# ON REFRACTION AND REFRACTION TABLES

by

INGÉNIEUR HYDROGRAPHE GÉNÉRAL P. DE VANSSAY DE BLAVOUS,

*Director.*

## I. — *ASTRONOMICAL REFRACTION*

*Astronomical refraction* is the angle made by the initial direction of a ray of light (outside the atmosphere) with that of its arrival at the surface of the ground. Its effect on heavenly bodies is to make them appear higher over the horizon than they actually are. It has exercised the minds of astronomers for a long time, because the knowledge of it is necessary for correcting their observations, although they endeavour to arrange these so as to minimize its influence.

The experimental determination of astronomical refraction is obtained by means of observations of circummeridian altitudes of circumpolar stars in their culmination. In his *Optics* Ptolemy (II° Century A.D.) already talked of the refraction suffered by rays of light in the terrestrial atmosphere. But the ignorance of its real cause resulted in the attribution of different values to refractions for the sun and the moon, on the one hand, and for the stars, on the other hand[1]. Great refractions amounting possibly to 4° were even suggested. The Rev. FOURNIER'S tables attributed, according to TYCHO-BRAHÉ (1546-1604)[2] a horizon refraction of 34' to the sun and the moon, and of 30' to the stars.

An excellent historical study by J.-H. DE LEEUW, was published in the *Hydrographic Review*, Vol. XII, N° 1, pp. 142-147, to which we refer our readers. We will merely add a few details.

The theoretical calculation of refraction depends essentially on density values and therefore on atmospheric temperature and pressure at various altitudes. Before enumerating some of the refraction tables, it seems advisable to consider the main hypotheses which were successively developped as to the constitution of atmosphere at different altitudes. The principal ones are[3]: that of G. Dominique CASSINI who regards the density of air as constant and next to unity.

---

(1) PIAZZI also surmised some difference between the values of these refractions in his work: *Del Reale Osservatorio di Palermo* (libro sexto), issued in 1806. (See: *Additions pour la Connaissance des Temps* de 1809, p. 452.)

(2) Dates of birth and death.

(3) Most of this information has been borrowed from a very comprehensive and interesting article published by P. PUISEUX, after the German article by A. BOMPARD, in the *Encyclopédie des Sciences mathématiques pures et appliquées*, tome VII, fascicule 1 (1st August 1913).

That of Tobias MAYER who admits a linear decrement of density $\delta$. Let h be the height of the air layer at the considered point, H the total height of the atmosphere, n a coefficient greater than unity, to be determined, we have :—

$$\frac{\delta}{\delta_0} = 1 - \frac{h}{H} n;$$

that of Isaac NEWTON, who assumes the air temperature to be constant. The pressure is then :—

$$p = p_0 \, e^{-K \frac{h}{a+h}},$$

where a is the earth's radius and where

$$K = \frac{a}{l\left(1 + \frac{t_0}{273}\right)},$$

$t_0$ being the centigrade temperature at the place of observation, l a coefficient, described as a reduced height of the atmosphere and equal to about 8 kilometers; that of P. S. LAPLACE :—

$$\frac{\delta}{\delta_0} = \left(1 + f\frac{u}{K}\right) e^{-\frac{u}{K}},$$

where f and k are physical constants,

$$u = \frac{h}{a+h} - \alpha\left(1 - \frac{\delta}{\delta_0}\right),$$

$\alpha$ being the constant of refraction(10);

that of F. W. BESSEL who agrees to the law of variation of temperature :—

$$\frac{t + 273}{t_0 + 273} = e^{-\frac{a}{K}\frac{h}{a+h}},$$

where k is a constant which would be equal to 228 according to BRADLEY'S observations. He infers therefrom the approximate law of variation of densities :—

$$\frac{\delta}{\delta_0} = e^{\beta \frac{h}{a+h}},$$

that of James IVORY (1765-1842) who in his *Memoir on Refractions* published in the "Philosophical Transactions of London" for 1823 and 1838, agrees to a linear ratio between temperature and density of the atmosphere :—

$$\frac{t + 273}{t_0 + 273} = 1 - f\left(1 - \frac{\delta}{\delta_0}\right),$$

f being a constant to be determined.

That of J. K. Ed. SCHMIDT (*Theorie der Astronomischen Strahlenbrechung,* Göttingen, 1828) :—

$$\frac{t + 273}{t_0 + 273} = 1 - \frac{h}{H} .$$

This hypothesis is best in agreement with the kilometric decrement of temperature, such as was observed in mountains.

The Author assumes :—

$$H = 49 \text{ km. } 1$$

that of H. GYLDEN, derived from Schmidt's, it agrees to :—

$$\frac{t + 273}{t_0 + 273} = \left(1 - \frac{\beta}{2} \cdot \frac{h}{a + h}\right)^2,$$

by taking :—

$$\beta = 120\ (1 + i),$$

where i received variable values of —0,40 to + 0,66 according to the month of the year. This formula which attempted to adapt refractions to those observed by V. FUSS(4), gives a kilometric decrement of temperature with the height following a behaviour opposite to that actually observed; that of M. KOVALSKI (*Researches on astronomical refraction,* Kazan, 1878):—

$$\frac{t_0 + 273}{t + 273} = 1 + K\left(1 - \frac{\delta}{\delta_0}\right)^{\frac{5}{7}},$$

where k is a constant.

The purpose of this hypothesis was to present a state of the atmosphere corresponding to the results collected from 1860 to 1870 by J. GLAISHER in his air voyages, such results, not being in harmony with those obtained more recently by L. TESSEREINC DE BORT.

That of M. Ernest ESCLANGON, who published in the *Comptes Rendus de l'Académie des Sciences* of 11th January 1943 a note on atmospheric refraction. He proposes to adopt between the indices of refraction $n_0$ at the place of observation and n at the considered point of the trajectory the ratio:

$$\frac{n - 1}{n_0 - 1} = e^{\mu^2 (n_0^2 r_0^2 - n^2 r^2)},$$

$r_0$ and r denoting the distances in kilometers to the centre of the earth, $\mu^2$ a coefficient to be determined by observation and approximating $10^{-5}$.

This law of the variation of refraction index permits an easy integration. It is consistant with meteorological data up to 15 kilometers. It is less so for higher layers whose importance is but small from the point of view of refraction.

---

(4) Mem. Acad. Petersb. (7) 18 (1872). Memoir N° 3.

To effect the integration, let us put :—

$$x = \mu\, n_0\, r_0 \cos \zeta_0,$$

and it is obtained for astronomical refraction the expression :—

$$R = 2\, \mu\, n_0\, r_0\, (n_0 - 1) \sin \zeta_0\, \psi\, (x),$$

$\psi$ is a function of which RADAU has given very full numerical tables in Vol. XVIII of the *Annales de l'Observatoire de Paris.* They allow a rapid calculation of refraction and give for horizontal refraction the value :—

$$\mu\, n_0\, (n_0 - 1) \sqrt{\pi}.$$

if $\zeta$ is less than 75°, we may write :—

$$\psi\, (x) = \frac{1}{2\, x}\left(1 - \frac{1}{2\, x^2}\right),$$

and consequently adopt for astronomical refraction the formula :—

$$R = (n_0 - 1) \text{ tang } \zeta_0 \left(1 - \frac{1}{2\, \mu^2\, n_0^2\, r_0^2} - \frac{tg^2 \zeta_0}{2\, \mu^2\, n_0^2\, r_0^2}\right).$$

Other physicists have also sought a formula connecting refraction to the directly observed properties of the atmosphere, namely through sounding-balloons. It is not certain whether there could be such a formula, equally well adapted to various seasons and various weather conditions.

Following a series of observations with the prismatic astrolabe, Admiral J.D. NARES pointed out in a note issued in the *Hydrographic Review,* Vol. IX, N° 2, pp. 121-123, that refraction seems sometimes different according to whether the path of the rays of light runs over sea or over land.

The following table shows values of refraction calculated according to various hypotheses :—

| $\zeta$ | CASSINI | MAYER | LAPLACE | BESSEL | IVORY | SCHMIDT | GYLDEN | KOVALSKI | RADAU |
|---|---|---|---|---|---|---|---|---|---|
| 45° | 60",3 | 60",3 | 60",5 | 60",3 | 60",0 | 60",5 | 60",3 | 60",3 | 60",04 |
| 70° | 164 ,5 | 164 ,5 | 165 ,1 | 164 ,5 | 165 ,2 | 165 ,0 | 164 ,6 | 164 ,5 | 163 ,78 |
| 80° | 330 ,8 | 331 ,0 | 332 ,3 | 331 ,1 | 332 ,9 | 332 ,3 | 331 ,4 | 331 ,4 | 329 ,8 |
| 85° | 607 ,9 | 611 ,8 | 617 ,7 | ( 615 ,8)(1)<br>( 613 ,9)(2) | 619 ,3 | 617 ,9 | 616 ,4 | 616 ,4 | 613 ,5 |
| 90° | 1287 ,0 | 1816 ,4 | 2106 ,0 | (2241 ,3) | 2200 ,6 | 2207 ,8 | 2210 ,2 | (2095 ,6) | 2196 ,0 |

(1) Theoretical.
(2) Observed.

Observatories dealing with variations of latitude must look after the very precise correcting of refraction. The question has arisen whether certain variations of latitude could not be accounted for by a daily periodical alteration of refraction. It was also found important [5] that temperature

(5) YAMAMOTO, *Memoirs of the College of Science, Kyoto Imperial University.* Ser. A, Vol. VI, N° 7 (1922-23).

and ventilation should be exactly the same at the North and South of each transit instrument, which only seems possible if the premises in which the instrument is placed are at least twenty meters long in a North-South direction.

## II. — *TABLES AND FORMULAE*

G. Dominique CASSINI published in 1662 at Modena : *Ephemerides novissima motuum coelestium ab anno* 1661 *ad anno* 1666, *additis ephemeridibus solis et tabulis refractionum.* In order to establish his refraction tables, he assumes, as we noticed before, that the density of air is independent of altitude. Under these conditions both pressure and temperature must decrease linearly with altitude.

The " Bureau des Longitudes ", in Paris, that has been publishing the *Connaissance des Temps* (Nautical Almanac) since 1679 gives in the year 1683 volume [6] a small table of astronomical refractions for apparent altitudes from degree to degree, up to 45° ; then in that of 1687 up to 60° and in that of 1689 up to 90°. These tables are substantially correct from 10° in apparent altitude, for lower altitudes, they show discrepancies from 1' to 6' with present tables. From 1689, numbers in the Table are slightly différent. The table has been enlarged in 1693.

In the *Connaissance des Temps* of 1702, p. 119, the table computed by Dominique CASSINI (1625-1712) [2] shows refraction values in minutes and seconds for apparent altitudes from degree to degree.

The " Bureau des Longitudes " never ceased to take an interest in this question; it published in the *Additions à la Connaissance des Temps,* from 1762 to 1796, notes by CASSINI, BRADLEY and LACAILLE, on this ame subject.

It follows from letters written to Flamsteed by Isaac NEWTON (1642-1727) [2] that the latter was the first in 1694-95 [7] to apply a mathematical theory to the problem of refraction, by using the method of parabolical quadratures. But he did not publish his work, neither did he publish the refraction table which he had drawn up in the case of a uniform temperature for the whole atmosphere layer. This table was published by HALLEY in the year 1721 volume of *Philosophical Transactions.* Biot reproduced it in the *Connaissance des Temps* for 1839, page 101 of the Additions.

Formulae by P. BOUGUER (1698-1758) [2], Thomas SIMPSON (1710-1761) [2] and James BRADLEY (1692-1762) [2] correspond to the same hypothesis ; numerical coefficients are computed from observations.

BOUGUER [8] in his memoir of 1729, studied astronomical refraction

(6) It has not been possible to consult an earlier volume.
(7) *Journal des Savants* for 1836.
(8) *Histoire générale de la Navigation,* by F. MARGUET, Paris, 1931.

and put forward a theory of the " solar ", a name which he gave to the curve described by a ray of light coming from the heavenly body to the observer. He reached some serviceable formulae.

In 1751, a new edition of a pamphlet by d'*Après de Mannevillette,* a Captain of the East India Company, published by BORY, contains a table of dips of the horizon [8].

James BRADLEY gave for astronomical refraction the formula :—

$$R = x \tan(\zeta - y R),$$

where x and y are constants.

The table in the *Connaissance des Temps* for 1789, in minutes, seconds and tenths of seconds was computed from this formula. The use of BRADLEY's formula is not as easy as its simplicity might lead to believe. One must proceed by a method of successive approximations, as it contains an unknown quantity R in both members. It has been found since, that the values employed by BRADLEY for x and y were too low.

BRADLEY's table was used for correcting altitudes of heavenly bodies taken during the voyage made in 1768-69, for the purpose of testing Ferdinand BERTHOUD's marine clocks; in Vol. II published in Paris in 1773 giving a narrative of this voyage, pp. 582-588, we find a reproduction of a more comprehensive and rather different refraction table, after Abbé DE LA CAILLE, which was given by DE LA LANDE, in the *Connaissance des Temps* for 1771, page 221.

Abbé DE LA CAILLE's table gives no correction for temperature or pressure. The one given in the *Connaissance des Temps,* after BRADLEY from 1800 is for a 10° temperature and a 28 inch barometer reading. It is slightly different from that of 1789.

The same Volume II, page 589, reproduces from Memoirs of the " Académie des Sciences " of 1739, p. 421, a table of astronomical refractions by BOUGUER for places at sea level, in the torrid zone. This table differs substantially from that of Abbé DE LA CAILLE.

T. SIMPSON showed that the influence of the earth's sphericity became appreciably negligible when the apparent altitude is more than 50°.

Tobias MAYER published in London in 1770, some refraction tables in his work :— *Tabulae motuum solis et lunae novae et correctae, quibus accedit methodus longitudinum promota.* His formula is based on the hypothesis of the linear decrement of density; which involves a uniform decrement of temperature.

J. Ch. BORDA (1733-1799) [2] through calculations and observations, and J.B.J. DELAMBRE (1749-1822) [2], through observations made at Bourges, found that BRADLEY's formula was only an approximation, liable to errors of from 10" to 15". This formula was however still employed for the computation of the *Connaissance des Temps* tables for 1798 ; DELAMBRE applied a correction to it to bring temperature down to 10° centigrade and the barometer reading to 28 inches. The table gives refraction in seconds for apparent altitudes varying from 10' to 10' for low altitudes.

BORDA seems to have employed for computing refractions a series which he did not publish and which was not found after his death.

Volumes of the *Connaissance des Temps* from 1801 to 1808 still employed BRADLEY's formula, bringing down temperature to 12°5 and barometric pressure to 760 mm.

Chr. KRAMP was the first to apply accurately the principle of small distance attractions in a variable density medium in his *Analyse des Réfractions astronomiques et terrestres,* Strasbourg, an VII, 1799. He adopted an exponential decrement of temperature, but lacked observation results to determine the physical constants of his formulae.

P.S. LAPLACE (1749-1827) [2] in Vol. IV, Book X, pp. 258-315 of his *Mécanique Céleste* (1799), gave a complete mathematical theory of refraction. He compiled its differential equation on the assumption of the sphericity of the earth and of the same density for atmospheric layers (he pointed out that flatness can be taken care of by substituting the curvature radius for the earth radius; which is not generally necessary). This equation holds good for any law of the variation of atmospheric density in terms of altitude. He first of all integrated it in the two following hypotheses; constant density or density decreasing in geometrical progression as altitude increases in arithmetical progression (which supposes a uniform temperature). He noted that the first hypothesis supplied him with refraction values much lower than the results obtained from experience, whilst the second supplied much greater ones and was not in agreement with observations made during GAY-LUSSAC's air ascent when the thermometer kept going down as the balloon rose up. But LAPLACE also showed that for apparent altitudes of more than 12°, expressions obtained for refraction were independent of any hypothesis on the constitution of atmosphere and was dependent only on barometer and thermometer readings at the observation spot.

In the case of apparent altitudes of less than 12°, he introduced a fairly complicated exponential expression for atmospheric density, for the purpose of rendering integration possible and of obtaining an intermediate result between those of the two preceding hypotheses.

He did not take into account the more or less great atmospheric moisture, but showed that its influence is very small and affords means of correcting refractions by adding the following increments :—

*Température*

| | |
|---|---|
| 15° | 0",563 tg ζ |
| 20° | 0 ,744 » |
| 25° | 0 ,977 » |
| 30° | 1 ,274 » |
| 35° | 1 ,651 » |
| 40° | 2 ,122 » |

These increments are computed on the assumption that the air is saturated with moisture; they should be multiplied by the ratio supplied by the hypsometer between the quantity of water vapour actually contained in the atmosphere and that corresponding to saturation.

They cannot apply in the case of great zenith distances :—

The *Connaissance des Temps* employed LAPLACE's formula in 1809 for computing astronomical refraction tables in determining constants from PIAZZI's (1746-1826) [2] and DELAMBRE's observations.

In volume I of *Verdun's voyage* (1771-72), Paris 1778, is expressed a desire that refraction tables (for Mariners use) instead of being computed for a mean state of the atmosphere in Paris or London, be computed for a warmer state of the atmosphere (15° to 16° Réaumur) or that, for observations requiring greater precision, should be employed corrections tables relating to thermometer degrees and various rises of the mercury in the barometer tube.

In 1810, the *Connaissance des Temps* gave for the first time, a table computed by DELAMARE for correcting mean refractions for temperature and pressure. It extends between pressures 710 and 789 and temperatures — 20° to + 30°. With this object in view, the table supplies two factors, by which mean refraction has to be multiplied.

These tables are taken from those published by the " Bureau des Longitudes ". The *Connaissance des Temps* reproduces them without alteration up to 1851 [9] from which date, the computation of LAPLACE'S formula is made by V. CAILLET (see Additions to the *Connaissance des Temps* for 1851) who only makes alterations of a few seconds to mean refraction for very small apparent altitudes and of a few thousandths to correction factors. However the correction table is expanded between temperatures of — 29° and + 50° ; then, from 1867, between pressures of 630 mm. and 789 mm. CAILLET adopted the value 60",616, for the constant of refraction [10].

We should also note that from 1820 to 1913 (inclusive) while ceasing

(9) In the *Cours d'Astronomie Nautique* by H. FAYE, Paris 1860, p. 65 is found a table deduced from that of the *Connaissance des Temps* of 1851, giving refractions for true zenith distances.

(10) $n$ being the air refraction index, the constant of refraction is: $\frac{n^2 - 1}{2n^2}$.

to give lunar distance tables in 1905, the *Connaissance des Temps* also gave without alterations, during that period, two tables supplying the logarithmic difference to 7 places between the cosine of the true altitude and that of the apparent altitude, for the sun and the stars, for apparent altitudes of 90° to 3°. The first takes refraction and the sun parallax into account, the second only refraction. These tables computed by J. Ch. BURCKHARDT (1773-1825) [2] were intended to facilitate the calculation of longiudes by lunar distances according to BORDA'S method and at, the same time to afford more precision than the refraction table. The refraction values from which they are deduced differ by a few seconds for low altitudes from the refraction tables to be found in another part of the volume.

They assume the barometer to be at 76 cm and the thermometer at 10° centigrade. It is pointed out that the table numbers should be reduced by 5 units for each degree of increase in temperature and by 16 units per centimeter below 76 cm.

We find in the *Additions à la Connaissance des Temps* for 1818, a French translation by DELAMBRE of a memoir by Stephen GROOMBRIDGE, of the Royal Society of London, extracted from the *Philosophical Transactions* for 1814, in which the author endeavoured to apply BRADLEY'S formula, by determining the coefficients x and y through star observations made by himself and Dr. BRINKLEY and by correcting refractions for temperature and pressure. In his remarks on this paper, DELAMBRE shows that the tables compiled by S. GROOMBRIDGE for mean refractions differ, but very slightly from those published by the *Connaissance des Temps*.

PIAZZI (1746-1826) [2] then F.W. BESSEL (1784-1846) [2] also compiled new refraction tables from their own observations and those made by BRADLEY from 1750 to 1762. BESSEL'S tables, published in 1818 and computed for a pressure of 751 mm 5 and a temperature of 9°3 centigrade were in very frequent use and are still, after altering the constants; although the law accepted by BESSEL for the decrement of temperature with altitude is not satisfactory (*Fundamenta astronomica pro anno* 1755, *auctore Frederico Wilhelmo Bessel,* Königsberg, 1818 and 1830).

BESSEL'S tables [11] use the formula :—

$$R = R_m \left(\frac{p}{p_0}\right)^A \left(\frac{1 + \beta t_0}{1 + \beta t}\right)^\lambda,$$

by means of values of $\lambda$ given with two places, and logarithms to 4 places :—

$$\log. \gamma = \log \frac{1 + \beta t_0}{1 + \beta t}$$

and

$$\log. B = A \log \frac{p}{p_0}$$

(11) *Tabulae Regiomontanae reductionum observationum astronomicarum, ab anno* 1750 *usque a.* 1850 *computatae*, Königsberg 1830.
See: Jordan EGGERT, Volume III, Book I, pp. 485-486 (65)-(68).

and of log a which permits to calculate:—

$$R_m = a \tan \zeta.$$

The apparent zenith distances vary from 0° to 80°, temperature from — 15° to + 35° centigrade, pressure from 680 to 780 mm.

An error of only a few hundredths of a second is committed if the unity be adopted as value for A.

If only an accuracy to one second of arc is sought, computation by logarithms can be dispensed with and BESSEL's tables can be arranged by giving the value of Rm for the different values of $\zeta$ and adding algebraically to Rm the value given by a table of correction whose arguments are Rm and t, then that given by a second table whose arguments are the value of Rm after applying the first correction and pressure. This is the form adopted by the *Nautisches Jahrbuch.*

The tables published in *Spherical and Practical Astronomy* by CHAUVENET, Vol. II, pp. 571-575 (Philadelphia, 1906) are also those of BESSEL. They give mean refractions for a pressure of 29.65 inches and a temperature of 48° 75 Fahrenheit and for apparent distances from 0° to 90°.

A first table gives logarithms to 5 places, of the coefficient a for zenith distances from 0° to 85°, as well as coefficients A and $\lambda$.

Let us put :—

$$\frac{p}{p_0} = BT,$$

and the tables give the logarithms to 5 places of B and T, the first in terms of the barometer reading between 725 and 785 mm., the second in terms of the internal thermometer reading [12] between ± 35° centigrade ; then the log $\gamma$ between the same readings of the external thermometer. (Pressures are also expressed in inches, temperatures in Fahrenheit and Réaumur degrees.) The barometer reading must be reduced to a temperature of 32° Fahrenheit.

The same tables give the values log a', A' and $\lambda'$, which correspond to quantities log a, A and $\lambda$ when the true zenith distance is known instead of the apparent distance.

Finally they give the values log a'', A'' and $\lambda''$ which correspond to true zenith distances and allow the corerction for refraction of micrometric measurements taken between two stars; without regard to their absolute positions.

For apparent zenith distances comprised between 85° and 90°, we find.

(12) BESSEL, after GROOMBRIDGE, distinguishes between the internal thermometer (attached to the telescope) and the external thermometer (outside the observation cupola). See: *Additions à la Connaissance des Temps,* 1818, p. 222 and 1821, pp. 348-351.

in the same work by CHAUVENET, Vol. I, p. 132, a table giving, to 5 places, the log of Rm, as well as A and λ to 4 places, to allow the computation of R by means of the formula :—

$$R = R_m \, (BT)^{A} \gamma.^{\lambda}$$

This table, based on the German astronomer ARGELANDER'S (1799-1875) [2] observations enables to calculate refraction values which can only be approximate for these great zenith distances.

Most *Nautical tables* contain refraction tables which are generally adapted to the relative precision of sextant measurements. We will also mention the Rev. James INMAN'S tables, London, 1913, which give (p. 110 and following) mean refractions in minutes and seconds for apparent altitudes from 0° to 90°, for a pressure of 30 inches and a temperature of 50° Fahrenheit, as well as correction tables which through an ingenious artifice, give only one correction according to the apparent altitude and the différence : barometer reading (in inches) less (—) $\frac{1}{18}$ of the temperature in Fahrenheit degrees.

We also find there, p. X, a small table giving for heights of the eye comprised between 0 and 120 feet, the correction to be made in the altitude, for differences in the temperature of the sea and air comprised between 1° and 40° Fahrenheit (+ correction when the sea is colder, — correction when it is warmer than air).

J.W. NORIE'S tables (London 1917), p. 190 give mean refractions in minutes and seconds for apparent altitudes from 0° to 90°, with the barometer at 29.6 inches and the thermometer at 50° Fahrenheit ; then a correction table for apparent altitudes from 2° to 90°, the thermometer being between 20° and 60° Fahrenheit and the barometer between 28.3 and 30.9 inches. The mean refraction table supplies the same values as those to be found in the Tables *Requisite to be used with the Nautical Ephemeris* for 1871.

The refraction tables of the *Almanaque Nautico* of San Fernando (Cadiz) are also based on BESSEL'S formula. Correction for temperature (from — 15° to + 35° centigrade) and for pressure (from 800 mm. to 720 mm.) are given in seconds with their sign. They must be added algebraically to mean refractions given in minutes and seconds for 10° centigrade and 760 mm. The English tables *Requisite to be used with the Nautical Ephemeris published* in London in 1871 give mean refractions in minutes and seconds for apparent altitudes varying from 0° to 89°. They seem to be in conformity with BRADLEY'S tables and are similar to those of the *Connaissance des Temps* for 1789.

In the English tables *to be used with the Nautical Almanach* for 1821, table XI gives the correction to be made to the apparent altitudes of the sun and stars. The first takes parallax into account, the second is solely a refraction table. This is also Dr BRADLEY'S table. In the first seven degrees however, some differences of 1" from the year 1781 table numbers, are to be found.

In the *Additions pour la Connaissance des Temps,* for 1819, DELAMBRE translated two memoirs by John BRINKLEY, Director of the Dublin Observatory, and ANDREWS, Professor of Astronomy, at the University of Dublin, which were read on 9th May 1814 and published in the Transaction of the Royal Irish Academy of 1815, Volume XII [13]. Their authors discussed therein the results of observations of stars made at Dublin Trinity College and the formulae proposed for calculating refraction, in particular the one put forward by LAPLACE. BRINKLEY submitted a new arrangement for refraction tables. A first table gives for each temperature a logarithm to 4 places to which must be added the logarithm of the barometric height (in inches) and that of tan $\zeta$. The sum of these 3 logarithms is the logarithm of a certain number of seconds of arc. From this number of seconds is deducted the number of seconds and tenths of seconds supplied by a second table in terms of $\zeta$ and barometric pressure. BRINKLEY has put his table in harmony with French tables and, in the curious following paragraph (page 407 of the English text) he makes a recommandation for uniformity which only seems possible when no more improvement can be expected for the correction of refraction:— " However as it (the result) does not differ considerably from the refraction of French tables, I see no objection to adopting these tables for the most delicate astronomical investigations.

" It is of much importance that the same tables of refraction should be used by astronomers and it will afford satisfaction to the author of this paper should it in any manner conduce to this desirable end. It cannot be doubted but that sooner or later the refractions as given by the French tables as far as 80°, or a very slight modification thereof will be generally used by astronomers. "

When the zenith distance is more than 80° the author considers (page 420) " that there is no reason to hope for an accurate and suitable method of computing refraction."

" It is not likely that the irregularities will be ever submitted to any law, and investigations respecting formulae for refractions, for zenith distances greater than about 80° may be considered more curious than useful. For less zenith distances the French tables as it has been the principal object of this paper to show, seem as accurate as can be desired."

---

(13) The Transactions of the Royal Irish Academy (Dublin 1815). Analytical investigations respecting Astronomical Refractions and the application thereof to the formation of Convenient Tables together with the results of observations of circumpolar stars, tending to illustrate the Theory of Refractions.

Various scientists who had endeavoured to work out a mathematical theory of astronomical refraction reached the same conclusion.

On September 5th 1836 (*Additions à la Connaissance des Temps* of 1839) Jean-Baptiste BIOT (1774-1862) [2] read before the " Académie des Sciences de Paris " an important *Mémoire sur les Réfractions Astronomiques.* He computes therein the action of the different atmospheric layers, which he assumes to be spherical, with due regard to pressure, temperature and humidity. The influence of the latter intermingles with a possible variation of the atmospheric chemical composition. He shows that it is sufficient, in practice, to know the state of the atmosphere at the observation spot. By integrating the differential equation, he proves that, if the apparent altitude is greater than 16°, the expression of refraction can be developed in a series of terms the first two of which are sufficient to supply the mean refraction without apreciable error. In this manner, he finds again BRADLEY'S formula and shows in which case it is inadequate. He points out that IVORY did not take into account the variation of gravity with altitude and that his law of atmospheric pressure variation p comes to writing:—

$$p = A\,\delta + B\,\delta^2$$

A and B denoting two constants to be determined by observations.

This law is a middle course, as suggested by LAPLACE, between that which assumes that pressure is proportional to $\delta$ (decrement of densities in geometrical progression) and that which assumes that pressure is proportional to $\delta^2$ (decrement of densities in arithmetical progression).

In this paper, BIOT gives his particular attention to our imperfect knowledge of the constitution of terrestrial atmosphere at different altitudes. Accordingly, he pursued his study in a *Mémoire sur la vraie constitution de l'Atmosphere terrestre, déduite de l'expérience, avec ses applications à la mesure des hauteurs par les observations barométriques et au calcul des réfractions* (" Paper on the true constitution of terrestrial atmosphere, derived from experience, with its application to the measuring of altitudes from barometric observations and to the computation of refractions ") read before the " Académie des Sciences " on April 2nd, 6th and 30th 1838 and published in the *Additions to the Connaissance des Temps* of 1841. He relies on GAY-LUSSAC's observations during his balloon ascent in Paris and those of HUMBOLDT in 1802, near the equator, from the plains situated at the foot of the Chimboraçao right up to its summit.

L. DE BALL published new refraction tables at Leipzig in 1906 in a very convenient form. Mean refractions are given for a temperature of 0° centigrade and a pressure of 760 mm. He takes the tension of water vapour into account.

In his *Lehrbuch der sphärischen Astronomie,* Leipzig 1912, he gives the following formula for mean refractions in seconds of arc for small values of zenith distances:—

$$R = [1,77888] \operatorname{tg} \zeta - [8,82337] \operatorname{tg}^3 \zeta,$$

the numbers between square brackets being logarithms of coefficients. For greater zenith distances other terms should be added to the development of R.

Following up R. RADAU's works and the tables which he published in Volumes XVI (1882), XVIII (1885) and XIX (1889) of the *Annales de l'Observatoire de Paris,* the *Connaissance des Temps* has been publishing since 1915, some refraction tables which are those of RADAU in an abridged form, except that they now adopt for the refraction constant [9] the value 60",154 an average resulting from modern observations, instead of 60"4455, as used by RADAU.

In these tables, mean refractions, called *Normal refractions* correspond to the following normal conditions:— Latitude 45°, altitude 0 m., temperature 0° centigrade, barometric pressure: 760 mm of mercury at 0°, tension of water vapor 6 mm.

First of all, barometric height H', observed at temperature t', should be reduced to that of air t, in latitude 45° (instead of $\varphi$) and at sea level (instead of h).

$$H = H' \; [1 - 0{,}00264 \cos 2\varphi \; (- 0{,}000\,000\,196\,h - 0{,}000\,163\,(t' - t)].$$

Normal refraction R is then corrected by algebraical application of the correction:—

$$R\,A\,\alpha\,\frac{1 + 0{,}00367\,t}{1 + Kt},$$

A, $\alpha$ and K are given in correction tables.

To refraction R' resulting therefrom is also applied correction:—

$$R'\,B\,\beta,$$

B and $\beta$ being given in correction tables.

The expressions of A and B are :—

$$A = -\frac{0{,}00383\,t}{1 + 0{,}00367\,t}, \qquad B = \frac{H}{760} - 1.$$

The factor:— $\dfrac{1 + 0{,}00367\,t}{1 + kt}$ differs from unity only for altitudes of less than 10°.

## III. — *DIP OF THE HORIZON*

The correction of observations made at sea, which are referred to the horizon necessitate also a knowledge of the dip of the horizon. The latter depends on the elevation *h* of the eye of the observer and the radius *a* of the earth.

Its expression is:—

$$D = \sqrt{(1 - K_0)\frac{2h}{a}},$$

$K_0$ being considered as a constant for which the value 0.16 is now generally adopted. The result being that if D is expressed in minutes of arc and h in meters, the formula becomes :—

$$D = 5', 6\sqrt{\frac{h}{10}}.$$

Let us observe that the dip of the horizon is not exactly the same in the various azimuths on account of the difference of curvature of the terrestrial ellipsoid. The flatness of the geoid could even be theoretically deduced from observation of the dip in two azimuths; but, in order to attain some accuracy, it should be necessary to observe at altitudes when the horizon being too distant would cease to be clear, which would render measurements illusory.

In volume II of the voyage made by order of the King in 1768 and 1769 to different parts of the world in order to test marine clocks, Paris 1773, p. 581, de FLEURIEU reproduced a table of the dip of the horizon for elevations of the eye in feet, inches and lines. It seems to be derived from the *Traité de Navigation* by BOUGUER (1759) it is more comprehensive but differs little from the following one.

For 1779 and some other years up to 1800 (inclusive) the *Connaissance des Temps* gave a table entitled :— *Table de l'inclinaison de l'horizon visuel avec l'horizon vrai.* It was computed by JEAURAT in terms of the height of the eye expressed in feet. It was not kept up in the *Connaissance des Temps* probably on account of its uselessness to observatories and its lack of precision. The dip of the horizon depends essentially on refraction in the lower atmospheric layers where it is practically impossible to foresee it with accuracy.

As, however, mariners are in constant need of knowing the dip of the horizon, the *Ephémérides Nautiques* give a table of the apparent dip of the horizon in minutes and tenths of minutes for elevations of the eye up to 50 meters. It is calculated by the above mentionned formula.

The *Tables requisite to be used with the Nautical Ephemeris,* for 1781, 1799 and 1802 also give a table of the dip of the horizon in minutes and seconds for elevations of the eye up to 100 feet. Those of 1821, for elevations of the eye up to 150 feet, with a few differences in the seconds.

We find in the *Connaissance des Temps* for 1827, p. 316, and in the second volume of the *Mémoires scientifiques d'Arago,* p. 662, the text of a paper on the *Dépression de l'horizon de la mer* in which ARAGO points out the dubiousness of the knowledge of the dip, a dubiousness which he estimates at $\pm$ 1'. He infers from a certain number of measurements that the dip given by tables is greater than the dip observed, in temperate climates, only when the temperature of air is higher than that of water.

He suggests the determination of the dip of the horizon from direct observations by measuring the angular distance from one point of the horizon to the point directly opposite. The excess of this distance over 180° is double the dip of the horizon, if atmospheric conditions are the same in both opposite directions.

This measurement can be taken either with a sextant, a surveying or BORDA Circle [14] specially arranged to that effect or a Borda and Wollaston dip measuring apparatus constructed for the purpose. More perfect modern apparatuses have been constructed since then and especially the Pulfrich [15] apparatus which seems to be satisfactory.

LAPLACE had dealt with the question of the dip of the horizon in the afore cited work p. 314, and suggested the above mentioned formula.

On 19th November 1838, BIOT read a paper before the " Académie des Sciences " of Paris :— *Sur la mesure théorique et expérimentale de la réfraction terrestre, avec son application à la détermination exacte des différences de niveau, d'après les observations des distances zénitales simples ou réciproques.* This paper was published in the *Additions à la Connaissance des Temps* of 1842, pp. 3 to 80 and completed in the volume of 1843, p. 67, it is a continuation of those by the same author, which we referred to and which are inserted in the *Additions aux Connaissances des Temps* for 1839 and 1841.

Refractions are studied therein on the basis of LAPLACE differential equation, in the case when the atmospheric layer passed through is fairly thin, the length of the trajectory comparatively short and the zenith distance somewhere about 90°. The formulae proposed by BIOT comprise coefficients which must be determined by means of pressures and temperatures at both ends of the trajectory of the ray of light. He gives several examples of calculation, based, as in the preceding papers, on observations made by GAY-LUSSAC and HUMBOLDT. He applies his formulae to the problem of the dip of the horizon for which it may be assumed that the trajectory of the ray of light is tangent to the sea's surface. Together with Claude-Louis MATHIEU (1783-1875) [2], BIOT made some observations on the dip of the horizon at Dunkirk (*Mémoires de l'Institut,* 1809) by placing himself successively at stations of diversified heights above the sea and whose absolute altitudes were measured directly. He found that the conditions of tangency of the ray of light trajectory and of the surface of the sea do not always obtain, particularly when the sea is much colder than the atmospheric layers situated

---

(14) Ingénieur hydrographe M.-P. ALLARD has shown that the measuring at an angle of about 180° with a surveying circle is always delicate and inaccurate.

See: *Annales Hydrographiques,* 1942, Paris.

See: *Hydrographic Review,* Vol. XX, p. 90.

(15) See: *Annalen der Hydrographie* 1904, pp. 84-85, 514-522, 1909; pp. 180-182, 277-279. See: *Hydrographic Review,* Vol. VIII, N° 1, p. 216.

somewhere above it. Still, his formulae can also be used in this case, provided that the temperature and pressure of the air be observed at the observation spot together with the sea temperature.

Many investigations have been made into depression. The conclusion reached was that the position of the horizon was sometimes appreciably shifted on account of weather conditions, apart from the mirage that alters it completely.

In particular, K. Koss [16] made many observations in 1887-88 during the expedition of the *Pola* to the Red Sea and the Mediterranean and in 1898-99 to Fort Verudella.

From these observations the following formula giving D in minutes of arc has been drawn :—

$$D = 1{,}82\sqrt{h} - 0{,}003\ h - 0{,}41\ \Delta;$$

in which h denotes the elevation of the eye in meters, and $\Delta$ the difference in degrees centigrade between the air temperature at the height of the eye and the water temperature. It has been found since that this formula gives too great a dip in a very light breeze or in the absence of wind; hot air can then gather in the upper layers.

On the initiative of Admiral YONEMURA, observations on the dip of the horizon were made in the *Kurosio* area. When making his report, Hydrographic engineer S. OGURA, of the Japanese Navy; published in the *Suiro Yohô* two very comprehensive and interesting articles which were translated and reproduced in the *Hydrographic Review,* Vol. VIII, N° 1, pp. 101-118. He reviewed therein the principal tables drawn up recently for the dip of the horizon, which makes it unnecessary to refer to them here. He considered that, under ordinary conditions, if the temperatures of air and water are not very different and if corrections are made for temperature and pressure, it is possible to record altitudes of heavenly bodies at sea between 0 and 5° without the mean error of the result exceeding one minute of arc. These articles were followed by an article by Commander A. SONE, who was instructed by Admiral YONEMURA to make observations in the same area.

In an altogether different area, the Atlantic Ocean, on the " Georges Bank ", L.S. HUBBARD, Hydrographic and Geodetic Engineer of the Coast and Geodetic Survey of the U.S.A., made some observations on the dip of the horizon in 1931. He reached the conclusion that a constant term :— — 0.40 should be added to Koss's formula. During a previous mission in the same area some observations on altitudes of stars had induced him to adopt the same constant but with $\Delta$ bearing a smaller coefficient [17] : 0,30.

(16) See: *Annalen der Hydrographie* 1901, pp. 162-167; 1903, pp. 533-554; 1904, p. 177; 1905, pp. 158-170; 1909, pp. 306-324; 1910, pp. 120-132, 160-171.
See: *Hydrographic Review,* Vol. VIII, N° 1, pp. 209-217; Vol. XI, N° 2, pp. 118-122.
(17) See: *Bulletin of the Association of Field Engineers,* June 1932, p. 58. — See: *Hydrographic Review,* Vol. XI, N° 1, pp. 115-16.

Ernest MAJO made many observations on the dip of the horizon in the Bay of Naples in 1926. He obtained refraction coefficients which differed according to the season, temperature, pressure and humidity [18].

The *Suiro Yôhô 12* (1933) when submitting a summary of the observations on the dip of the horizon effected by the Japanese Navy, gave the observation results obtained on the *Kasuga* [19] in the seas of China in 1932. The author inferred therefrom the formula :—

$$D = 1',776 \sqrt{h} - 0',08\, \Delta .$$

In the Review *Der Seewart* [20], Hamburg, 2 (1934), H.C. FREIESLEBEN reports on several investigations made into the dip of the horizon, and in particular on those made by Rear-Admiral Dr CONRAD [21] on board the *Hunte*.

He points out that in a heavy sea the horizon is less clear, the height of the eye variable and consequently imperfectly known; the result is generally a heightening of the horizon.

CONRAD adopts the formula :—

$$D = 1',92 \sqrt{h} - 0',35\, \Delta .$$

In the *Annalen der Hydrographie,* of May 15th 1934, pp. 208-214, Dr Hubert MICHLER [22] reports on the measurements made with a Pulfrich apparatus on the steamer *Arucas* in 1933. He observed in various azimuths and noted the influence of swell, in accordance with previous observations:— HESSEN, THORADE [23] and CONRAD.

The correction of the dip for the barometric pressure would be :—

$$+ 0'03\ (760\ \text{mm} - \text{barometric reading}).$$

It is generally negligible.

The one depending on the difference of temperature between air and water appeared to him to have been very small; it seems that its influence grows as the elevation of the eye diminishes. The coefficients of $\Delta$ proposed by those who made measurements vary between 0.'04 (HESSEN) and 0.'57 (BREITFUSS). The effect of the difference of temperature in air and in water seems to depend to a great extent on the more or less still state of the atmosphere; it may be considerable in calm weather or light wind. The *Meteor,* during her 1925-27 campaign, in the tropics, experienced variations of as much as 3'.

(18) See: *Rivista Marittima,* June 1927. — See: *Hydrographic Review,* Vol. V, N° 1, page 272.
(19) See: *Hydrographic Review,* Vol. XI, N° 2, page 171.
(20) See: *Hydrographic Review,* Vol. XI, N° 2, pp. 118-123.
(21) *Astronomische Ortesbestimmung und Kimmtiefenmessung auf See,* Berlin 1933.
(22) See translation in the *Hydrographic Review,* Vol. XII, N° 1, pp. 127-134.
(23) *Kimmtiefenmessungen an Bord von Schiffen der Reichsmarine,* Berlin 1930.

The coefficient of $\sqrt{h}$ in the dip formula is more accurately determined, it varies from 1.'28 (Kohlschütter) to 1.'92 (Conrad) [24].

## IV. — *GEODETIC REFRACTION* [25]

In Tome I of the voyage made in 1771 and 1772 (Paris 1778) (mentioned above) the author, in connection with the peak of Teneriffe (pages 124-125), refers to a method of correcting heights of mountains regarding the effect of refraction, viz. geodetic refraction. This method, which is approximate, is based on the use of astronomical refraction tables and on a supposed approximately known height of the mountain.

The mathematical problem of geodetic refraction between two land points was dealt with by LAPLACE in his *Mécanique Céleste,* Vol. IV, Book X, p. 310, who adopted the same hypothesis on the Constitution of the atmosphere as for the computation of astronomical refraction.

BIOT's paper (*Addition à la Connaissance des Temps* for 1842) mentioned above, supplementing LAPLACE's study makes an important contribution in the solution of the problem. BIOT, on the basis of available observations, shows that the trajectory of the ray of light is exactly a conic section and that the ratio between atmospheric densities and pressures is of parabolic form, which allows easy integrations in function of weather conditions. It is customary to use the simplified formula giving the difference of height h between two land points whose horizontal distance is d and between which a zenith distance $\zeta$ has been observed :—

$$h = d \cot. \zeta + (0{,}5 - n) \frac{d^2}{R},$$

n denoting the refraction coefficient and R the earth's curvature radius in this azimuth. n varies with the different states of the atmosphere, with the time of the day and climate.

In order to obtain accurate values of h, reciprocal simultaneous observations for the determination of n must be made.

We noticed that the hypothesis put forward by E. ESCLANGON on the distribution of air densities would represent particularly well the state of the lower atmospheric layers (under normal conditions); it must therefore be suitable for calculating geodetic refraction. The formula of refraction becomes then [26]

$$R = 2\,\mu\, n_0\, r_0 \sin \zeta_0 \left[ (n_0 - 1)\,\psi(x) - (n - 1)\,\psi\left(\sqrt{x^2 - \log_e \frac{n_0 - 1}{n - 1}}\right)\right];$$

$n_0$ and n denoting the indices of refraction at the two points.

(24) See also: *Annalen der Hydrographic* 1912, p. 34 to 45, p. 187 to 192; 1927, p. 219 to 227; 1935, p. 340 to 349.

(25) See the Bibliography of Astronomical and Geodetic refraction up to 1926 in *Jordan-Eggert Zweiter Band, Zweiter Halbband,* Stuttgart, 1933, p. 173 to 175.

(26) See: *Comptes Rendus de l'Académie des Sciences,* 18th January 1934, pp. 137-139.

Reciprocal simultaneous sightings may be used for checking.

Let h be the difference of height in meters, $\tau$ the decrement of temperature in degrees centigrade per kilometer (a value of $4.^{\circ}9$ is generally assumed for $\tau$ ) we have :—

$$\frac{n - 1}{n_0 - 1} = \left(1 - \frac{h\,\tau\,10^{-3}}{t_0 + 273}\right)^{\frac{273}{8\,\tau} - 1},$$

$$h = 1000\,\frac{t_0 + 273}{\tau}\left[1 - \left(\frac{n - 1}{n_0 - 1}\right)^{1 - \frac{273}{8\,\tau}}\right]$$