NEW TABLE FOR COMPUTING SUMNER LINE FROM D.R. POSITION

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Introduction to a New Nagivation Table for computing sumner line from D.R. position.

Object and characteristics.

The object of this table is to calculate the altitude of a celestial body, from the D.R. position, in terms of the time by means of logarithms, using a single trigonometric function i.e. : the Haversine only, after a suitable transformation of the fundamental formula for the altitude.

Such process has the following advantages :—

1) The calculation formula is strictly symmetrical ;
2) The number of logarithms required is reduced to four ;
3) Any special rule for signs is eliminated ;
4) The calculating equipment is reduced to a single table of a few pages only, consisting of a single function whose variations are clearly shown by the arrangement of the table ;
5) Interpolating obligations, unavoidable when using tables, are reduced to a minimum.

Basic formula.

The fundamental formula for H_e the D.R. altitude is

$$\sin H_e = \sin \varphi_e \sin D + \cos \varphi_e \cos D \cos P_e \quad (1)$$

P_e is the hour angle for the D.R. position

$P_e =$ HA, when the heavenly body is West,
$P_e = 24 -$ HA, when the heavenly body is East.

By adopting this method of notation, a similar way is used as in the modern anglo-saxon tables the argument of which is the hour angle expressed in degrees of arc.

Let us put

$$X = \cos^2 \frac{P}{2}, \qquad Y = \sin^2 \frac{P}{2}$$

Subtracting from 1 each terms of formula (1)

$$1 - \sin H_e = 1 - [(X + Y) \sin \varphi_e \sin D + (X - Y) \cos \varphi_e \cos D]$$

is obtained because :—

$$\begin{cases} X + Y = 1 = \cos^2 \frac{P}{2} + \sin^2 \frac{P}{2} \\ X - Y = \cos P = \cos^2 \frac{P}{2} - \sin^2 \frac{P}{2} \end{cases}$$

Simplifying and substituting $90 - \zeta_e$ for H_e (ζ_e = zenith distance) ; we have

$$1 - \cos \zeta_e = 1 - [X \cos (D - \varphi) - Y \cos (D + \varphi)]$$

substituting $X + Y$ for 1

$$1 - \cos \zeta_e = X [1 - \cos (D - \varphi)] + Y [1 + \cos (D + \varphi)]$$

By putting :—

$$\begin{cases} Y = \text{Haversine } P = \frac{1 - \cos P}{2} \\ X = \text{Haversine } (180 - P) = \frac{1 + \cos P}{2} \end{cases}$$

or co Haversine P,

$$\begin{cases} y = \text{Haversine } (D - \varphi) = \dfrac{1 - \cos (D - \varphi)}{2} \\ x = \text{Haversine } (180 - (D + \varphi)) = \dfrac{1 + \cos (D + \varphi)}{2} \end{cases}$$

or co Haversine $(D + \varphi)$.

formula (1) may be written as follows :—

$$\text{Haversine } \zeta_e = Xy + Yx \qquad (2)$$

The Haversines and co-Haversines functions vary from 0 to + 1 and are always positives as well as all the terms of the formula. Besides :—

$$\text{Haversine } N^\circ = \text{co-Haversine } (180 - N^\circ)$$

D-φ / D+φ / P	65° 00' log co Hav x X	log Hav y Y	Hav A B	65° 30' log co Hav x X	log Hav y Y	Hav A B	
0'	9,85 206	9,46043	28 869	9,84 963	9,46626	29 265	30
1'	9,85 198	9,46 063	28 882	9,84 955	9,46 655	29 279	29
2'	9,85 190	9,46 083	28 895	9,84 947	9,46 675	29 292	28
3'	9,85 182	9,46 103	28 909	9 84 939	9,46 694	29 305	27
4'	9,85 174	9,46 123	28 922	9,84 931	9,46 714	29 318	26
5'	9,85 166	9,46 142	28 935	9,84 923	9 46 733	29 332	25
6	9 85 158	9,46 162	28 948	9,84 914	9,46 753	29 345	24
7	9,85 149	9,46 122	28 961	9,84 906	9,46 773	29 358	23
8	9,85 141	9,46 202	28 975	9 84 898	9,46 792	29 371	22
9	9,85 133	9,46 222	28 988	9,84 890	9,46 812	29 385	21
10	9,85 125	9,46 241	29 001	9 84 882	9,46 831	29 398	20
11	9,85 117	9 46261	29 014	9,84 874	9 46 851	29 411	19
12	9,85 109	9,46 281	29 027	9,84 866	9,46 871	29 424	18
13	9,85 101	9,46 301	29 041	9,84 857	9 46 890	29 438	17
14	9,85 093	9,46 320	29 054	9,84 849	9,46 910	29 451	16
15	9,85 085	9,46 340	29 067	9,84 841	9,46 929	29 464	15
16	9,85 077	9,46 360	29 080	9,84 833	9,46 949	29 477	14
17	9,85 069	9 46 380	29 093	9,84 825	9 46 968	29 491	13
18	9,85 061	9,46 399	29 107	9,84 817	9 46 988	29 506	12
19	9,85 052	9 46 419	29 120	9,84 808	9 47 007	29 517	11
20	9,85 044	9,46 439	29 133	9,84 800	9,47 027	29 530	10
21	9,85 036	9,46 458	29 146	9,84 792	9,47 046	29 544	9
22	9,85 028	9,46 478	29 160	9,84 784	9,47 066	29 557	8
23	9,85 020	9,46 498	29 173	9,84 776	9 47 085	29 570	7
24	9,85 012	9,46 517	29 186	9,84 767	9 47 105	29 584	6
25	9,85 004	9,46 537	29 199	9,84 759	9,47 124	29 597	5'
26	9,84 996	9,46 557	29 212	9,84 751	9,47 144	29 610	4'
27	9,84 988	9,46 576	29 226	9,84 743	9,47 163	29 623	3'
28	9,84 979	9,46 596	29 239	9,84 735	9,47 183	29 637	2'
29	9,84 971	9,46 616	29 252	9,84 726	9,47 202	29 650	1'
30	9,84 963	9,46 635	29 265	9,84 718	9,47 222	29 663	0'
	Y y log Hav	X x log co Hav	Hc 24° 30'	Y y log Hav	X x log co Hav	Hc 24° 00'	P D+φ D-φ
	114° 30'			114° 00'			

arc	Hav	log Hav	log co Hav
0,'1	1	1,5	0,5
0,15	2	3,1	1
0,25	3	4,6	2
0,3	4	6,2	2,5
0,4	5	7,7	3
0,45	6	9,2	3,5
0,55	7	10,8	4,5
0,6	8	12,3	5
0,7	9	13,9	5,5
0,75	10	15,4	6
0,85	11	16,9	6,5
0,9	12	18,5	7,5
1	13	20	8

Operating method.

Formula (2) is calculated by means of a semi-logarithmic method using four logarithms :

$$\text{Haversine } \zeta_e = A + B$$

$$\log A = \log X + \log y$$

$$\log B = \log Y + \log x$$

Each page of the table concerns one degree (of arc) reading from the top and one degree from the bottom. Data relating to acute angles are listed on the left hand side from the top ; those relating to obtuse angles are listed on the right hand side from the bottom.

On each page three columns are given : log Haversine, log co-Haversine and Natural Haversine. Natural numbers are thus found by the side of their logarithms, thus avoiding a special table for logarithms.

The data, by which altitude H_e is obtained, is found on the right hand side at bottom of page, this avoiding reference to the zenith distance ζ_e.

The selected tabulation interval is one minute of arc, this being the final approximation possible to be obtained at sea.

The interpolation, negligible in many cases, is readily obtained if required by means of a proportional parts table appearing on each page. This table is given for a single tabular difference, usually the maximum tabular difference in the page. The error resulting of such approximation is less than 5/100 of a minute of arc.

The hour angle entries are shown in the same units of arcs as $D - \varphi$ and $D + \varphi$, this being the only rational system for obtaining simplicity in the arrangement of table. The method adopted enables one to read X and Y at the same time for a single entry with P. x and y are thus obtained simultaneously.

To simplify the printing, numbers have been multiplied by 10^{10}, i. e. : 10 has been added to each logarithm characteristic. This means that 9, 8, 7, etc., have been substituted for the negative characteristics $\bar{1}$, $\bar{2}$, $\bar{3}$, etc. For very small angles or for angle very close to 180°, only the necessary number of decimals has been retained, i. e. : the one which determines the 5th place of decimals of the corresponding haversine.

COMPUTING THE AZIMUTH.

Principle.

The five elements formula for azimuth in terms of altitude and time is :—

$$\cos H \cos Z - \sin \varphi \cos D - \sin D \cos \varphi \cos P.$$

This may be written :

$$l = \cos H \cos Z = \sin(D - \varphi) \cos^2 \frac{P}{2} + \sin(D + \varphi) \sin^2 \frac{P}{2}$$

also, by adopting similar notations as above $l = \alpha X + \beta Y$.

By putting $\alpha = \sin (D - \varphi)$ $\quad\quad \beta = \sin (D + \varphi)$,

$$X = \cos^2 \frac{P}{2} \quad\quad Y = \sin^2 \frac{P}{2}$$

$l = \cos H \cos Z$ is obtained by means of a trigonometric diagram with two parallel axes with divisions for α and β.

From l, H being known, the value of Z is easily obtained. The latter being very well determined in the vicinity of the first vertical.

Method of using the diagram.

The azimuth diagram for solving the above formula is given at the end of the book. The bearing of the heavenly body is readily obtained as follows :—

Enter the left hand side scale with $D - \varphi$, the right hand side scale is entered with $D + \varphi$ from the bottom if positive, from the top if negative. Draw the straight line from A to B, or use a strait edge. Let M be the intersection of AB with the ordinate of P., the hour angle, as read off from the horizontal scale. Take the intersection of the horizontal line through M with one of the concentric circles corresponding to the altitude.

The azimuth is read on the external circle divided into degrees from 0° to 180° from North.

EXAMPLE

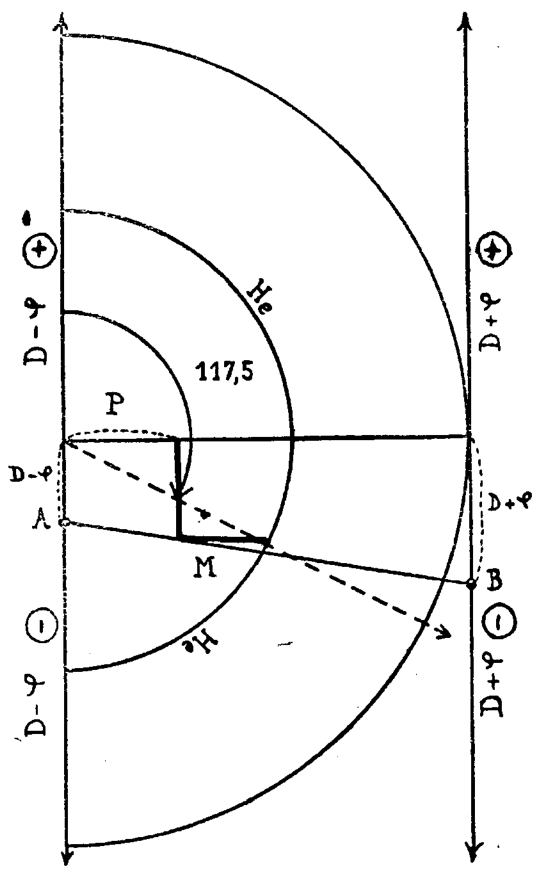
$D = 21°\ 36'$ S $\qquad \varphi = 1°\ 40'$ S

$D - \varphi = 19°\ 56'$ S $\qquad D + \varphi = 23°\ 16'$ S

$P_e = 45°\ 37'$ E $\qquad H_e = 41°\ 27'$

$Zv = N\ 117{,}5\ E$

$R^t_v = 117{,}5$



Method of calculation.

Data :— $\varphi_e = 27°28',5$ N. $D = 38°03', 5$ N. $P_e = 65°08'$ W.

(1) $D - \varphi = 10°35'$ N.	(2) $\log y = 7{,}92\ 770$	(1) $D + \varphi = 65°32'$ N. $\log x = 9{,}84\ 947$
$P_e = 65°08'$ W.	(3) $\log X = 9{,}85\ 141$	(3) $\log Y = 9{,}46\ 202$
	(4) $\log A = 7{,}78\ 111$	(4) $\log B = 9{,}31\ 140$
	(5) $A = 0{,}00\ 604$ $B = 0{,}20\ 487$	(6) $H_e = 35°19'4$
	(6) Hav $\zeta_e = 0{,}21\ 091$	(7) $Z = $ N. $61°{,}2$ W. (Diagram) $R^t_v = 299$

Accuracy of table.

The solution of the standard fundamental formula by means of Friocourt table using only five places of decimals gives a lesser degree of accuracy than that obtained with the present table as shown by the following figures :—

	$H_e = 60°$	$H_e = 75°$	$H_e = 84°$
Standard (Friocourt) Method.....	= 0',5	= 0',9	= 2,'2
Haversines Table................	= 0',4	= 0',7	= 1',7

With a small volume and a quicker calculation the accuracy is higher using the same number of decimals.

Conclusion.

The table established on the above principle is in the press. The advantages of its simplicity and symmetry will be noted on ship bridges where the important equipment of triple-entry tables is not always available, although the superiority of the latter is well known as compared with all other tables based on splitting the spherical triangle into two rectangular ones.

The proposed booklet will consist of about 50 pages easy to use furnishing at sea a useful aid to navigators.

APPENDIX I.

Calculation of the probable error of He when using the Haversine table.

The following is extracted from a discussion concerning the use of the symmetrical formula $\sin H_e = Xy + xY$ and the probable error by Ingeneer hydropher George.

First, errors due to second differences are quite negligible and will not be taken into account ; only errors due to linear interpolations will be considered.

They involve for each logarithm to 5 places of decimals a probable error of: $1{,}5 \times 10^{-5}$. If [log Xy] is the probable error of $\log A = \log Xy$:—

$$[\log Xy] = 3 \times 10^{-5} \times \log A = a$$

If log A be known with error a : then the error of A is

$$\frac{Aa}{\log e}$$

(1) Algebraic difference and sum $D - \varphi$, $D + \varphi$.
(2) log y and log x (entry with $D - \varphi$ and $D + \varphi$).
(3) log X and log Y (single entry with P.).
(4) Arithmetic sum (2) + (3).
(5) A and B are obtained from log A and log B by using columns log Hav and Hav.
(5) Arithmetic sum A + B.
(6) Get out of table bottom in column H_e, right hand side from bottom.
(7) Use diagram.

On the other hand, the use of the table for determining A introduces a new error :

$$1{,}5 \times 10^{-5} \frac{(1 + A)}{\log e}$$

Consequently the total error of A is

$$[A] = \frac{Aa}{\log e} + 1{,}5 \times 10^{-5} \frac{(1 + A)}{\log e} \qquad (2)$$

From (1) :—

$$[A] = [Xy] = 1{,}5 \times 10^{-5} + \frac{4{,}5 \times 10^{-5}}{\log e} Xy$$

$$[B] = [Yx] = 10^{-5} (1{,}5 + \frac{4{,}5 \; Yx}{\log e})$$

and the total error of Hav ζ is :

$$[\text{Hav } \zeta] = 3 \times 10^{-5} + \frac{4{,}5 \times 10^{-5}}{\log e} (Xy + Yx)$$

$$= 10^{-5} (3 + \frac{4{,}5}{\log e} \text{Hav } \zeta)$$

Now, if Hav ζ is known with error ε, the resulting error of ζ' is :

$$\frac{2 \varepsilon}{\sin \zeta \sin 1'}$$

The use of table for determining ζ introduces a new error :

$$\left(\frac{1}{20} + \frac{3 \times 10^{-5}}{\sin \zeta \sin 1'} \right)$$

and the final error of ζ is

$$[\zeta]' = \frac{1}{20} + \frac{3 \times 10^{-5}}{\sin \zeta \sin 1'} + \frac{6 \times 10^{-5}}{\sin \zeta \sin 1'} + \frac{4{,}5 \times 10^{-5}}{\log e \sin 1'} \frac{\sin \frac{\zeta}{2}}{\cos \frac{\zeta}{2}}$$

$$[\zeta]' = 0{,}05 + 0',31 \text{ cosec } \zeta + 0',36 \text{ tg } \frac{\zeta}{2}$$

The following table is obtained from the above formula for various altitudes :

Altitude..........	0°	32°	60°	70°	80°	84°
Error.............	0,7	0,6	0,8	1	1',9	3',2

This is in conformity with the results given above, the error being the double of probable error of H_e with reference to the altitude mean value.

APPENDIX II.

Graphical interpretation of Table.

When transforming the fundamental formula, the following intermediary form has been obtained :—

$$\sin H_e = \cos \zeta_e = X \cos (D - \varphi) - Y \cos (D + \varphi)$$

the latter suggests the idea of a vectorial product or of the double of the surface of a triangle the vertices of which would be :—

1) The axes origin o.

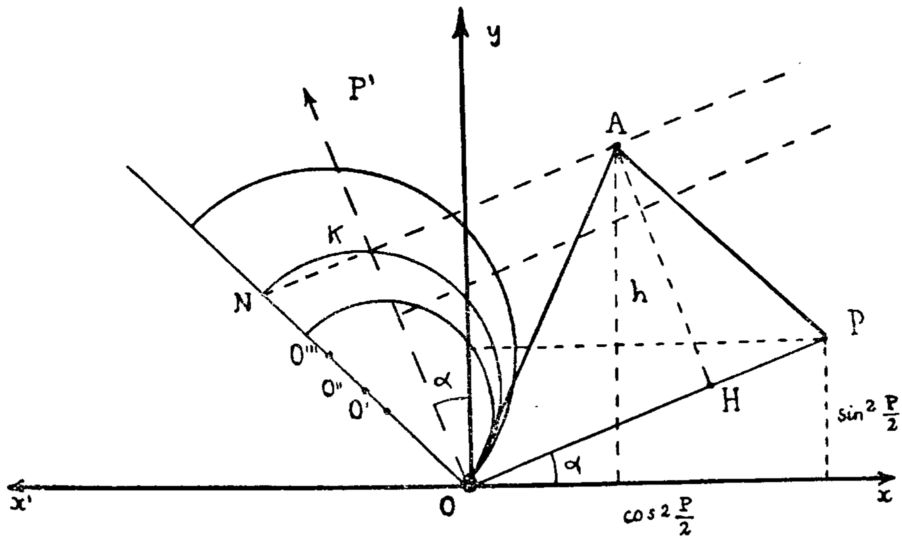
2) Point A of rectangular coordinates :

$$A \begin{cases} x = \cos (D + \varphi) \\ y = \cos (D - \varphi) \end{cases}$$

3) Point P of rectangular coordinates :

$$P \begin{cases} u = \cos^2 \frac{P}{2} = X \\ v = \sin^2 \frac{P}{2} = Y \end{cases}$$

when $u + v = X + Y = 1$.



For a given point P, we have :

$$\sin H_e = 2\ S = \rho \times h = \overline{PO} \times h$$

with $\rho = \overline{OP}$ and $h = \overline{AH}$, the perpendicular, and

$$\rho = \sqrt{u^2 + v^{2'}}$$

Let us write that the surface is a constant

$$2\ S = \rho \times h = uy - vx = \sin H_e \quad (1)$$

This definite the locus of A for a given point P, as the equation of straight line Δ parallel to OP, of which the original ordinate is OM. For any given point P, a selected point A will define a distance h = OK; and it suffices to graduate OK in terms of surface i.e.: in terms of H_e, the altitude of the heavenly body. It is to be noted that all points P are situated onto a straight line $u + v - 1 = 0$. However it is of more interest to note that **axis** OP is defined by its angular coefficient :

$$\tan \alpha = \tan^2 \frac{P}{2} = \frac{v}{u}$$

The intersection of straight line (1) with OP' perpendicular to OP, i. e. : point R, is obtained by associating to (1) the equation of OP

$$\frac{y}{x} = \frac{-u}{v} \qquad (2)$$

together with the particular equation

$$u + v = 1 \qquad (3)$$

By eliminating u and v from the 3 equations, the locus of points K for various P angles is obtained.

From (1) and (3) we have :

$$(1 - v)\ y - vx - \sin H_e = 0 \quad (4)$$

From (2) and (3) we have :

$$\frac{y - x}{x} = - \frac{(u + v)}{v} = - \frac{1}{v} \text{ from (2) and (3)}$$

Transferring to (4)

$$y^2 + x^2 - \sin H_e (y - x) = 0 \quad (5)$$

By putting $\sin H_e = K$ let us write (5) under the following form :

$$\left(y - \frac{K}{2}\right)^2 + \left(x + \frac{K}{2}\right)^2 - \frac{K^2}{2} = 0 \quad (6)$$

These are equations of circles, the center of which of coordinates

$$\begin{cases} \alpha = - \dfrac{K}{2} \\ \beta = + \dfrac{K}{2} \end{cases}$$

is on the bissector of yox', and radius $R = \dfrac{K}{\sqrt{2}}$.

From the above the construction of the diagram is easily conceived. The right hand side angle xoy of the 2 rectangular axes permits to plot point A using its cartesian coordinates

$$\begin{cases} x = \cos (D + \varphi) \\ y = \cos (D - \varphi) \end{cases}$$

these two axes being graduated in degrees of (L — D) and (L + D).

The left hand side, angle x'oy is splited by means of radial lines through the origin of angular coefficient $\tan^2 \frac{P}{2}$ reckoned from OY. It is not necessary to draw the system of circles of which the radii are $\frac{\sin H_e}{\sqrt{2}}$ because AK, the perpendicular to OP', intersects the bissector precisely at point N, the extremity of the diameter of a circle, i. e. : at a distance ON from O

$$ON = 2\frac{\sin H_e}{\sqrt{2}} = \sqrt{2} \sin H_e$$

Consequently it suffices to graduate the bissector straight off in terms of altitude of the heavenly body.

In the attached example, the axes of the left hand side angle x'oy have been divided straight off in terms of Hour Angle. After having plotted point A on the right hand side by means of its coordinates $D + \varphi$ and $D - \varphi$ it suffices to drop AH, the perpendicular from A onto OP' and the value of the D.R. (estimated) altitude is read at H on the bissector of the second quadrant. It is noted that AH, the perpendicular of the plane triangle OAP in lieu of the spherical triangle, is given by :

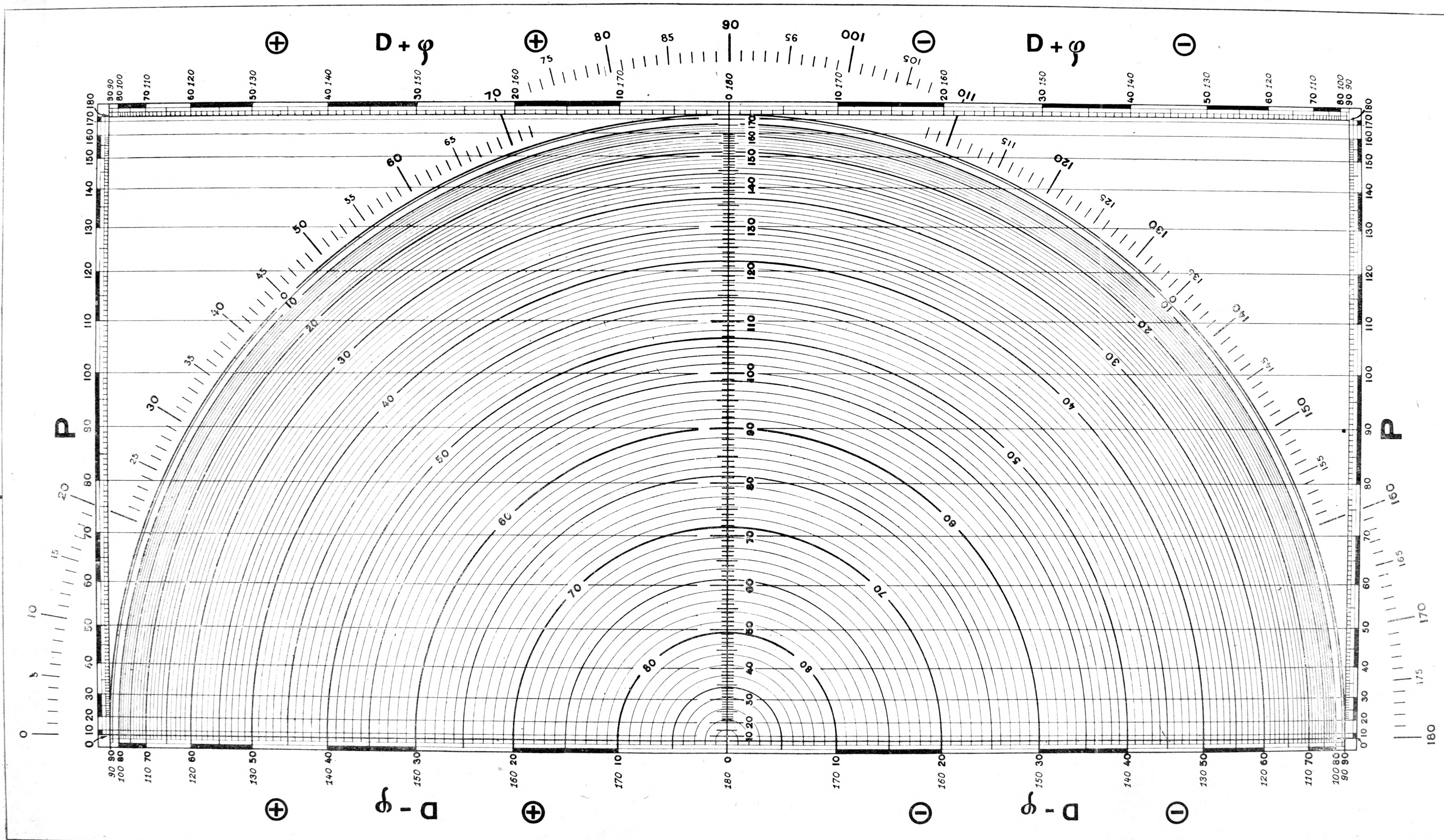
$$OK = OH \cos (45 - \alpha) = \sqrt{2} \sin H_e \cos (45 - \alpha)$$

when

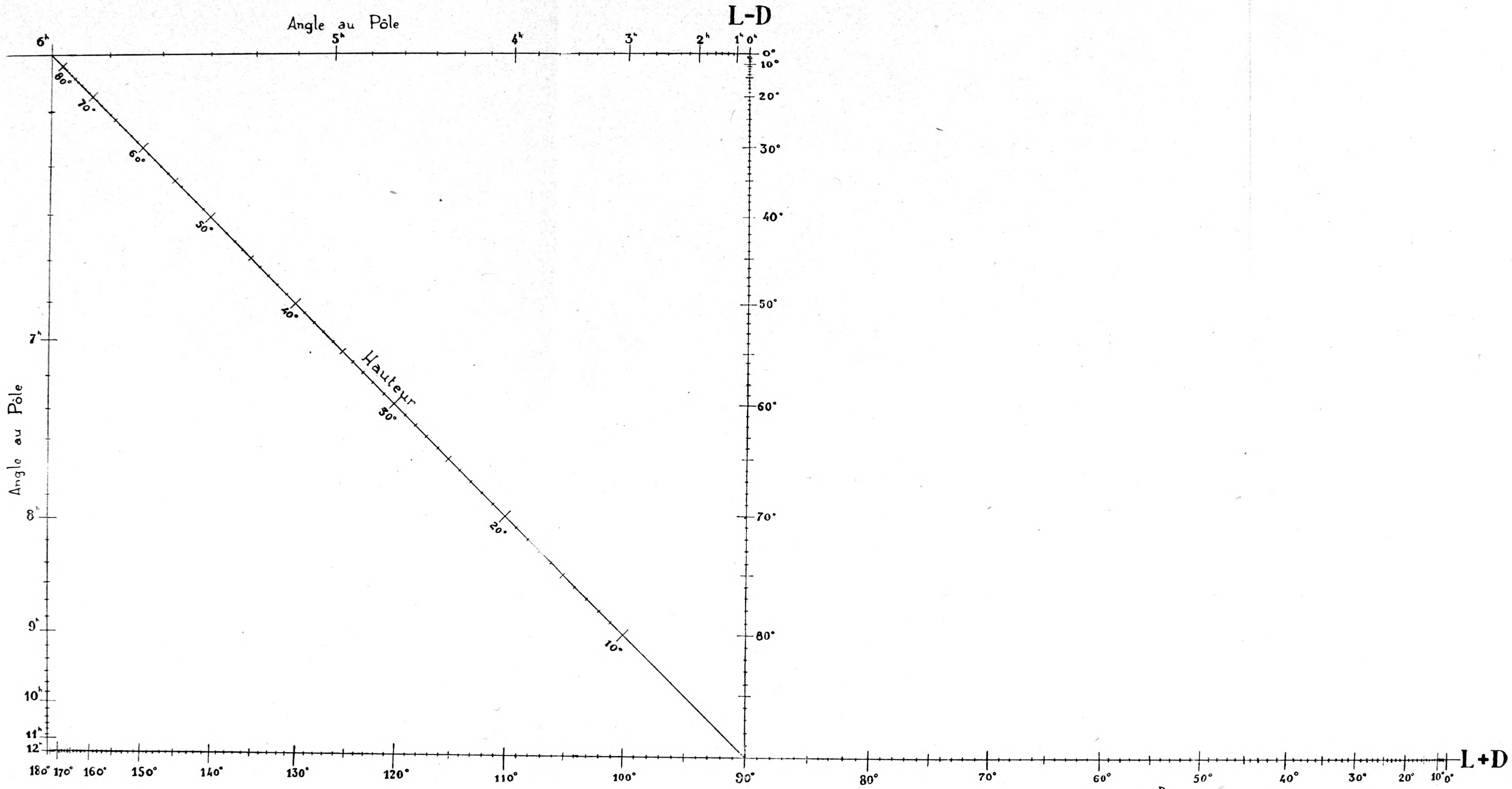
$$\tan \alpha = \tan^2 \frac{P}{2} = \frac{Y}{X}$$

Owing to the restricted dimensions of the diagram, this graphic illustration of the property of the transformed formula has not, under its present form which gives an accuracy to one degree only, a large practical interest. However it might provide interesting results under the form of an apparatus fitted with a movable alidade and a divided limb.

ABAQUE D'AZIMUT



ABAQUE DE HAUTEUR



Soient L la latitude du lieu, D la déclinaison et P l'angle au pôle de l'Astre   Calculer $L-D$ et $L+D$

1°.– Joindre l'origine des axes au point de l'echelle "Angle au Pôle" marqué P

2°.– Construire le point A dont les projections sur les axes sont marquées $L+D$ et $L-D$

3°.– Mener de A la perpendiculaire sur OP et le prolonger jusqu'en H sur l'echelle des hauteurs

La hauteur cherchée est marquée en H