrift für Vermessungswesen* of 15th November 1943, pp. 238-244, points out the methods of calculation which seem to be most rapid, safest and best adapted to the use of calculating machines.

We will endeavour to set them forth briefly, with due regard to various publications on the same subject, while adopting the notations of Jordan-Eggert treatise.

I

Let us take n equations of condition, also described as error equations (*Fehlergleichungen;* after Helmert, *Ausgleichungsrechnung,* Leipzig 1872).

$$
(1) \quad
\begin{aligned}
v_1 &= a_1 x_1 + b_1 x_2 + c_1 x_3 + \ldots\ldots - l_1 \\
v_2 &= a_2 x_1 + b_2 x_2 + c_2 x_3 + \ldots\ldots - l_2 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
v_n &= a_n x_1 + b_n x_2 + c_n x_3 - \ldots\ldots - l_n
\end{aligned}
$$

By writing that, for the various x, we have $\frac{\delta\,[v\,v]}{\delta\,x} = 0$, and adopting Gauss's well-known symbol : $[v\,v] = v_1^2 + v_2^2 + v_3^2 + \ldots\ldots + v_n^2$, and noting that $[v\,v] = \frac{\delta\,[v\,v]}{\delta\,x}\,\Sigma\frac{x}{2} - [a\,l]\,x_1 - [b\,l]\,x_2 - [c\,l]\,x_3 - \ldots\ldots + [l\,l]$,

we obtain p *normal equations* :—

$$(2)\quad \begin{array}{l} [a\,a]\,x_1 + [a\,b]\,x_2 + [a\,c]\,x_3 + \ldots\ldots - [a\,l] = 0 \\ [a\,b]\,x_1 + [b\,b]\,x_2 + [b\,c]\,x_3 + \ldots\ldots - [b\,l[= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

Calculating [vv] permits the addition of equation :—

$$(3)\quad -[a\,l]\,x_1 - [b\,l]\,x_2 - [c\,l]\,x_3 \ldots\ldots + [l\,l] = [v\,v],$$

$x_1, x_2 \ldots x_p$ may be calculated from the normal equations. Calculation by ordinary methods of elimination was systematized and adapted to the use of calculating machines by the French Major CHOLESKY, of the Colonial Artillery, killed in action, on 31st August 1918.

Taking advantage of the fact that the determinant of equations (2) is symmetrical about the diagonal and eliminating successively each x unknown quantity from these equations, we obtain with Gauss's symbols *final equations* (*Endgleichungen*) from which all unknown quantities may be drawn :—

$$(4)\quad \begin{array}{r} [a\,a]\,x_1 + [a\,b]\,x_2 + [a\,c]\,x_3 + \ldots\ldots - [a\,l] = 0 \\ [bb.\,1]\,x_2 + [bc.\,1]\,x_3 + \ldots\ldots - [bl.\,1] = 0 \\ [cc.\,2]\,x_3 + \ldots\ldots - [cl.\,2] = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ [ll.\,p] = [v\,v] \end{array}$$

Abridged descriptions [bb. 1], [cc. 2]... frequently used for calculating determinants, were introduced for the first time by Gauss in 1810 in his paper. "*Disquisitio de elementis ellipticis Palladis, etc...*".

They have the following significations :—

$$[bb.\,1] = [b\,b] - \frac{[a\,b]}{[a\,a]}[a\,b],\quad [bl.\,1] = [b\,l] - \frac{[a\,b]}{[a\,a]}[a\,l],\quad [bc.\,1] = [b\,c] - \frac{[a\,b]}{[a\,a]}[a\,c],$$

$$[cc.\,2] = [c\,c] - \frac{[a\,c]}{[a\,a]}[a\,c] - \frac{[bc.\,1]}{[bb.\,1]}[bc.\,1],\quad [cl.\,2] = [c\,l] - \frac{[a\,c]}{[a\,a]}[a\,l] - \frac{[bc.\,1]}{[bb.\,1]}[bl.\,1],$$

$$[ll.\,p] = [v\,v] = [l\,l] - \frac{[a\,l]}{[a\,a]}[a\,l] - \frac{[bl.\,1]}{[bb.\,1]}[bl.\,1] - \frac{[cl.\,2]}{[cc.\,2]}[cl.\,2] - \ldots\ldots.$$

By dividing all the coefficients of the unknown quantities, of the equations by $\sqrt{[a\,a]}$, the final equations take the form :—

$$(5)\quad \begin{array}{r} (a\,a)\,x_1 + (a\,b)\,x_2 + (a\,c)\,x_3 + \ldots\ldots - (a\,l) = 0 \\ (b\,b)\,x_2 + (b\,c)\,x_3 + \ldots\ldots - (b\,l) = 0 \\ (c\,c)\,x_3 + \ldots\ldots - (c\,l) = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ [ll.\,p] = [v\,v] \end{array}$$

The symbols () have the following significations :—

$$(a\,a) = \sqrt{[a\,a]},\ (a\,b) = \frac{[a\,b]}{\sqrt{[a\,a]}} = \frac{[a\,b]}{(a\,a)},\ (a\,c) = \frac{[a\,c]}{(a\,a)},\ (a\,l) = \frac{[a\,l]}{(a\,a)},$$

$$(b\,b) = \sqrt{[bb.\,1]} = \sqrt{[b\,b] - (a\,b)\,(a\,b)},$$

$$(b\,c) = \frac{[bc.\,1]}{(b\,b)} = \frac{1}{(b\,b)}\left\{[b\,c] - (a\,b)\,(a\,c)\right\},\ (b\,l) = \frac{1}{(b\,b)}\left\{[b\,l] - (a\,b)\,(a\,l)\right\}$$

$$(c\,l) = \frac{1}{(c\,c)}\left\{[c\,l] - (a\,c)\,(a\,l) - (b\,c)\,(b\,l)\right\}$$

$$[ll.\,p] = [l\,l] - (a\,l)\,(a\,l) - (b\,l)\,(b\,l) - (c\,l)\,(c\,l)\ \ldots\ldots$$

$$(m\,n) = \sqrt{[n\,n] - (a\,n)\,(a\,n) - (b\,n)\,(b\,n) - \ldots\ldots - (m\,n)\,(m\,n)},$$

$$(n\,q) = \frac{1}{(n\,n)}\left\{[n\,q] - (a\,n)\,(a\,q) - (b\,n)\,(b\,q) - \ldots\ldots - (m\,n)\,(m\,q)\right\}.$$

each coefficient may thus be deduced from the one just calculated.

VERIFICATIONS

Calculations may be set forth in tabular form in which operations are always the same. Major CHOLESKY provided for many verifications. By adding the sum — s of all the coefficients of one and same equation (1) and treating these numbers s as ordinary coefficients, we shall have each time, verifications which insure the accuracy of the preceding results.

We actually obtain :—

$$(6)\quad \begin{aligned} &a_1 + b_1 + c_1 + \ldots\ldots - l_1 + s_1 = 0\\ &a_2 + b_2 + c_2 + \ldots\ldots - l_2 + s_2 = 0\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &a_n + b_n + c_n + \ldots\ldots - l_n + s_n = 0 \end{aligned}$$

and likewise :—

$$(7)\quad \begin{aligned} &[a\,a] + [a\,b] + [a\,c] + \ldots\ldots\ldots - [al] + [as] = 0\\ &[a\,b] + [b\,b] + [b\,c] + \ldots\ldots\ldots - [bl] + [bs] = 0\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &-[a\,l] - [b\,l] - [c\,l] - \ldots\ldots + [l\,l] - [l\,s] = 0\\ &\quad [a\,s] + [b\,s] + [c\,s] + \ldots\ldots - [l\,s] + [s\,s] = 0 \end{aligned}$$

and also

$$(8)\quad \begin{aligned} &[bb.\,1] + [bc.\,1] + \ldots\ldots\ldots - [bl.\,1] + [bs.\,1] = 0\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &-[bl.\,1] - [cl.\,1] - \ldots\ldots + [ll.\,1] - [ls.\,1] = 0\\ &\quad [bs.\,1] + [cs.\,1] + \ldots\ldots - [ls.\,1] + [ss.\,1] = 0 \end{aligned}$$

and so on until :—

$$[ll.\,p] = [vv] = [ls.\,p] = [ss.\,p].$$

We may by this method take up the resolution of equations with from 40 to 50 unknown quantities. We may also apply this method to the resolution of linear equations in equal number with that of the unknown quantities. In the latter case, calculations are not less long than by ordinary methods, but they have the advantage of being systematic and affording many verifications.

BIBLIOGRAPHY

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4) *Zeitschrift für Vermessungswesen,* 15 Nov. 1943, pamphlet 11, pp. 236-244. Note by Bertil HALLERT, Stockholm: " On a few methods for solving normal equations ".