LATITUDE AND AZIMUTH DETERMINATION BY THE OBSERVATION OF A SINGLE UNKNOWN STAR

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1. *Object of the present paper.*

Mr. Sanjib K. Ghosh, of the *Survey of India,* has recently described in these pages a method in geodetic astronomy which, although not of wide application, appears to have possible uses in field operations (1). As Mr. Ghosh has only given a direct solution of the problem, it would appear of value to adopt the method of using a first approximation to the correct answer (method of approximate position), the more so as the set of observation equations can be treated by a graphical method which not only results in a gain in the simplicity and rapidity of calculation but also enables the conditions under which the method is applicable to be examined easily.

2. *Principle of the method.*

The method proposed by Mr. Ghosh enables one to determine the latitude of a station and the azimuth of one ray from that station by means of three observations of a single star whose coordinates need not be known. These three observations, which should be taken at intervals of some hours with a theodolite, consist in the measurement of the altitude of the star above the horizon and in a simultaneous reading of the horizontal circle.

The unknowns in the problem are not only the latitude of the station and the azimuth of the zero of the horizontal circle but also the declination of the star. The given quantities, which for each observation are the true altitude of the star and the reading on the horizontal circle, are related to the unknowns by an observation equation deduced by an application of spherical trigonometry. Three such equations suffice to provide the solution.

In his paper, Mr. Ghosh is mainly preoccupied with the solution of this set of three trigonometrical equations, just as Gauss was in 1808, when he proposed the method of equal altitudes which led similarly to a set of analogous equations containing two principal unknowns and one auxiliary unknown, or like Delambre in 1812 when he discussed the method proposed by Gauss without perceiving the use that it might have in positional astronomy.

3. *Notes on the method of approximate position.*

It is curious that Gauss, who was also responsible for developing the method of least squares, the application of which necessitates the observation equations being made linear by the use of an approximate solution, did not think of improving the method of equal altitudes by adopting the same device which conside-

(1) « Determination of Azimuth and Latitude from Observations of a Single Unknown Star by a New Method », by Sanjib K. Ghosh. *Empire Survey Review.* xii, 87, 17-26. January, 1953.

rably simplifies the solution of a set of observation equations when the number of these is equal to the number of unknowns and which makes it easy to treat the case of additional observations when, in order to get rid of the accidental errors that may affect the data and the measured quantities, one uses more observations than there are unknowns.

So far as the method of equal altitudes is concerned, Knorre in 1832 and Anger in 1835 proposed methods of indirect solution applicable to any number of observations. But the most important improvement was in 1890, and was due to Admiral Perrin, who, by an ingenious extension of the method of using an approximate auxiliary point as employed in navigation and hydrographic work, proposed a graphical soltution which not only led to a notable simplification in the computations but also made it possible to investigate the accuracy of the observations and the conditions under which the method of equal altitudes can be applied.

In general, the method of the assumed approximate position when solved graphically is suitable for all problems involving only two unknowns. When there are three unknowns, the graphical solution is only possible when the observation equations are of a particular type and when, reduced to linear form, they represent planes enveloping a cone of revolution ; the observations are then represented graphically by lines, which, instead of meeting as in the case of two unknowns, touch a circle the position of whose centre and the value of whose radius give the solution of the problem. In astronomy, this particular case is not only that of the method of equal altitudes but more generally that of all problems in three unknowns in which the observed stars are situated on a single circle of the celestial sphere, for example meridian observations with a meridian transit (1) or the method of using « azimuth lines » of constant altitude (2).

We shall now examine how the method proposed by Mr. Gosh raises the same question and leads to the same method of solution.

4. *Formation and transformation of the observation equation.*

Recalling that the unknowns in the problem are :

φ = latitude of the station.

V = azimuth of the zero of the horizontal circle.

δ = declination of the observed star.

The data are simply the observed quantities :

h = true altitude of star (observed altitude corrected for refraction).

L = reading on the horizontal circle.

Considering the astronomical triangle P (pole), Z (zenith of station), A (star), we see at once that the known and unknown quantities are related by the observation equation :

$$\sin \delta = \sin \varphi \sin h + \cos \varphi \cos h \cos (V+L)$$

(1) « Sur une représentation graphique rationnelle des observations de passage à la lunette Méridienne », by A. GOUGENHEIM, *Bulletin Géodésique*, No. 58, avril-mai-juin 1938.

(2) « Une Méthode Nouvelle d'Astronomie Géodésique : la Méthode des Droites d'Azimut », by André GOUGENHEIM, *Annales Hydrographiques*, 4e Série, Tome II, 1951 ; *International Hydrographic Review*, Vol. XXIX, No. 2, Nov. 1952 (English translation).

Let us choose approximate values φ_0 and V_0 for the unknowns φ and V. We can then calculate an approximate value δ' for the declination by the formula :

$$\sin \delta' = \sin \varphi_0 \sin h + \cos \varphi_0 \cos h \cos (V_0 + L)$$

Let $\Delta\varphi = \varphi - \varphi_0$, $\Delta V = V - V_0$.

If these quantities are so small that we can neglect their squares and product we can then obtain $\delta - \delta'$ by a simple differentiation of the last equation, thus obtaining a new form for the observation equation :

$$(\delta - \delta') \cos \delta' =$$
$$[\cos \varphi_0 \sin h - \sin \varphi_0 \cos h \cos (V_0 + L)] \; \Delta\varphi - \cos \varphi_0 \cos h \sin (V_0 + L) \; \Delta V$$

If we denote by H' the angle at P corresponding to the approximate values V_0 and δ' which is given by :

$$\sin H' = \frac{\cos h \sin (V_0 + L)}{\cos \delta'}$$

the observation equation is simplified and becomes :

$$\delta - \delta' = \cos H' \; \Delta\varphi - \sin H' \cos \varphi_0 \; \Delta V$$

Let us complete the approximate solution by choosing an approximate value δ_0 for the declination δ : it then follows that :

$$\delta_0 - \delta' = \cos H' \; \Delta\varphi - \sin H' \cos \varphi_0 \; \Delta V - (\delta - \delta_0)$$

which is a linear equation in terms of the three new unknowns $\Delta\varphi$, ΔV and $\delta - \delta_0$

Each observation then results in a corresponding equation characterised by the values of δ' and H' and the next step is to solve the set of observation equations.

If there are only three observations, the solution is simple but if there are more than three it is necessary to apply the method of least squares. We shall now see that in both cases the solution can be arrived at graphically.

5. *Graphical solution of the system of observation equations.*

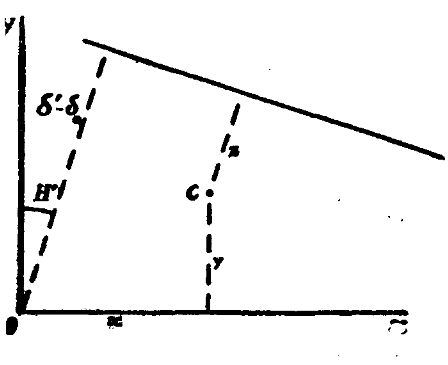
Let us take as new unknowns the quantities :

$$x = \cos \varphi_0 \; \Delta V \qquad y = - \Delta\varphi \qquad z = \delta - \delta_0$$

The observation equation can then be written :

$$x \sin H' + y \cos H' + z = \delta' - \delta_0.$$

In a system of rectangular co-ordinates X O Y let us set off a line of bearing H' + 90° distant $\delta' - \delta_0$ from the origin. Consideration of the observation equation shows that this line passes at distance z from the point C whose co-ordinates are x and y, and from this we can obtain the unknowns by drawing the circle to which the lines arising from the several observations are most nearly tangential and by scaling off from the plot the co-ordinates of its centre and the value of its radius.



The choice of the scale for the plot depends on the precision of the instrument used in making the observations. If, on the adopted scale, the centre of the circle is too far from the origin, this means that the solution differs too greatly from the approximate solution ; in consequence, the assumption made as to the smallness of $\Delta\varphi$ and ΔV is not valid and it is then necessary to repeat the reduction by means of a new approximate solution suggested by the initial graph and plotted to a smaller scale.

6. *Arrangement of computations.*

In order to plot the graph, the δ' and H' for each observation must be calculated. The approximate declination δ' is given by the formula already mentioned, viz :

$$\sin \delta' = \sin \varphi_0 \sin h + \cos \varphi_0 \cos h \cos (V_0 + L)$$

or by an equivalent expression. The precision of this quantity should be of the same order as that obtained in the measurement of h, which itself depends on the accuracy sought for the latitude. Next, an approximate value for the hour angle H' is calculated from the formula :

$$\sin H' = \frac{\cos h \sin (V_0 + L)}{\cos \delta'}$$

but it is not necessary to calculate H' to any great accuracy ; a tenth of a degree will do, since H' serves solely to orientate the lines on the plot while δ' serves to fix their position (1).

If a good approximate value for the declination cannot be obtained *à priori*, a round value for δ_0 in the neighbourhood of the mean of all the values of δ' obtained from the different observations can be taken, or rather a round value adjusted to this mean in such a manner that the circle to be traced on the graph will have a radius of about 4 to 5 centimetres on the scale adopted.

7. *Geometrical interpretation of the method of observation and of the graphical solution.*

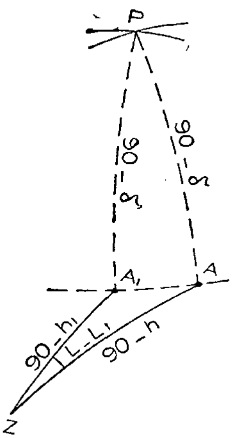
The problem concerned, to determine the latitude of the place and the direction of the meridian at that place, really amounts to setting the pole in its position relative to the place since the pole is situated on the meridian at angular distance 90° - φ from the point considered.

When approaching the problem from this viewpoint we shall see that we can arrive directly by geometrical means at the solution which we have obtained by a process of calculation.

(a) Let us consider on the celestial sphere the zenith Z of the place of observation and the position A, at any given instant, of the star whose observed zenith distance from Z is 90° - h. If we knew the declination δ of the star, we could obtain an exact geometrical locus of the pole as it traces out the small circle with A and spherical radius 90° - δ. A second observation corresponds to a second position A_1 of the star and this we can plot with reference to the first

(1) The true hour angle is equal to — *H'*.

since we have measured the altitude h_1 and the difference of azimuth $L - L_1$. As before, if δ were known, this observation would give us a second locus of the pole.

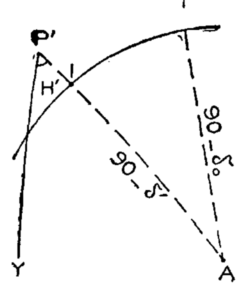


Other observations of the star lead to similar loci which should meet together at the pole P.

But, if we plot these loci with a different spherical radius $90^\circ - \delta_0$, they do not intersect at P. We may note, however, that they are all tangential to a small circle whose centre is P and spherical radius $\delta - \delta_0$: if, starting with the pseudo-loci so traced, we determine the centre of the small circle we obtain the position of the pole and from the radius $\delta - \delta_0$ we deduce the value of the declination δ.

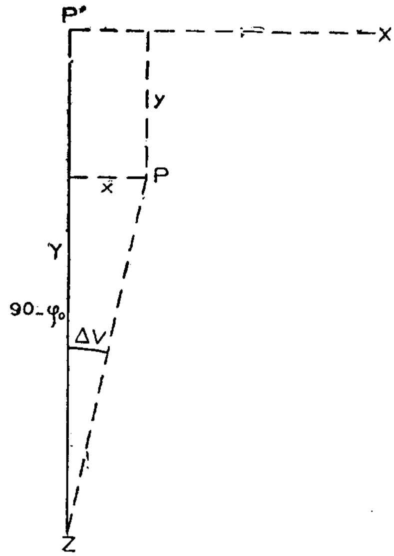
(b) The problem then is to plot the geometrical pseudo-loci : for this purpose we use an approximate position P' for the pole, corresponding to an approximate solution in terms of φ_0, V_0. We can then calculate the side $90^\circ - \delta'$ of the triangle P'AZ on the celestial sphere ; the small circle with centre A and spherical radius $90^\circ - \delta_0$ passes at distance $\delta' - \delta_0$ from P' reckoned as following the direction of the star.

If our approximations are sufficiently close, we can replace the celestial sphere by its tangent plane at P'. The small circle with centre A and spherical radius $90^\circ - \delta_0$ is then represented by its tangent, as obtained by prolonging from P' the intercept $P'I = \delta' - \delta_0$ in the direction of the star ; that is to say on bearing H' if one adopts as the axis of reference P'Y, the approximate direction P'Z of



the meridian of the station. This is the construction for representing the observations that we have already found. Paragraph (a) above shows us how we can obtain the exact position P of the pole with the aid of the circle to which the pseudo-loci are tangential.

(c) It remains to obtain the values of the unknowns. We have already stated that the radius of the enveloped circle represents the correction $\delta - \delta_0$ to the declination. One also sees at once that the ordinate y of the point P is the correc-



tion, $-\Delta\varphi$, to the approximate co-latitude and that the abscissa x of P represents the displacement of the pole caused by the correction ΔV to the approximate azimuth, that is to say $\Delta V \cos \varphi_0$.

8. *Discussion.*

The circle which yields the solution of the problem is best determined when the lines which represent the different observations and which should envelop it cover a wide extent in bearing. As the latter is dependent on the value of the hour angle at the moment of observation, it is advisable to spread the observations over as long an interval as possible.

It is thus advisable to choose a star which to the observer remains above the horizon at the place considered during the whole night. In order to diminish the uncertainty as regards refraction, preference should be given to stars which are at upper transit during the night (1).

It is thus only in very low latitudes that there is any advantage in using the equatorial stars recommended by Mr. Ghosh.

Nevertheless, because of daylight and of the necessity for not observing too close to the horizon, difficulties arise in these regions in the case of observations

(1) In the case of a circumpolar star, particularly when h differs only slightly from ϕ_0, it is advisable to determine δ' from the well-known relation

$$\sin^2\left(45 - \frac{\delta'}{2}\right) = \sin^2\left(\frac{\phi_0 - h}{2}\right)\cos^2\left(\frac{V_0+L}{2}\right) + \cos^2\left(\frac{\phi_0+h}{2}\right)\sin^2\left(\frac{V_0+L}{2}\right).$$

spread over 12 hours so that the lines which represent them scarcely envelop a semi-circle, and this is not very favourable to a good determination of the centre or of the radius ; this same difficulty arises also in the determination of time (and of instrumental azimuth) by transit observations, the stars observed being distributed there also over less than a semi-circle.

It is only in winter and at middle and high latitudes that, in the method which we are studying, stars can be observed over more than 180° of hour angle and in which, consequently, the latitude and azimuth can be satisfactorily determined.

Note that if the uncertainty in the result is particularly apparent in the graphical solution, there is no means of determining an accurate solution by computational methods ; it is sufficient to make one of the measured quantities vary very slowly to prove that this causes an important alteration in the result.

However, if two circumpolar stars are observed whose right ascensions differ but slightly from twelve hours and which transit towards midnight, the one at upper transit and the other at lower transit, it is possible to plot on the graph two circular arcs of different radii on either side of a common centre and the determination is then considerably improved (see example at end).

Let us add in conclusion that, for various reasons, the method of determining latitude and azimuth by the observation of a single star whose co-ordinates are unknown cannot be a very precise one ; the instrument is made vertical by means of a level, the observations necessitate readings on graduated circles and corrections for refraction, it is possible that there exist instrumental flexures which vary with the elevation of the star, the method assumes that the instrument is stable in azimuth and elevation over a long period of time, and finally, and above all, it is impossible to make precise pointings at the same time on both the horizontal and vertical hairs of the reticule.

Nevertheless, this method is of some interest for field operations ; unfortunately, if one wishes to determine the third element needed in geodetic astronomy, the local hour angle defined by the clock-correction of a chronometer, the happy idea of having recourse to a star of unknown co-ordinates does not apply, as the solution of the system only furnishes the difference between the clock-correction and the right ascension of the star.

9. *Example.*

The example which follows concerns the solution of a series of observations (fictitious) comprising four points to a star whose upper transit occurs at night, and four pointings to a second star, somewhat nearer to the pole and differing by nearly twelve hours in right ascension from the other. There is about an eight-hour interval between the extreme pointings on each star.

Observed values.

Star No. 1

True elevation h	38°44'22"	50°57'28"	52°11'17"	42°34'09"
Reading of horizontal circle L.	256 22 04	245 26 09	222 46 27	209 29 21

Star No. 2

True elevation h	30°18'12"	29°13'25"	29°57'42"	32°14'19"
Reading of horizontal circle L.	229 04 15	232 00 36	235 00 26	236 52 36

Assumed approximate values.

Latitude of station .... φ_0 = 33°11'30" N.
Azimuth of zero of horizontal circle .. V_0 = 127°42'30"

Calculated values.

	Star No. 1			
Calculated declination δ'	69°48'22"4	69°48'04"1	69°47'50"5	69°47'46"1
Hour angle H'	67°2	24°5	342°9	304°3

	Star No. 2			
Calculated declination δ'	86°01'12"1	86°01'28"7	86°01'38"5	86°01'36"2
Hour angle H'	224°3	183°5	143°7	102°6

Construction of Graph.

Assumed approximate declination ...... Star No. 1 : δ_0 = 69°47'40"
Star No. 2 : δ_0 = 86°00'50"

Adopted scale .... 2 mm. to 1" of arc.

Graphical solution.

Abscissa of centre .... $x = \Delta V \cos \varphi_0$ = 38.5 mm. = 19"25
whence ΔV = 23"0

Ordinate of centre .... $y = -\Delta\varphi$ = - 32 mm. = - 16"

Radius of the enveloped circle .... Star No. 1 : $z = \delta - \delta_0$ = 62 mm. = 31"

Star No. 2 : $z = \delta - \delta_0$ = 48 mm. = 24"

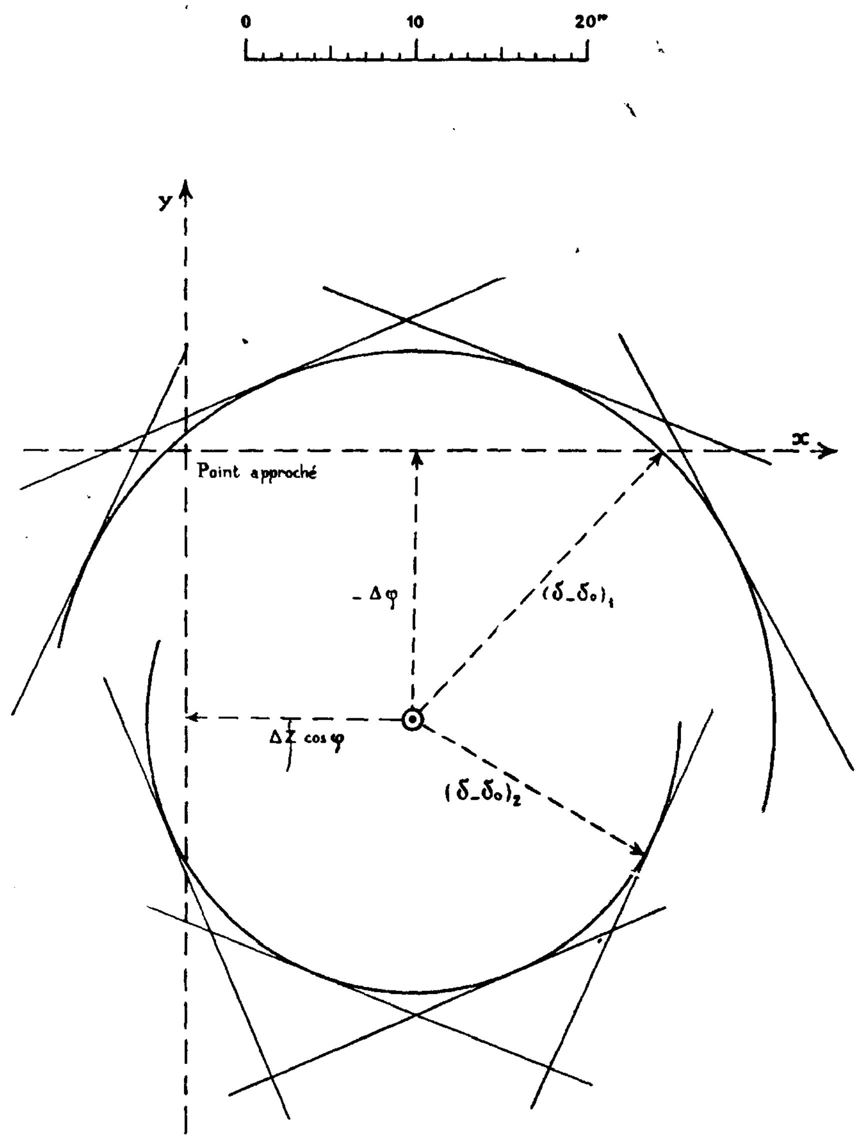
Results.

Latitude .... $\varphi = \varphi_0 + \Delta\varphi$ = 33° 11' 46" N

Azimuth of zero of circle .... $V = V_0 + \Delta V$ = 127°42'53"

Declination of stars .... Star No. 1, δ = 69°48'11"

Star No. 2, δ = 86°01'14"



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