CHART OF EQUAL AZIMUT

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Charts of Equal Azimuth are plane representations of the system of parallels and meridians of the terrestrial sphere in which all the straight lines (∞^2) of the figure plane are representations of curves of equal azimuth in this system. If the coordinates of the sphere grid be λ for longitude and φ for latitude, the equation for a curve of equal azimuth is :

$$(1) \quad \cot a + \sin \varphi \cot (\lambda - \lambda_0) - \tan \delta \cos \varphi \operatorname{cosec} (\lambda - \lambda_0) = 0$$

where φ and λ are the current coordinates of the point of the curve $\lambda = \lambda_0$ and $\varphi = \delta$ the coordinates of the terminal point of the curve of equal azimuth, in the constant azimuth of (λ, φ) towards $(\lambda_0\ \delta)$ the equation (1) must be transformed into the equation for a plane straight line.

$$(2) \quad y - x \tan \beta - c = 0$$

so that *y* and *x* be functions of φ and λ, while β and *c* be suitably contingent on the constants of equation (1). The latter presenting 3 parameters : a, δ, and λ, there exists on the sphere a triple diversity (∞^2) a complex of equal azimuth.

As the plane comprises only ∞^2 of straight lines according to the two parameters β and *c*, one can, however, from this complex choose only one ∞^2 of curves of equal azimuth and represent them by straight lines on a chart. As we generally wish to avail ourselves of all the values of a for the azimuths, the choice is fixed by a relationship between λ_0 and δ, which means that the terminal points ∞^1 of these ∞^2 curves of equal azimuth be on a curve of the sphere. It will be practical to choose for this curve a meridian which shall be taken as initial number. λ_0 will then be nil in the equation (1). The advantage of this procedure is that any point of the earth may be taken as terminal point. It will only be necessary in this case to relinquish the representation of continents and be content with a schematic chart, or diagram of azimuth with a system of meridians and parallels on which the terminal point is plotted every time on the meridian $\lambda = 0$ and where λ is considered as the difference in longitude in relation to the longitude of the terminal point.

The equation (1) simplified by making $\lambda = 0$ becomes

$$(1\,a) \quad \cot a + \sin \varphi \cot \lambda - \tan \delta \cos \varphi \operatorname{cosec} \lambda = 0$$

it may be transformed in 3 different ways into equation (2)

$$y - x \tan \beta - c = 0$$

at the same time as it is transformed for a useful division so that either its first or second or third term contains the constant *c*.

A) Let us first consider the first term cot a as the constant — c ; the equation of transformation will then be :

$$(3\,a) \quad y = \sin \varphi \cot \lambda, \qquad x = \cos \varphi \operatorname{cosec} \lambda$$

and

$$\tan \beta = \tan \delta$$

and by eliminating from (3*a*), once φ, another time λ, we find, to represent the meridians λ, the ellipses,

$$\frac{x^2}{\operatorname{cosec}^2 \lambda} + \frac{y^2}{\cot^2 \lambda} = 1$$

and for the parallels, the hyperbolae

$$\frac{x^2}{\cos^2 \varphi} - \frac{y^2}{\sin^2 \varphi} = 1$$

G. Prüfer suggested this system of equal azimuth in the *Annalen der Hydrographie* of 1941, p. 331. The grid for the whole of the earth covers the plane twice, in the following way :—

The two poles are represented by the full y axis which bears a scale for λ. On this, the origin of the coordiantes corresponds to $\lambda = \pm 90°$ and the meridians $\lambda = \pm 90°$ finally join the x axis as far as the equator which is represented by the prolongation of the x axis. The zero meridian $\lambda = 0$ (or $\lambda = 180°$) is a straight line of the infinite of the plane. The representation is not conformal although the meridians and parallels form a system of orthogonal ellipses and hyperbolae. The fact that the points of origin are at ∞ on the meridian $\lambda_0 = 0$ makes rather peculiar the reading δ of the latitudes of the points of origin. In order to place the straight line of equal azimuth of a ship's position (φ, λ) towards a point of origin of latitude δ rejected to infinity, one must draw the straight line from the point (φ, λ) so that it meets the x axis on the angle $\beta = \delta$. The azimuth is then read off on the cot division of the y axis.

Prüfer has already pointed out that the grid has the same aspect as Weir's well-known azimuth diagram, with this difference that Prüfer's meridian ellipses $\lambda = 90°$, from $\lambda = 0$, correspond to the latitude ellipses $\varphi = 0$ up to $\varphi = 90°$ and that Prüfer latitude hyperbolae $\varphi = 0$ up to $\varphi = 90°$ correspond to the meridian hyperbolae $\lambda = 0$ up to $\lambda = 90°$. In fact, Prüfer's system is already known if his ellipses are considered as meridians and his hyperbolae as parallels. It is the conformal representation given by Fiorini [1] in which parallels and meridians are inverted, as pointed out by Prüfer, so that the origin of co-ordinates be the meeting point of the equator $\varphi = 0$ with the meridian $\lambda = 0$. Being thus numbered, the chart is conformal but it is no longer a chart of equal azimuth curves.

B) If we now set out the equation (1*a*) so that its second term be the constant *c*, we must multiply it by $\sin \lambda \sec \varphi$. The equation will then take the form (1 *b*) $\cot a \sin \lambda \sec \varphi - \tan \delta + \tan \varphi \cos \lambda = 0$ it may be transformed into the equation $y - x \tan \beta - c = 0$ by the equations of transformation (3 *b*) $y = \tan \varphi \cos \lambda$, $x = \sin \lambda \sec \varphi$ where $\tan \beta = -\cot a$ and $c = \tan \delta$. By eliminating in (3*b*) once φ, another time λ, we find, for representing the meridians λ the hyperbolae $\frac{y^2}{\sin^2 \lambda} - \frac{x^2}{\cos^2 \lambda} = 1$ and, for representing the parallels φ the ellipses $\frac{x^2}{\tan^2 \varphi} + \frac{y^2}{\sec^2 \varphi} = 1$.

The grid is that of Weir's well-known azimuth diagram. Prüfer called it Littrow-Lambert's system. It is only right that this projection should be designated by the name of Littrow who suggested the equation (3 *b*) of representation for a conformal projection in his "Chorography" of 1833, p. 142, but who realized their signification to such a small extent that he pointed out that the representations of parallels as well as of meridians were hyperbolae.

On the contrary, Lambert had no idea that this system of homofocal hyperbolae and ellipses could give a conformal representation of the earth parallels and meridians. For the first time, in 1918, Wedemeyer showed that in this system, the great and small circles of a certain equatorial diameter formed a system of Lambert circles. I reproduced and discussed the first cartographie representation in this projection for part of the earth in 1905 in the *Annalen der Hydrographie* and for a planisphere in the form of a Riemann double plane in *Petermanns Mitteilungen* (1911). This planisphere was also reprinted in *Petermanns Mitteilungen Ergänzungs heft* 221 (1935) in tables VIa and VIb. I have certainly no wish to belittle Lambert's effort whose remarkable work in cartography, in German, French and

(1) For further information concerning Fiorini's projection and its relationship with Littrow's projection, see my paper « Ebene Kugelbilda » in *Petermans Mitteilungen, Ergânzungs heft 221*, 1935, pp. 64-66.

English I spoke highly of in 1931 (1), but it should be sufficient to name this cartographic projection after Littrow who was at least the first to furnish equations of transformation.

C) Let us now set out the equation (1*a*) so that its third term becomes the constant *c* ; this will be done by dividing by cot a sin φ cot λ.

We then obtain the form

$$(1\,c)\quad \tan\lambda\cos\varphi - \tan\delta\tan a\cot\varphi\sec\lambda + \tan a = 0$$

and the equation of transformation

$$(3\,c)\quad y = \tan\lambda\ \mathrm{cosec}\ \varphi, \qquad x = \cot\varphi\sec\lambda,$$

in which : $c = -\tan a$ and $\tan\beta = \tan\delta\tan a$.

If in (3 *c*) φ is first eliminated, then λ we find for representing the meridians λ, the hyperbolae :

$$\frac{x^2}{\sec^2\lambda} - \frac{y^2}{\tan^2\lambda} = 1,$$

and for representing the parallels φ, the hyperbolae :

$$\frac{x^2}{\cot^2\varphi} - \frac{y^2}{\mathrm{cosec}^2\varphi} = 1.$$

The representation of the whole earth also covers, for this chart of equal azimuth twice the plane, like for the other two. As in the case of Prüfer's system, the full *y* axis is the representation of the two poles, but in such a way that the origin P of the coordinates lies at the extremity of the terminal meridian $\lambda = 0$ (and on the other side of the meridian $\lambda = 180°$), which is repeated on the *x* axis, P.M., on the cot φ scale. We may also draw a straight line of equal azimuth from the ship's position (φ, λ) to the origin $(\delta, \lambda = 0)$ by joining these two points, the azimuth will be read off on the scale PR of the *y* axis tangents.

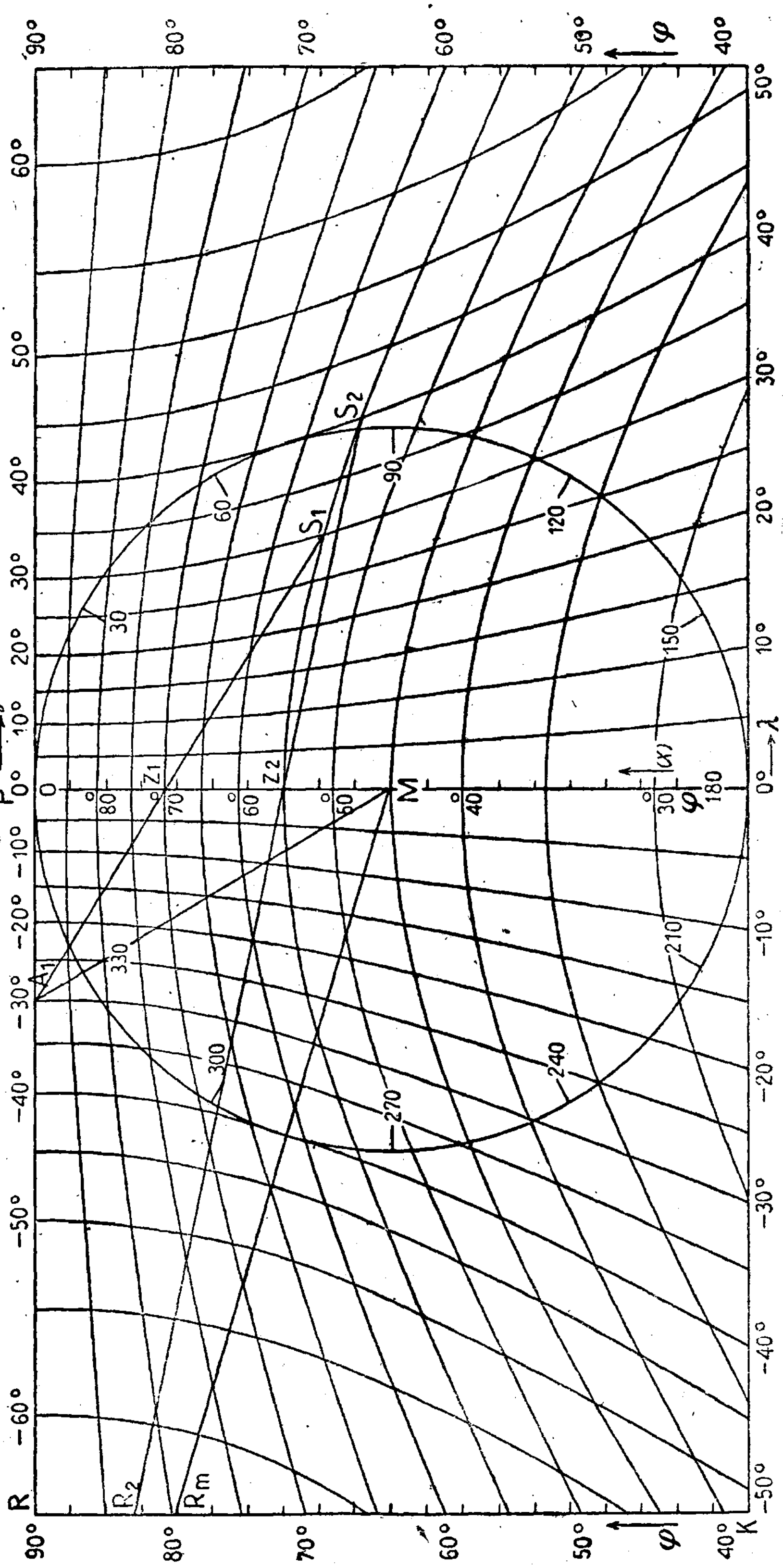
Figure 1 gives a diagram of half ($\lambda = -90°$ to $\lambda = +90°$ passing through $\lambda = 0$) a similar chart of equal azimuth for regions of high latitudes.

The other half ($\lambda = 90°$ to $\lambda = 270°$ passing through $\lambda = 180°$) would be the mirror reflection of the other side of the *x* axis. The straight line resects the *y* axis at the point A1 $\lambda = -30°$). The azimuth of the straight line of equal azimuth is then $a = -30°$ or $+330°$. Instead of reading off the azimuth on the *y* axis on the unequal scale of the tangents, it may also be read with a regular division on the circle having for its centre the point M ($\lambda = 0$, $\varphi = 45°$) and by drawing the straight line MA. As PA1 = tan *a* and PM = 1, the angle $\widehat{PMA} = a$. It is this reading off the circle (the divisions are indicated from 30 to 30 degrees) which must be employed when the straight line of azimuth no longer meets the portion of the *y* axis which is represented. The straight line must then be drawn from M to the inaccessible point, as shown on the figure for the straight line running from the ship's position S2 ($\varphi = 55°, \lambda = 40$) to the origin Z2 ($\varphi = 55°, \lambda = 0$). The straight line S2Z2 meeting the curve RK at the point R2, the point R_M of the frame is found through the relation :

$$RR_M = \frac{RR_2}{PZ_2}\ PM.$$

The straight line MR_M then gives the desired azimuth on the circle. In order to make the operation easier, there is on the frames on both sides of the *x* axis a division which represents 100 unities for the length PM (it is marked at every 10 units on the diagram). It is quite sufficient to read only the lengths RR2 and PZ2 off these divisions.

(2) *Zeitschrift d. ges. f. Erdkunde*, Berlin 1931 and *Hydrographic Review*, Monaco, vol. VIII, n° 1, 1931.



φ
λ (y)
P
R
K
M
S1
S2
Z1
Z2
A1
R2
Rm
(x)
0° → λ
30
60
90
120
150
180
210
240
270
300
330

FIG. 1

Their quotient $q = \frac{RR2}{PZ2}$ furnishes subsequently $RR_M = 100$ q. The straight line MR_M may then be drawn and permits reading the azimuth $-71°,5 = +288°,5$ off the circle.

The curve of equal azimuth may be given either by the ship's position S1 and the point of origin Z1 or by the ship's position (point S1) and the azimuth (point A1), or thirdly by the point of origin Z1 and the azimuth (point A1). In every case the straight line may be drawn directly by means of two points and the desired value will be read off at the third point so long as the point A is accessible. When this is not the case, one of the points on the frame such as R2 or R_M is always available, the other being easily obtained by means of the relation $(RR_M) \times (PZ2) = (RR2)\ 100$. Such a chart of equal azimuth for high latitudes should be preferred to that recommanded by Prüfer in *Seewart* 1941, page 46 and to the one which I had supplied myself in 1931 (1) to the dirigible balloon "Graf Zeppelin" for the flight over the Arctic. It is easier to work with curves of equal azimuth in the form of straight lines rather than of circles. It is also better in some respects than the grid which is now suggested by Prüfer in the *Annalen der Hydrographie* 1941, p. 331 (see paragraph A) because it is much clearer to have before one's eyes the point of origin of the curves of equal azimuth on zero meridian than to assume it to be at an infinite distance.

The drawback of my chart in which the difference of longitude λ between the ship's position and that of the origin goes on diminishing and for a low latitude, the angle at which the meridians and parallels intersect departs more and more from the right angle, makes it unserviceable for high values of λ and of $(90 - \varphi)$ which are therefore to be avoided. Still, these two charts of equal azimuth, that of Prüfer for high values of λ and mine for low values of λ, are fortunately complementary to one another for navigation in high latitudes.

D) There is another possibility of choosing, from amongst the three methods, a second one for representing the curves of equal azimuth by straight lines when taking for all the points of origin all the curves of a same azimuthal angle.

In the basic equation (1), *a* is no longer a parameter but a constant for the whole chart ; λ_0 is no longer a constant but a parameter which may assume any value. We then have no longer a mere grid of meridians and parallels but a regular chart.

We multiply the equation (1) by $\sin(\lambda - \lambda_0) \sec\varphi$ and we replace $\cos(\lambda - \lambda_0)$ by $\cos\lambda \cos\lambda_0 + \sin\lambda \sin\lambda_0$ and $\sin(\lambda - \lambda_0)$ by $\sin\lambda \cos\lambda_0 - \sin\lambda_0 \cos\lambda$ and we obtain :

$$(1\ d)\quad (\cot a \sin\lambda + \cos\lambda \sin\varphi) \sec\varphi - (\cot a \cos\lambda - \sin\varphi \sin\lambda) \sec\varphi \tan\lambda_0 - \tan\delta \sec\lambda_0 = 0.$$

This equation will be identified with the equation for the straight lines :

$$(2)\quad y - x \tan\beta - c = 0$$

by the equations :

$$(3\ d)\quad \begin{aligned} y &= \sec\varphi (\cot a \sin\lambda + \cos\lambda \sin\varphi), & \beta &= \lambda_0 \\ x &= \sec\varphi (\cot a \sin\lambda - \sin\lambda \sin\varphi), & c &= \tan\delta \sec\lambda_0 \end{aligned}$$

By eliminating λ, we obtain the equation for circles of latitude :

$$(4\ d)\quad x^2 + y^2 = \sec^2\varphi \cot^2 a + \tan^2\varphi.$$

In order to eliminate φ, we first form $(x \cos\lambda + y \sin\lambda) = \cot a \sec\varphi$ and we shall draw $\sec\varphi$ from the equation (4 *d*) by $\sec^2\varphi = \sin^2 a\ (x^2 + y^2 + 1)$. The equations for the meridians will then be :

$$(x \cos\lambda + y \sin\lambda)^2 = \cos^2 a\ (x^2 + y^2 + 1)$$

The parallels are therefore all concentric circles and the pole a straight line to infinity.

When $a = 90°$, the equation $x^2 + y^2 = \tan^2\varphi$ of the equator shows that the equation

(1) *Sonderheft II der Marine-Luftflotten-Rundschau* 1932, page 21.

becomes reduced to the point $x = y = 0$. The meridians become the straight lines $x = -y \tan\lambda$, passing through the point of origin and each curve of equal azimuth with the point of origin $(\lambda=\lambda_0, \varphi=\delta)$ will be the straight line meeting the circle $x^2 + y^2 = \tan^2\delta$ at the point $(\lambda=\lambda_0, \varphi=\delta)$ As for other values of a, the equator is a finished circle of diameter cot a, the meridians are hyperbolae.

In the particular case of $a = 45°$, the circles of latitude have the diameter $\sec\varphi \sqrt{1+\sin^2\varphi}$ and the equator the diameter 1 while the meridian equation takes the form :

$$(x^2 - y^2) \cos 2\lambda + 2xy \sin 2\lambda = 1.$$

It furnishes the following hyperbolae :

for $\lambda = 0$	$x^2 - y^2 = 1$
— $\lambda = 45°$	$2xy = 1$
— $\lambda = 90°$	$y^2 - x^2 = 1$

The preceding formulae of charts of equal azimuth are not the only possible ones ; any collinear transformation of one of them will furnish a chart of equal azimuth, as through such transformations every straight line is transformed into a straight line. But the types so obtained will generally be inconvenient on account of occuring distortions, so that only the preceding formulae should be taken into consideration as azimuth charts with straight lines representing the curves of equal azimuth.