# PRECISE ASTRONOMIC POSITIONS FROM PROJECTED STAR POSITIONS ANALYTICALLY PROCESSED

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Introduction: Precise astronomic positions are normally arrived at from star observation data by computation procedures based principally — and logically — on the spherical trigonometry, considering that such positions involve determination of the angular values of spherical arcs and/or angles of one sort or another.

There appears at first glance to be scant reason to consider a different approach to the computation problem. For one thing, the resources of the spherical trogonometry have certainly not been found inadequate. Be that as it may, these articles undertake to demonstrate that if the resources of the spherical trigonometry are adequate, so also are the resources of the analytic, -and of the *plane* analytic geometry at that.

Certain background details of the origin of the new (so far as is known to this writer and to many consultees) method may be of general interest :

The method was first conceived of for processing data observed with the prismatic astrolabe. Each star observed with such an instrument (or with its first cousin, a pendulum-type astrolabe) is on or very close to the circumference of a small circle on the celestial sphere, this following, of course, from the principle of the instrument.

One of the many useful and interesting things about this small circle on the celestial sphere is that its stereographic projection is also a circle. And inasmuch as a stereographic projection is always onto some plane or other, the circle can be expressed mathematically thus :

$$x^2 + y^2 + Dx + Ey + F = O$$

if and when appropriate rectangular axes are selected, and if and when x, y coordinates relating to various points on its circumference are arrived at. These activities are carried out in Part I of these articles.

The promising results obtained in the test computation described in Part I led to consideration of other applications of the projection and analytical processing for determination of precise astronomic positions. The following possibilities soon evolved :

a. A two-star method employing intersecting equal-altitude circles.

b. A method employing successive short-interval observations of a single star.

c. A method employing simultaneous (or near-simultaneous) observations of a number of stars.

d. A method employing constant-azimuth star sights.

e. Two emergency position-finding methods not involving measurement of altitudes and/or azimuths. The term emergency refers to the situations of individuals, particularly downed fliers, who are restricted so far as equipment is concerned, and who may conceivably be without formal training in position-finding techniques.

These possibilities are described in Part 2 of these articles.

The opinions or assertions contained in these articles are the private ones of the writer and are not to be construed as official or reflecting the views of the Navy Department or the Naval Establishment. The writer wishes to express his appreciation of much valuable advice and comment, and for certain computation simplifications (in Part 2 now in preparation), to Mr Julius L. Speert, Mr. Alfred D. Sollins, and to Mr. Herbert L. Norman, all of whom are in the U.S. government service and are of recognized professional standing.

## PART I

# THE EMPIRIC EQUATION METHOD FOR PROCESSING ASTROLABE DATA FOR PRECISE ASTRONOMIC POSITIONS

The observation and computation procedures for determining precise astronomic positions from star data obtained with an astrolabe, either prismatic type or pendulum type, have been described authoritatively by a number of writers (1). The observation procedures are much the same, consisting as they do in noting the exact instants of time when known stars attain an unvarying altitude above the horizon, this altitude being fixed by the arrangement and/or type of the optical components of the instrument.

The computation procedures in most general use also appear to be more-or-less standard consisting as they do in the solution of astronomical triangles, a graphic plot and a final least squares adjustment to eliminate the guesswork and personal judgment factors almost inevitable in any graphic plot or solution. The standard procedure requires that a position be assumed for computation purposes, and also the calculation of theoretical or computed zenith distances which are compared with those observed with the instrument and therefore considered as true.

An entirely different method of processing astrolabe data has been developed by this writer, and tested using actual field data. This method does not require the solution of astronomical triangles, involves no assumptions regarding the position of the observation site for computation purposes, does not necessarily involve a graphic plot, and employs a least squares adjustment procedure that has been acknowledged as less complicated than that used in the so-called standard procedure. The test computation results support a statement that maximum accuracy in the final position can be expected with the method.

The method considers the observation circle (the small circle on the celestial sphere described by the line of sight of the observing instrument as it swings through 360° of azimuth about its vertical axis, and determined by the star positions on it) as being stereographically projected onto the plane of the celestial equator. Figure 1 illustrates the central idea. This projection is mathematically legitimate in that the star positions that determine the unique observation circle in space retain their identities after projection as points determining the same unique circle, — this following, of course, from the well-known principles of the stereographic projection.

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(1) « Geographic Positions by the Prismatic Astrolabe » : Lt. J.L. Speert, U.S. Force, Washington, D.C., 1943.

« The Prismatic Astrolabe »: Messrs. Claude and Driencourt: « The Hydrographic Review » (International Hydrographic Bureau), Vol. VII, No. 1, May 1930.

« Use of the Prismatic Astrolabe for Astronomic Positions by the Hydrographic Office » : L.M. Samuels : American Geophysical Union Transactions of 1942.

« Technical Instructions for 60° Pendulum Astrolabe » : Engineer School, Fort Belvoir, Va., January 1945.

One result of this projection useful for computation purposes is that projected star positions can then be expressed in the familiar x, y coordinates of the plane, or 2-dimensional, analytic geometry, whereas before projection they are expressed in the spherical coordinates (hour angle, declination) of the 3-dimensional sphere.

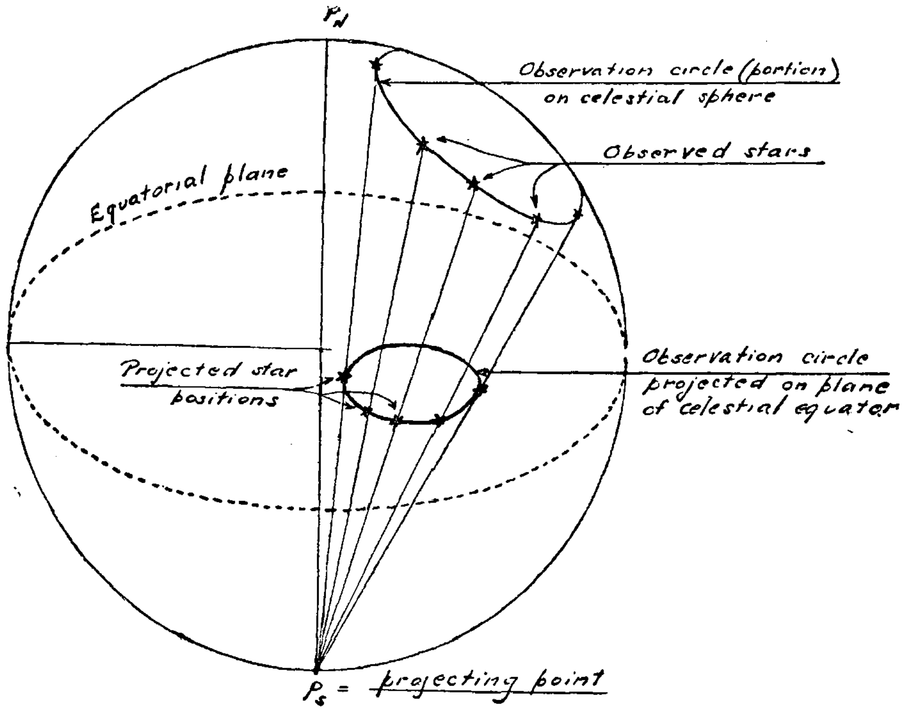



Figure 1
The projection principle

This article undertakes to demonstrate that this stereographic projection leads to a simpler computation process. An analytical description of the method is first presented, and later a description of actual computation details. Two reasons are given for this handling; first, an analytical description free of process details spotlights the purely mathematical framework of the method as a sort of target for discussion and criticism; secondly, the systematizing and/or streamlining of the mechanics of the computation have probably concealed much of the framework just mentioned.

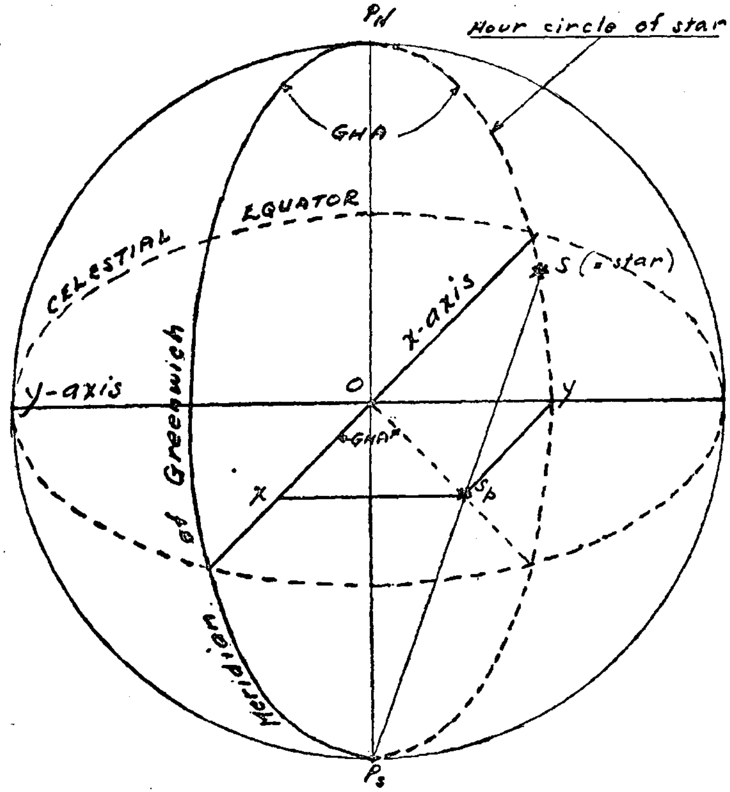
Here, then, is an analytical description of the method :

1. Determine the x, y coordinates of each stereographically projected star position by the use of the standard projection formulas :

$$x = \tan 1/2\ (90 \backsim D) \cos GHA$$
$$y = \tan 1/2\ (90 \backsim D) \sin GHA$$

Figure 2 shows how these formulas are arrived at.

(An observer in the northern hemisphere would take the south celestial pole as the point of projection. In such case, $(90 \backsim D)$ becomes $(90 - D)$ for stars of north declination, and $(90 + D)$ for stars of south declination. An obser-




Arc $P_N S = (90° - \text{Dec.} *)$

$\angle SP_s P_N = \frac{1}{2}(90° - \text{Dec} *)$

$OS_p = \tan \frac{1}{2}(90° - \text{Dec} *)$, when the radius of celestial sphere = 1

$\frac{OX}{OS_p} = \cos GHA$ , or

$OX = \tan \frac{1}{2}(90° - \text{Dec.} *) \cos GHA$

$\frac{OY}{OS_p} = \sin GHA$ , or

$OY = \tan \frac{1}{2}(90° - \text{Dec} *) \sin GHA$

Thus, the projected star position has as its coordinates in the plane of the celestial equator:

$$\begin{cases} x = \tan \frac{1}{2}(90° - D) \cos GHA \\ y = \tan \frac{1}{2}(90° - D) \sin GHA \end{cases}$$

Figure 2
Derivation of Projection Formulas

ver in the southern hemisphere of the earth would reverse these conventions. GHA's are taken as positive or negative according as they are measured westward or eastward of the reference meridian).

2. Substitute each pair of x, y values so obtained in a general equation of a circle :

$$x^2 + y^2 + Dx + Ey + F = O$$

(Thus, if n stars were observed, this step yields a system of n equations in 3 unknowns, D, E, and F).

The least squares solution of the system arrived at by step 2 is as follows:

3. Multiply each equation of the system by the coefficient of D in it.

4. Multiply each equation of the system by the coefficient of E in it.

5. Add individually the three sets of equations arrived at by steps 2, 3 and 4. The result will be three simultaneous equations, which can be solved by an elementary algebraic procedure. Replacing D, E, and F of the general equation with their numerical values just determined yields an equation which is a mathematical statement of the unique circle in space observed with the astrolabe; it is the best and truest statement possible from the observed data by reason of the proven least squares adjustment processing. (It is called an *empiric equation* in the sense that it was derived from observational data).

6. The empiric equation arrived at in step 5 will be of the form :

$$x^2 + y^2 + A + By + C = O$$

in which A, B, and C are, of course numerical quantities. The equation is related to the observer's position thus :

$$\text{Latitude of observer} = 90^\circ - \text{arc tan} \frac{\sqrt{A^2 + B^2}}{1 - C}$$

$$\text{Longitude of observer} = \text{arc tan} \frac{B}{A}$$

(The latitude will be named north or south according to the hemispheric assumption of step 1; the longitude will be named west or east according as quantity $\frac{B}{A}$ is positive or negative, respectively).

With the analytical description now before us, we can take up a description of the mechanics of the computation process;

a. A Form 1 card, illustrated in figure 3, is completed for each star of the observation set. The top line of the card and the left-hand portion down to and including the line « tan. 1.17387687 » is filled in from the observation data, star data tabulations, and tables of natural functions to 8 decimal places.

(This much of the computation process is more-or-less the same with both the method under description and the standard method. Estimates of computation time for the method under description do not apply to this stage).

b. Determine x and y values from the formulas on lines 1 and 2 of the right-hand portion of each Form 1 card. Insert these values on lines 1 and 2 and

Star: #234 i Cet. Az.: 130° 31' Card No. 4

| | | | |
|---|---|---|---|
| Chro : | 2 h. 46 m. | 52.75 s. | |
| C.C. : | (+) | 11 | 57.65 |
| GST : | 2 | 58 | 50.40 |
| RA* : | 0 | 16 | 27.34 |
| GHA* : | 2 | 42 | 23.06 |
| : | 40° | 35' | 45.40" |
| sin. : | .6507 2231 | | |
| cos. : | .7593 1580 | | |
| Dec.* : | 09° | 08' | 45.97" S. |
| (90+D) : | 99 | 08 | 45.97 |
| $\frac{1}{2}$(90+D) : | 49 | 34 | 22.98 |
| tan. : | 1.1738 7687 | | |

| | | |
|---|---|---|
| 1 | $x$ = .8913 4325 | (= tan·cos) |
| 2 | $y$ = .7638 6787 | (= tan. sin) |
| 3 | $xy$ = .6808 6847 | |
| 4 | $x^2$ = .7944 9279 | |
| 5 | $y^2$ = .5834 9412 | |
| 6 | $(x^2+y^2)$ = 1.3779 8691 | |
| 7 | $x(x^2+y^2)$ = 1.2282 5933 | |
| 8 | $y(x^2+y^2)$ = 1.0525 9442 | |

Residual: + 0.0000 0063 [Form No. 1]

Figure 3

compute and insert the indicated combinative values on the remaining 6 lines of the cards.

(This part of the computation process, a major part of the least squares processing, can be carried out rapidly on a desk-calculator, and, moreover, by an individual of a technical training level no higher than that of desk-calculator operator. The same is true of steps c and e to be described. This possibility of freeing personnel of advanced technical training (surveyers, engineers, computers) from major portions of the computation mechanics may be of interest in certain organizations, or under certain operational circumstances).

(This step can be completed for 50 cards (i. e., arbitrarily assuming an observation set of 50 stars) in from one hour forty minutes to two hours thirty minutes, according to estimates (1) supplied to the writer, by an operator of reasonable proficiency. These time estimates are based on the use of a 10-column manual machine, which, in one nationally distributed model, weighs no more than 25 pounds with carrying case. The manual-operation and weight data are supplied as of interest when astrolabe computations must be carried out in the field, away from a source of electric power).

c. After completing a Form 1 card for each star, algebraically add the line 1 quantities on all the cards. Enter this sum in the coefficient-boxes labeled « Σ 1 » on a Form 2 card, which is illustrated in figure 4. Carry out the same procedure for the other 7 lines on the Form 1 cards, entering the sums in the coefficient-boxes « Σ1 Σ2....Σ8 ». Fill in the uppermost right-hand coefficient-box on the Form 2 card with the number of cards (sights) in the set. (This would be

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(1) By Mr. R.D. Bryan, Education Director, Monroe Calculating Machine Co., Orange, N.J. The writer hereby expresses his appreciation of Mr. Bryan's interest and assistance in the computation mechanics.

« 50 » for a set of 50 stars; it is 6 in the Form 2 card illustrated, which is a test sub-computation). (Time estimate : one hour forty-five minutes).

| Equation No. | (Type equation: $x^2+y^2+Dx+Ey+F=0$) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | $\boxed{\Sigma 6}$ +4.8528 3889 | $\boxed{\Sigma 1}$ +2.0676 8154 | D | $\boxed{\Sigma 2}$ +4.1914 4554 | E | $\boxed{n}$ + 6 | F | = 0 |
| 2 | $\boxed{\Sigma 7}$ +1.8687 2321 | $\boxed{\Sigma 4}$ +1.4389 6617 | D | $\boxed{\Sigma 3}$ +1.1958 7895 | E | $\boxed{\Sigma 1}$ + 2.0676 8154 | F | = 0 |
| 3 | $\boxed{\Sigma 8}$ +3.9921 0612 | $\boxed{\Sigma 3}$ +1.1958 7895 | D | $\boxed{\Sigma 5}$ +3.4138 7272 | E | $\boxed{\Sigma 2}$ +4.1914 4554 | F | = 0 |
| First equation: | | (A) | | (B) | | (C) | | |
| 4 | $x^2+y^2$ | −.8416 5071 | x | −1.6697 5837 | y | + .6476 8826 | | = 0 |
| Adjusted (final) equation: | | | | | | | | |
| 5 | $x^2+y^2$ | −.8416 5071 | x | −1.6697 5837 | y | + .6476 8826 | | = 0 |

$$\text{Lat.} = 90° - \arctan \frac{\sqrt{A^2+B^2}}{1-C} = 10°40'12.91''\text{ N.}$$

$$\text{Long.} = \arctan \frac{B}{A} = 63°14'57.70''\text{ W.}$$

[Form No. 2]

d. Steps a, b, and c eventuate in the formation (practically automatic) of equations nos. 1, 2, and 3 on the Form 2 card. Equation no. 4 on that card is, of course, arrived at by the simultaneous solution of equations nos. 1, 2, and 3. (Time estimate, including verification of solution : twenty minutes).

e. Compute the residuals substituting each pair of x and y values in equation no. 4 in succession. Note that values of x, y and ($x^2 + y^2$) already shown on each Form 1 card on lines 1, 2, and 6, respectively, facilitate this operation. Enter each residual as it is arrived at on the indicated line at the bottom of the Form 1 card (Time estimate : two hours).

f. Study the residuals. To do this, arrange the Form 1 cards in the order of their azimuths. Any card showing an extremely large residual indicates that the star to wich it refers was either mis-identified or grossly mistimed. Discard such cards, after algebraically adding their lines 1 through 8 and applying the sums as adjustments to the coefficients of D, E, and F in equations nos. 1, 2, and 3 of the Form 2 card. Solve these equations again; the resultant equation should be entered as equation no. 5 on the Form 2 card.

g. Compute the observer's position from equation no. 5 on the Form 2 card (or from equation no. 4 if examination of the residuals disclosed a normal pattern of residuals). This completes the computation process.

Now that analytical description and the description of the mechanics of the computation have been completed, we can note more fully certain basic differences between this method and the standard method :

1. With this method it is not necessary to know or assume an initial observer's position for computation purposes. His position does not even have to be known even approximately. This fact is reflected in part in the projection formulas, which make use of the GHA of each star at the instant of collimation, rather than an LHA.

2. The fixed angle integral with the observing instrument does not have to be known, even approximately, in that its value does not enter into the computation process. This is equivalent to saying, of course, that zenith distances or the radius of the observation circle are not a part of the progress.

3. Azimuths, either computed by formula or measured on the instrument at the time of observation, are used in the standard method as an indispensable part of the graphic plot or solution. Although this method contemplates a reading of the azimuth at the time of observation, this azimuth does not enter into any part of the computation process. Arranging the cards in the order of azimuth values when they are taken up for the study of residuals enables the discovery to be made that at times the observation « circle » is in reality an ellipse, as a result of unusual atmospheric refraction in two opposite directions at the observation site. This deformation, is of course, indicated by the pattern of the residuals. When this has been noted, or is suspected, a graphic plot, using the observed azimuths, the residual quantities, and a suitable plotting scale, would no doubt lead to a clearer understanding of the extent of the deformation. Individual sights that should be discarded by reason of star mis-identification or gross timing error, and revealed by an unnaturally large residual have been already considered.

4. With this method the least squares adjustment of the data can be carried out almost mechanically in that the forms used with the method indicate an unvarying sequence of operations. This can hardly be said of the standard method, which appears to require the use of additionnal table entries, as well as $i(=\sqrt{-1})$ terms. The participation in the computation process, in particular with that portion of it involving the least squares adjustment procedure, by personnel of limited technical training does not appear practical with the standard method, as it does with the method under discussion.

Incidentally, the selection of the pole of the celestial hemisphere opposite to that containing the observer, together with the fact that the observation circle is always a small, rather than a great, circle of the celestial sphere, completely excludes the possibility of the observation circle ever passing through the point of projection, and thereby stereographically projecting as a straight line instead of a circle.

5. This method has been adjudged particularly well-adapted to processing on automatic high-speed computing machines. Something more than 90 % of the computation can be carried out in a matter of minutes, once suitable arrangements have been made on such equipment, according to one authority. This is not cited as a basic difference between the two methods, however; so far as is known to this writer, no study has been made of the standard method in connection with automatic high-speed computing equipment.

In conclusion, the all-important matter of the accuracy of the final results gotten with this method and with the standard method ought to be considered, so far as is possible. One way not to arrive at a realistic conclusion is simply to compare final positional results gotten with the two methods using the same field data. (Incidentally, in the case of the writer's test computations, such results disagreed only by a few hundredths of a second of arc). Such comparison apparently is not mathematically decisive; the most realistic conclusion regarding final accuracy must apparently be arrived at only by way of the most rigid examination of the mathematical principles, and their application, of the two methods.