NEW TABLE FOR COMPUTING SUMNER LINE
FROM D.R. POSITION

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Introduction to a New Navigation Table for computing
sumner line from D.R. position.

Object and characteristics.

The object of this table is to calculate the altitude of a celestial body, from the D.R.
position, in terms of the time by means of logarithms, using a single trigonometric function
i.e. : the Haversine only, after a suitable transformation of the fundamental formula for
the altitude.

Such process has the following advantages :
1) The calculation formula is strictly symmetrical ;
2) The number of logarithms required is reduced to four ;
3) Any special rule for signs is eliminated ;
4) The calculating equipment is reduced to a single table of a few pages only, consisting
of a single function whose variations are clearly shown by the arrangement of the table ;
5) Interpolating obligations, unavoidable when using tables, are reduced to a minimum.

Basic formula.

The fundamental formula for $H_e$ the D.R. altitude is

$$\sin H_e = \sin \varphi_e \sin D + \cos \varphi_e \cos D \cos P_e \quad (1)$$

$P_e$ is the hour angle for the D.R. position

$P_e = HA$, when the heavenly body is West,
$P_e = 24 - HA$, when the heavenly body is East.

By adopting this method of notation, a similar way is used as in the modern anglo-saxon
tables the argument of which is the hour angle expressed in degrees of arc.

Let us put

$$X = \cos^2 \frac{P}{2}, \quad Y = \sin^2 \frac{P}{2}$$

Subtracting from 1 each terms of formula (1)

$$1 - \sin H_e = 1 - [(X + Y) \sin \varphi_e \sin D + (X - Y) \cos \varphi_e \cos D]$$

is obtained because :

$$\begin{align*}
X + Y &= 1 = \cos^2 \frac{P}{2} + \sin^2 \frac{P}{2} \\
X - Y &= \cos P = \cos^2 \frac{P}{2} - \sin^2 \frac{P}{2}
\end{align*}$$

Simplifying and substituting $90 - \zeta_e$ for $H_e$ ($\zeta_e$ = zenith distance) ; we have

$$1 - \cos \zeta_e = 1 - [X \cos (D - \varphi) - Y \cos (D + \varphi)]$$

substituting $X + Y$ for 1

$$1 - \cos \zeta_e = X [1 - \cos (D - \varphi)] + Y [1 + \cos (D + \varphi)]$$

By putting :

$$\begin{align*}
Y &= \text{Haversine } P = \frac{1 - \cos P}{2} \\
X &= \text{Haversine } (180 - P) = \frac{1 + \cos P}{2}
\end{align*}$$

or $\cos \text{Haversine } P$,.
\[
\begin{align*}
y &= \text{Haversine} \left( D - \varphi \right) = \frac{1 - \cos \left( D - \varphi \right)}{2} \\
x &= \text{Haversine} \left( 180 - (D + \varphi) \right) = \frac{1 + \cos \left( D + \varphi \right)}{2}
\end{align*}
\]

or co-Haversine \( (D + \varphi) \).

Formula (1) may be written as follows:

\[
\text{Haversine} \, \varphi = X y + Y x \quad (2)
\]

The Haversines and co-Haversines functions vary from 0 to +1 and are always positives as well as all the terms of the formula. Besides:

\[
\text{Haversine} \, N^\circ = \text{co-Haversine} \, (180 - N^\circ)
\]
Operating method.

Formula (2) is calculated by means of a semi-logarithmic method using four logarithms:

\[ \text{Haversine } \zeta_e = A + B \]
\[ \log A = \log X + \log y \]
\[ \log B = \log Y - (-\log x) \]

Each page of the table concerns one degree (of arc) reading from the top and one degree from the bottom. Data relating to acute angles are listed on the left hand side from the top; those relating to obtuse angles are listed on the right hand side from the bottom.

On each page three columns are given: \( \log \text{Haversine} \), \( \log \text{co-Haversine} \) and \( \text{Natural Haversine} \). Natural numbers are thus found by the side of their logarithms, thus avoiding a special table for logarithms.

The data, by which altitude \( H_e \) is obtained, is found on the right hand side at bottom of page, this avoiding reference to the zenith distance \( \zeta_e \).

The selected tabulation interval is one minute of arc, this being the final approximation possible to be obtained at sea.

The interpolation, negligible in many cases, is readily obtained if required by means of a proportional parts table appearing on each page. This table is given for a single tabular difference, usually the maximum tabular difference in the page. The error resulting of such approximation is less than 5/100 of a minute of arc.

The hour angle entries are shown in the same units of arcs as \( D - \varphi \) and \( D + \varphi \), this being the only rational system for obtaining simplicity in the arrangement of table. The method adopted enables one to read \( X \) and \( Y \) at the same time for a single entry with \( P \). \( x \) and \( y \) are thus obtained simultaneously.

To simplify the printing, numbers have been multiplied by \( 10^{10} \), i.e.: 10 has been added to each logarithm characteristic. This means that 9, 8, 7, etc., have been substituted for the negative characteristics 1, 2, 3, etc. For very small angles or for angle very close to 180°, only the necessary number of decimals has been retained, i.e.: the one which determines the 5th place of decimals of the corresponding haversine.

COMPUTING THE AZIMUTH.

Principle.

The five elements formula for azimuth in terms of altitude and time is:

\[ \cos H \cos Z - \sin \varphi \cos D - \sin D \cos \varphi \cos P. \]

This may be written:

\[ l = \cos H \cos Z = \sin(D - \varphi) \cos^2 \frac{P}{2} + \sin(D + \varphi) \sin^2 \frac{P}{2} \]

also, by adopting similar notations as above \( l = \alpha X + \beta Y \).

By putting \( \alpha = \sin (D - \varphi) \)
\( \beta = \sin (D + \varphi) \),

\[ X = \cos^2 \frac{P}{2} \quad Y = \sin^2 \frac{P}{2} \]

\( l = \cos H \cos Z \) is obtained by means of a trigonometric diagram with two parallel axes with divisions for \( \alpha \) and \( \beta \).

From \( l \), \( H \) being known, the value of \( Z \) is easily obtained. The latter being very well determined in the vicinity of the first vertical.
Method of using the diagram.

The azimuth diagram for solving the above formula is given at the end of the book. The bearing of the heavenly body is readily obtained as follows:

Enter the left hand side scale with $D - \varphi$, the right hand scale is entered with $D + \varphi$ from the bottom if positive, from the top if negative. Draw the straight line from A to B, or use a straight edge. Let M be the intersection of AB with the ordinate of P, the hour angle, as read off from the horizontal scale. Take the intersection of the horizontal line through M with one of the concentric circles corresponding to the altitude.

The azimuth is read on the external circle divided into degrees from 0° to 180° from North.

**Example**

\[
\begin{align*}
D &= 21^\circ 36' \ S \\
\varphi &= 1^\circ 40' \ S \\
D - \varphi &= 19^\circ 56' \ S \\
D + \varphi &= 23^\circ 16' \ S \\
\phi_e &= 45^\circ 37' \ E \\
H_e &= 41^\circ 27' \\
Zv &= N \ 117,5 \ E \\
R^i &= 117.5
\end{align*}
\]
Method of calculation.

Data: \( \varphi_e = 27^\circ 28',5 \) N. \( D = 38^\circ 03', 5 \) N. \( P_e = 65^\circ 08' \) W.

(1) \( D - \varphi = 10^\circ 35' \) N. \( (2) \log y = 7,92 \) 770

(3) \( \log X = 9,85 \) 141

(4) \( \log A = 7,78 \) 111

(5) \( \log B = 9,31 \) 140

(6) \( \varphi_e = 0,21 091 \) (7) \( Z = N. 61^\circ 2 \) W. (Diagram)

\( \rho_i = 299 \)

Accuracy of table.

The solution of the standard fundamental formula by means of Friocourt table using only five places of decimals gives a lesser degree of accuracy than that obtained with the present table as shown by the following figures:

<table>
<thead>
<tr>
<th></th>
<th>( H_e = 60^\circ )</th>
<th>( H_e = 75^\circ )</th>
<th>( H_e = 84^\circ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard (Friocourt) Method</td>
<td>( 0',5 )</td>
<td>( 0',9 )</td>
<td>( 2',2 )</td>
</tr>
<tr>
<td>Haversines Table</td>
<td>( 0',4 )</td>
<td>( 0',7 )</td>
<td>( 1',7 )</td>
</tr>
</tbody>
</table>

With a small volume and a quicker calculation the accuracy is higher using the same number of decimals.

Conclusion.

The table established on the above principle is in the press. The advantages of its simplicity and symmetry will be noted on ship bridges where the important equipment of triple-entry tables is not always available, although the superiority of the latter is well known as compared with all other tables based on splitting the spherical triangle into two rectangular ones.

The proposed booklet will consist of about 50 pages easy to use furnishing at sea a useful aid to navigators.

APPENDIX I.

Calculation of the probable error of \( H_e \) when using the Haversine table.

The following is extracted from a discussion concerning the use of the symmetrical formula \( \sin H_e = X_y + xY \) and the probable error by Engineer hydropher George.

First, errors due to second differences are quite negligible and will not be taken into account; only errors due to linear interpolations will be considered.

They involve for each logarithm to 5 places of decimals a probable error of: \( 1,5 \times 10^{-5} \).

If \( [\log X_y] \) is the probable error of \( \log A = \log X_y := [\log X_y] = 3 \times 10^{-5} \times \log A = a \)

If \( \log A \) be known with error \( a \) : then the error of \( A \) is

\[ \frac{Aa}{\log e} \]

(1) Algebraic difference and sum \( D - \varphi \), \( D + \varphi \).
(2) \( \log y \) and \( \log x \) (entry with \( D - \varphi \) and \( D + \varphi \)).
(3) \( \log X \) and \( \log Y \) (single entry with \( P \)).
(4) Arithmetic sum \( (2) + (3) \).
(5) \( A \) and \( B \) are obtained from \( \log A \) and \( \log B \) by using columns \( \log Hav \) and \( Hav \).
(6) Arithmetic sum \( A + B \).
(7) Get out of table bottom in column \( H_e \), right hand side from bottom.
(7) Use diagram.
On the other hand, the use of the table for determining $A$ introduces a new error:

$$1.5 \times 10^{-5} \frac{(1 + A)}{\log e}$$

Consequently the total error of $A$ is

$$[A] = \frac{Aa}{\log e} + 1.5 \times 10^{-5} \frac{(1 + A)}{\log e}$$

(2)

From (1):

$$[A] = [XY] = 1.5 \times 10^{-5} + \frac{4.5 \times 10^{-5}}{\log e} XY$$

$$[B] = [YX] = 10^{-5} (1.5 + \frac{4.5 YX}{\log e})$$

and the total error of $\text{Hav} \, \zeta$ is:

$$[\text{Hav} \, \zeta] = 3 \times 10^{-5} + \frac{4.5 \times 10^{-5}}{\log e} (XY + YX)$$

$$= 10^{-5} \left(3 + \frac{4.5}{\log e} \text{Hav} \, \zeta\right)$$

Now, if $\text{Hav} \, \zeta$ is known with error $\varepsilon$, the resulting error of $\zeta'$ is:

$$2 \varepsilon \frac{2}{\sin \zeta \sin \iota'}$$

The use of table for determining $\zeta$ introduces a new error:

$$\left(\frac{1}{20} + \frac{3 \times 10^{-5}}{\sin \zeta \sin \iota'}\right)$$

and the final error of $\zeta$ is

$$[\zeta]' = \frac{1}{20} + \frac{3 \times 10^{-5}}{\sin \zeta \sin \iota'} + \frac{6 \times 10^{-5}}{\sin \zeta \sin \iota'} + \frac{4.5 \times 10^{-5}}{\log e \sin \iota'}$$

$$[\zeta]' = 0.05 + 0.31 \cos \zeta + 0.36 \tan \frac{\zeta}{2}$$

The following table is obtained from the above formula for various altitudes:

<table>
<thead>
<tr>
<th>Altitude</th>
<th>0°</th>
<th>32°</th>
<th>60°</th>
<th>70°</th>
<th>80°</th>
<th>84°</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error</td>
<td>0.7</td>
<td>0.6</td>
<td>0.8</td>
<td>1.0</td>
<td>1.9</td>
<td>3.2</td>
</tr>
</tbody>
</table>

This is in conformity with the results given above, the error being the double of probable error of $H_e$ with reference to the altitude mean value.

**APPENDIX II.**

**Graphical interpretation of Table.**

When transforming the fundamental formula, the following intermediary form has been obtained:

$$\sin H_e = \cos \zeta_e = X \cos (D - \varphi) - Y \cos (D + \varphi)$$

the latter suggests the idea of a vectorial product or of the double of the surface of a triangle the vertices of which would be:

1) The axes origin $0$.

2) Point $A$ of rectangular coordinates:

$$A \begin{cases} x = \cos (D + \varphi) \\ y = \cos (D - \varphi) \end{cases}$$
3) Point $P$ of rectangular coordinates:

$$
P \left\{ \begin{align*}
\frac{u}{2} &= \cos^2 \frac{P}{2} = X \\
\frac{v}{2} &= \sin^2 \frac{P}{2} = Y
\end{align*} \right.
$$

when $u + v = X + Y = 1$.

For a given point $P$, we have:

$$
\sin H_e = 2 \cdot S = \rho \times h = FO \times h
$$

with

$$
\rho = OP \quad \text{and} \quad h = AH, \quad \text{the perpendicular, and}
$$

$$
\rho = \sqrt{u^2 + v^2}
$$

Let us write that the surface is a constant

$$
2 \cdot S = \rho \times h = uy - vx = \sin H_e \quad (1)
$$

This defines the locus of $A$ for a given point $P$, as the equation of straight line $\Delta$ parallel to $OP$, of which the original ordinate is $OM$. For any given point $P$, a selected point $A$ will define a distance $h = OK$; and it suffices to graduate $OK$ in terms of surface i.e.: in terms of $H_e$, the altitude of the heavenly body. It is to be noted that all points $P$ are situated onto a straight line $u + v - 1 = 0$. However it is of more interest to note that axis $OP$ is defined by its angular coefficient:

$$
\tan \alpha = \tan^2 \frac{P}{2} = \frac{v}{u}
$$

The intersection of straight line (1) with $OP'$ perpendicular to $OP$, i.e.: point $R$, is obtained by associating to (1) the equation of $OP$

$$
\frac{y}{x} = \frac{-u}{v} \quad (2)
$$

together with the particular equation

$$
u + v = 1 \quad (3)
$$

By eliminating $u$ and $v$ from the 3 equations, the locus of points $K$ for various $P$ angles is obtained.

From (1) and (3) we have:

$$
(1 - v) \cdot y - vx - \sin H_e = 0 \quad (4)
$$

From (2) and (3) we have:

$$
\frac{y - x}{x} = -\frac{(u + v)}{v} = -\frac{1}{v} \quad \text{from (2) and (3)}
$$
Transferring to (4)

\[ y^2 + x^2 - \sin H_e (y - x) = 0 \quad (5) \]

By putting \( \sin H_e = K \) let us write (5) under the following form:

\[ (y - \frac{K}{2})^2 + (x + \frac{K}{2})^2 - \frac{K^2}{2} = 0 \quad (6) \]

These are equations of circles, the center of which of coordinates

\[ \begin{align*}
\alpha &= -\frac{K}{2} \\
\beta &= +\frac{K}{2}
\end{align*} \]

is on the bissector of yox', and radius \( R = \frac{K}{\sqrt{2}} \).

From the above the construction of the diagram is easily conceived. The right hand side angle xoy of the 2 rectangular axes permits to plot point A using its cartesian coordinates

\[ \begin{align*}
x &= \cos (D + \varphi) \\
y &= \cos (D - \varphi)
\end{align*} \]

these two axes being graduated in degrees of \((L - D)\) and \((L + D)\).

The left hand side, angle x'oy is split by means of radial lines through the origin of angular coefficient \( \tan^2 \frac{P}{2} \) reckoned from OY. It is not necessary to draw the system of circles of which the radii are \( \sin \frac{H_e}{\sqrt{2}} \) because AK, the perpendicular to OP', intersects the bissector precisely at point N, the extremity of the diameter of a circle, i.e.: at a distance ON from O

\[ ON = \frac{2 \sin H_e}{\sqrt{2}} = \sqrt{2} \sin H_e \]

Consequently it suffices to graduate the bissector straight off in terms of altitude of the heavenly body.

In the attached example, the axes of the left hand side angle x'oy have been divided straight off in terms of Hour Angle. After having plotted point A on the right hand side by means of its coordinates \( D + \varphi \) and \( D - \varphi \) it suffices to drop AH, the perpendicular from A onto OP' and the value of the D.R. (estimated) altitude is read at H on the bissector of the second quadrant. It is noted that AH, the perpendicular of the plane triangle OAP in lieu of the spherical triangle, is given by:

\[ OK = OH \cos (45 - \alpha) = \sqrt{2 \sin H_e \cos (45 - \alpha)} \]

when \( \tan \alpha = \tan^2 \frac{P}{2} = \frac{Y}{X} \)

Owing to the restricted dimensions of the diagram, this graphic illustration of the property of the transformed formula has not, under its present form which gives an accuracy to one degree only, a large practical interest. However it might provide interesting results under the form of an apparatus fitted with a movable alidade and a divided limb.
ABACUS DE HAUTEUR

1°. Passer l'origine des axes au point de l'échelle "Angle au Pôle" marqué P.
2°. Construire le point A dont les projections sur les axes sont marquées L+D et L-D.
3°. Aire de A la perpendiculaire sur OP et de prolonger jusqu'à H sur l'échelle des hauteurs.
La hauteur cherchée est marquée en H.