CHANGE OF PROJECTION BY PROJECTIVE TRANSFORMATION

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Projective transformation consists in replacing ; in the equations for the coordinates of a projection, x by $\frac{P}{R}$, y by $\frac{Q}{R}$, without changing the axes and putting :—

$$P = ax + by + c \, ;$$
$$Q = dx + ey + f \, ;$$
$$R = mx + ny + p.$$

We will assign Index 1 to the coordinates of the first projection in order to distinguish them from those of the second which shall bear no index. This transformation cannot raise the degree of the curves drawn on the first projection, but the circles will be generally transformed into conics.

The first projection was defined by relations of the form :—

$$x_1 = \varphi\,(L, G), \qquad y_1 = \psi\,(L, G),$$

in which L represents latitude and G longitude.

The points of the first projection lying on the axis $x_1 = 0$, will be, in the second, on the straight line $P = 0$. Similarly, the points on the axis $y_1 = 0$, will be placed, in the second projection, on the straight line $Q = 0$. Lastly, the points that were at infinity in the first projection will be placed on the straight line $R = 0$, in the second. This latter property makes it possible to obtain a second projection representing at a convenient distance, an area which was put outside the limits of the chart on the first projection.

I.—CONDITIONS OF CONFORMITY.

The point x, y of the second projection, corresponding to the point x_1, y_1, of the first will be defined by :

$$P = R\,x_1, \qquad Q = R\,y_1.$$

Deriving these equations in relation to L, we shall have :—

$$a\frac{\delta x}{\delta L} + b\frac{\delta y}{\delta L} = (m\frac{\delta x}{\delta L} + n\frac{\delta y}{\delta L})\,x_1 + R\frac{\delta x_1}{\delta L},$$

$$d\frac{\delta x}{\delta L} + e\frac{\delta y}{\delta L} = (m\frac{\delta x}{\delta L} + n\frac{\delta y}{\delta L})\,y_1 + R\frac{\delta y_1}{\delta L}.$$

And similarly, deriving in relation to G, and applying the conditions of conformity assumed to be obtained for both projections :—

$$a\frac{\delta y}{\delta L} - b\frac{\delta x}{\delta L} = (m\frac{\delta y}{\delta L} - n\frac{\delta x}{\delta L})\,x_1 + R\frac{\delta y_1}{\delta L},$$

$$d \frac{\delta y}{\delta L} - e \frac{\delta x}{\delta L} = (m \frac{\delta y}{\delta L} - n \frac{\delta x}{\delta L}) y_1 - R \frac{\delta x_1}{\delta L}.$$

By eliminating $\frac{\delta x_1}{\delta L}$ and $\frac{\delta y_1}{\delta L}$ between these four equations, we obtain :—

$$\frac{\delta x}{\delta L} (mx_1 - ny_1 + e - a) + \frac{\delta y}{\delta L} (nx_1 + my_1 - b - d) = 0,$$
$$\frac{\delta x}{\delta L} (nx_1 + my_1 - b - d) - \frac{\delta y}{\delta L} (mx_1 - ny_1 + e - a) = 0.$$

These two equations can only co-exist if we have :—

$$(mx_1 - ny_1 + e - a)^2 + (nx_1 + my_1 - b - d)^2 = 0,$$

and therefore :—

$$mx_1 - ny_1 + e - a = 0,$$
$$nx_1 + my_1 - b - d = 0.$$

From which we deduce :—

$$(1) \quad x_1 = \frac{m (a - e) + n (b + d)}{m_2 + n^2}, \quad y_1 = \frac{m (b + d) - n (a - e)}{m^2 + n^2}.$$

At this point only of the first projection, the second remains conformal if the first were so. This point shall be undeterminated only if $m = n = 0$. In this case, we shall indeed see that the second projection remains conformal throughout if the first was so.

II.—Retaining the circular form.

This retaining of the circular form may be of interest in case a geometric locus, such as a curve of equal azimuth, had a circular form on the first projection.

Let us take a circle on the first projection, whose centre coordinate are : α_1, β_1 and radius r_1. Its equation was therefore :—

$$(x_1 - \alpha_1)^2 + (y_1 - \beta_1)^2 = r_1^2.$$

It will become by transformation :—

$$(1^a) \quad (P - \alpha_1 R)^2 + (Q - \beta_1 R)^2 = r_1^2 R^2;$$

whose terms of the second degree are :—

$$[ax + by - \alpha_1 (mx + ny)]^2 + [dx + ey - \beta_1 (mx + ny)]^2 - r_1^2 (mx + ny)^2.$$

In order that the conic represented by this equation may be a circle, it is necessary that :—

$$(2) \quad \begin{aligned} &(a - \alpha_1 m)^2 + (d - \beta_1 m)^2 - r_1^2 m^2 = (b - \alpha_1 n)^2 + (e - \beta_1 n)^2 - r_1^2 n^2, \\ &(a - \alpha_1 m)(b - \alpha_1 n) + (d - \beta_1 m)(e - \beta_1 n) - mn\, r_1^2 = 0, \end{aligned}$$

or

$$(2^a) \quad \begin{aligned} &(\alpha_1^2 + \beta_1^2 - r_1^2)(m^2 - n^2) - 2\alpha_1(am - bn) - 2\beta_1(dm - en) + a^2 + d^2 - b^2 - e^2 = 0, \\ &(\alpha_1^2 + \beta_1^2 - r_1^2)mn - \alpha_1(bm + an) - \beta_1(em + dn) + ab + de = 0. \end{aligned}$$

By eliminating $\alpha_1^2 + \beta_1^2 - r_1^2$ between these two equations, we obtain the equation (3) :—

$$(3) \quad \alpha_1(bm-an)+\beta_1(em-dn)=\frac{(ab+ed)(m^2-n^2)-(a^2+d^2-b^2-e^2)mn}{m^2+n^2},$$

which is the question of the chord common to both circles represented by the equations *2 a* where α_1 and β_1 are considered as current coordinates ; it no longer contains r_1. The points α_1, β_1, which are concerned, are therefore on the same straight line and at the intersection of this line with one of the circles defined by the equations *2 a.* These points being always real, as we shall show later, there will be for each value of r_1 two points which may be taken as centres of circles so as to still obtain circles after transformation. There will be nevertheless only one point if the straight line (3) is tangential to the circles (*2a*).

The square of the radius of the second circle (*2a*) is equal to :—

$$\frac{(bm-an)^2+(em-dn)^2}{4\,m^2\,n^2}+r_1^2.$$

On the other hand, the distance from the centre of this circle to the straight line (3) has for its square :—

$$\left(\frac{m^2-n^2}{m^2+n^2}\right)^2\frac{(bm-an)^2+(em-dn)^2}{4\,m^2\,n^2},$$

a quantity which is obviously less than the preceding one and which will be equal to it only if we have both : $r_1 = o$, $\frac{a}{b}=\frac{d}{e}=\frac{m}{n}$.

The two points α_1, β_1 are symmetrical on the straight line (3) with respect to the point :—

$$x=\frac{am+bn}{m^2+n^2}, \qquad y=\frac{dm+en}{m^2+n^2}.$$

The coordinates will be :—

$$\alpha_1=\frac{ma+nb\pm(me-nd)\,\xi}{m^2+n^2}, \quad \beta_1=\frac{md+ne\mp(mb-na)\,\xi}{m^2+n^2},$$

(the higher and lower signs are corresponding).

Transferring these values to the second equation (*2a*), we find :—

$$\xi^2 = 1 + \frac{r_1^2\,(m^2+n^2)}{(mb-na)^2+(me-nd)^2};$$

we shall see later that the negative value must be taken for ξ.

The radius of the circle on the second projection will not be null, as a rule, and there will be no conformity of the projection at the point α_1, β_1. Instead of determining α_1 and β_1 their values may be chosen arbitrarily and two of the coefficients of the transformation may be determined by means of the equations (*2 a*). The determination of m and n would generally lead to equations whose resolution might be fairly arduous.

It will be easier to determine a and d or b and e. For the first coefficients will be found two values symmetrical with respect to the point :—

$$x = m\,\alpha_1 + mn\,r_1^2\,\frac{b - \alpha_1 n}{(b - \alpha_1 n)^2 + (e - \beta_1 n)^2},$$

$$y = m\,\beta_1 + mn\,r_1^2\,\frac{e - \beta_1 n}{(b - \alpha_1 n)^2 + (e - \beta_1 n)^2},$$

on the straight line :—

$$x\,(b - \alpha_1 n) + y\,(e - \beta_1 n) = m\,(b\,\alpha_1 + e\,\beta_1) - mn\,(\alpha_1^2 + \beta_1^2 - r_1^2).$$

The values of a and d will be —

$$a = m\,\alpha_1 + mn\,r_1^2\,\frac{b - \alpha_1 n}{(b - \alpha_1 n)^2 + (e - \beta_1 n)^2} + \xi\,(e - \beta_1 n),$$

$$d = m\,\beta_1 + mn\,r_1^2\,\frac{e - \beta_1 n}{(b - \alpha_1 n)^2 + (e - \beta_1 n)^2} - \xi\,(b - \alpha_1 n)$$

with

$$\xi^2 = 1 + r_1^2\,\frac{m^2 - n^2 - \dfrac{r_1^2\,m^2\,n^2}{(b - \alpha_1 n)^2 + (e - \beta_1 n)^2}}{(b - \alpha_1 n)^2 + (e - \beta_1 n)^2}$$

For ξ the positive value must be taken.

b and e would be calculated likewise.

$$b = n\,\alpha_1 + mn\,r_1^2\,\frac{a - m\,\alpha_1}{(a - m\,\alpha_1)^2 + (d - m\,\beta_1)^2} + \zeta\,(d - m\,\beta_1),$$

$$e = n\,\beta_1 + mn\,r_1\,\frac{d - m\,\beta_1}{(a - m\,\alpha_1)^2 + (d - m\,\beta_1)^2} - \zeta\,(a - m\,\alpha_1).$$

with

$$\zeta^2 = 1 - r_1^2\,\frac{m^2 - n^2 + \dfrac{r_1^2\,m^2\,n^2}{(a - m\,\alpha_1)^2 + (d - m\,\beta_1)^2}}{(a - m\,\alpha_1)^2 + (d - m\,\beta_1)^2}$$

For ζ the negative value must be taken.

In the equations (*2a*) if the signs of α_1, a and b, or β_1, d and e, are changed at the same time, the equations do not change.

These equations furnish therefore two values for α_1, and β_1; which correspond to two values of a and b, or d and e, which are equal and of opposite signs. If d and e or a and b are given, there will then be only one value to be adopted for α_1 and β_1.

If, on the contrary, a and b are deduced from the equations (*2a*) two values will be found for each of these co-efficients, according to whether the positive or negative number will be adopted for ξ, from the value of ξ^2.

Likewise for b and e.

Special cases.

I.—If $r_1 = 0$, ξ^2 will have the value 1; α_1 and β_1 will have the values :—

$$\alpha_1 = \frac{ma + nb \pm (me - nd)}{m^2 + n^2}, \qquad \beta_1 = \frac{md + ne \mp (mb - na)}{m^2 + n^2}.$$

The circle on the second projection will have as equation :—

$$x^2 + y^2 - 2\,\alpha x - 2\,\beta y + \alpha^2 + \beta^2 - \rho^2 = 0,$$

with

$$\alpha = \frac{(\alpha_1 p - c)(a - \alpha_1 m) + (\beta_1 p - f)(d - \beta_1 m)}{(a - \alpha_1 m)^2 + (d - \beta_1 m)^2}$$

$$\beta = \frac{(\alpha_1 p - c)(b - \alpha_1 n) + (\beta_1 p - f)(e - \beta_1 n)}{(a - \alpha_1 m)^2 + (d - \beta_1 m)^2}$$

$$\rho^2 = \frac{(\alpha^2 + \beta^2)[(a - \alpha_1 m)^2 + (d - \beta_1 m)^2] - [(c - \alpha_1 p)^2 + (f - \beta_1 p)^2][(a - \alpha_1 m)^2 + (d - \beta_1 m)^2]}{[(a - \alpha_1 m)^2 + (d - \beta_1 m)^2]^2}$$

or

$$\rho^2 = \frac{[(\alpha_1 p - c)(b - \alpha_1 n) + (\beta_1 p - f)(e - \beta_1 n)]^2 - [(\alpha_1 p - c)(d - \beta_1 m) - (\beta_1 p - f)(a - \alpha_1 m)]^2}{[(a - \alpha_1 m)^2 + (d - \beta_1 m)^2]^2}$$

We have also, with the above values of α_1 and β_1 :—

$$b - \alpha_1 n = \pm (d - \beta_1 m),$$

$$a - \alpha_1 m = \mp (e - \beta_1 n),$$

Consequently ρ will be null for those values of α_1 and β_1, and the second projection will be conformal at this point if the first was so.

With respect to the values of α_1 and β_1, the higher and lower signs $\pm$ correspond. We have therefore :—

$$\alpha_{11} = \frac{m\,(a + e) + n\,(b - d)}{m^2 + n^2}, \qquad \beta_{11} = \frac{m\,(d - b) + n\,(a + e)}{m^2 + n^2},$$

and

$$\alpha_{12} = \frac{m\,(a - e) + n\,(b + d)}{m^2 + n^2}, \qquad \beta_{12} = \frac{m\,(d + b) - n\,(a - e)}{m^2 + n^2}.$$

It will be noted that the values of α_{12} and β_{12} are identical with those which we indicated for x and y in Chapter I.

If e is changed to — e and d to — d, we obtain the value α_{11} and β_{11} As, on the other hand, the constant f is not included in these formulae, α_{11} and β_{11} will correspond to the transformation :—

$$x_1 = \frac{ax + by + c}{mx + ny + p}, \qquad y_1 = -\frac{dx + ey + f}{mx + ny + p},$$

That is to the projection symmetrical to the preceding one about the x_1 axis, α_{12} and β_{12} being the only values to be adopted.

a and d or b and e might also be calculated for arbitrary values of α_1 and β_1.

(4) $a = m\,\alpha_1 - \beta_1 + e$

$d = m\,\beta_1 + n\,\alpha_1 - b$

or

(5) $b = n\,\alpha_1 + m\,\beta_1 - b$

$e = n\,\beta_1 - m\,\alpha_1 + a$

In the first case the transformation will be :—

$$x_1 = \frac{(m\,\alpha_1 - n\,\beta_1 + e)\,x + by + c}{mx + ny + p}, \quad y_1 = \frac{(m\,\beta_1 + n\,\alpha_1 - b)\,x + ey + f}{mx + ny + p},$$

In the second :—

$$x_1 = \frac{ax + (n\,\alpha_1 + m\,\beta_1 - d)\,y + c}{mx + ny + p}, \quad y_1 = \frac{dx + (n\,\beta_1 - m\,\alpha_1 + a)\,y + f}{mx + ny + p},$$

We shall thus have, if we determine a and d :—

$$a - \alpha_1\,m = e - n\,\beta_1,$$
$$d - \beta_1\,m = -(b - n\,\alpha_1),$$

and consequently the square of the radius of the circle on the second projection will be :—

$$\rho_2 = \frac{[(\alpha_1 p - c)(b - \alpha_1 n) + (\beta_1 p - f)(e - \beta_1 n)]^2 - [(\alpha_1 p - c)(b - \alpha_1 n) + (\beta_1 p - f)(e - \beta_1 n)]^2}{[(b - \alpha_1\,n)^2 + (e - \beta_1\,n)^2]^2}.$$

We shall again have $\rho = 0$.

It may also be noted that if the values a and d or b and e satisfy these equations, the coordinates α_1 and β_1 may be chosen arbitrarily and the values which might have been deduced from the equations (1) will be found for a and d or b and e.

Therefore, if the first projection is conformal at the point adopted for x_1, y_1, the second will also be conformal at the point x, y, resulting from the transformation, if the equations (1) are verified ; particularly if, x_1 and y_1, being chosen arbitrarily, a and d or b and e, or m and n satisfy these equations.

We thus find the values :—

$$m = \frac{\alpha_1\,(a - e) + \beta_1\,(b - d)}{\alpha_1^2 + \beta_1^2}, \qquad n = \frac{\alpha_1\,(b + d) - \beta_1\,(a - e)}{\alpha_1^2 + \beta_1^2},$$

which also verify the equations (2*a*).

We also find for m and n the values 0, which give finite values for m and n only if :

$$a = e \text{ and } b = -d.$$

The transformation will then be :—

$$x_1 = \frac{ax + by + c}{p}, \qquad y_1 = \frac{-bx + ay + f}{p};$$

and we shall have :—

$$x_1 + iy_1 = \frac{(a - bi)\,(x + iy) + c + fi}{p};$$

which shows that, in this case, the second projection will be conformal at all the points where the first one was.

II.—If $m = 0$, the equations (2) and (3) give :—

$$\alpha_1 = \frac{b}{n} + \frac{d}{n}\sqrt{1 + \frac{n^2 r_1^2}{a^2 + d^2}}$$

$$\beta_1 = \frac{e}{n} - \frac{a}{n}\sqrt{1 + \frac{n^2 r_1^2}{a^2 + d^2}}$$

On the second projection, after transformation, whatever r_1, may be, these points will be on a straight line parallel to the y axis :—

$$X = \frac{p\,(ab + ed) - n\,(ac + fd)}{n\,(a^2 + d^2)}$$

We may also take α_1 and β_1 arbitrarily and calculate a and d or b and e. We shall have :—

$$\begin{aligned} a &= (e - \beta_1 n)\,\xi, \\ d &= (b - \alpha_1 n)\,\xi, \end{aligned} \quad \text{with } \xi^2 = 1 - \frac{r_1^2 n^2}{(b - \alpha_1 n)^2 + (e - \beta_1 n)^2};$$

by taking the positive value for ξ.

It will be necessary that :—

$$(b - \alpha_1 n)^2 + (e - \beta_1 n)^2 \geqslant r_1^2 n^2;$$

or

$$\begin{aligned} b &= n\,\alpha_1 + d\,\zeta, \\ e &= n\,\beta_1 - a\,\zeta, \end{aligned} \quad \text{with } \zeta^2 = 1 + \frac{r_1^2 n^2}{(a - \alpha_1 m)^2 + (d - \beta_1 m)^2}$$

by taking the negative value for ζ.

III.—If $n = 0$, we shall have likewise :—

$$\alpha_1 = \frac{a}{m} - \frac{e}{m}\sqrt{1 + \frac{m^2 r_1^2}{e^2 + b^2}},$$

$$\beta_1 = \frac{d}{m} + \frac{b}{m}\sqrt{1 + \frac{m^2 r_1^2}{e^2 + b^2}};$$

and all these points will lie, whatever r_1 may be, on the second projection, on the parallel to the X axis :—

$$Y = \frac{p\,(ab + ed) - m\,(bc + fe)}{m\,(e^2 + b^2)}.$$

The equations (2*a*) become :—

$$(\alpha_1^2 + \beta_1^2 - r_1^2)\,m^2 - 2m\,(a\alpha_1 + d\beta_1) + a^2 + d^2 - b^2 - e^2 = 0,$$
$$(\alpha_1 b + \beta_1 e)\,m = ab + de$$

m may be drawn therefrom, if α_1 and β_1 be considered as known :—

$$m = \frac{ab + de}{\alpha_1 b + \beta_1 e}.$$

On the other hand, if m is eliminated between the two equations (2*a*), there comes, after simplification :—

$$(\alpha_1 d - \beta_1 a)^2 - (\alpha_1 b + \beta_1 e)^2 = r_1^2\,\frac{ab + de}{e^2 + b^2},$$

which equation shows that, if any value be taken for m ; the point α_1, β_1 must be selected on a hyperbola whose equation is independent of m, and at the intersection of this hyperbola with the straight line :—

$$\alpha_1 b + \beta_1 e = \frac{ab + de}{m}.$$

So, we shall have :—

$$m\,\alpha_1 = a - e\sqrt{1 + \frac{r_1^2\, m^2}{e^2 + b^2}},$$

$$m\,\beta_1 = d + b\sqrt{1 + \frac{r_1^2\, m^2}{e^2 + b^2}}.$$

If α_1 and β_1, be arbitrarily known, a value of m may be deduced therefrom. If, on the contrary, m be considered as known, α_1 and β_1, may be deduced therefrom.

We might also calculate a and d and b and e for arbitrary values of α_1 and β_1 as in the previous case. We shall find :—

$$a = m\,\alpha_1 + e\sqrt{1 + \frac{r_1^2\, m^2}{e^2 + b^2}},$$

$$d = m\,\beta_1 - b\sqrt{1 + \frac{r_1^2\, m^2}{e^2 + b^2}};$$

and likewise for e and b.

IV.—If m = n = o, the equations (2) will be reduced to :—

$$a^2 + d^2 = b^2 + e^2,$$
$$ab + ed = 0,$$

whose solution is :—

$$d = -\,b, \qquad e = +\,a.$$

It may be verified that in this case the transformation will be :—

$$x_1 + iy_1 = \frac{(x + iy)(a - bi) + c + fi}{p}.$$

The second projection remains therefore conformal if the first were so. Only a change of origin, axes and scale, has actually been effected.

V.—If a = d = o, the equations (*2a*) and (3) may be written :—

$$\alpha_1^2 + \beta_1^2 - r_1^2 = \frac{b^2 + e^2}{m^2 + n^2},$$

$$\alpha_1 b + \beta_1 e = n\,\frac{b^2 + e^2}{m^2 + n^2},$$

if m and n $\neq$ o,

from which are deduced :—

$$\alpha_1 = \frac{b\,n}{m^2 + n^2} - \frac{e}{m^2 + n^2}\sqrt{m^2 + r_1^2\frac{(m^2 + n^2)^2}{b^2 + e^2}},$$

$$\beta_1 = \frac{e\,n}{m^2 + n^2} + \frac{b}{m^2 + n^2}\sqrt{m^2 + r_1^2\frac{(m^2 + n^2)^2}{b^2 + e^2}}.$$

The circle will then have for equation on the second projection :—

$$m^2(x^2+y^2)(b^2+e^2) - 2mx\left[n(bc+ef) - p(b^2+e^2) - (ec-bf)\sqrt{m^2+r_1^2\frac{(m^2+n^2)^2}{b^2+e^2}}\right]$$

$$+2y\left[m^2(bc+ef)+n(ec-bf)\sqrt{m^2+r_1^2\frac{(m^2+n^2)}{b^2+e^2}}\right]+(m^2+n^2)(c^2+f^2)+p^2(b^2+e^2)$$

$$-2p\left[n(bc+ef)-(ec-bf)\sqrt{m^2+r_1^2\frac{(m^2+n^2)^2}{b^2+e^2}}\right] = 0.$$

It is impossible to have at the same time $a = d = b = e = 0$, because x_1, y_1 ought then to lie on the straight line :

$$fx_1 - ey_1 = 0.$$

VI.—If we have besides $n = 0$; therefore if $a = d = n = 0$:—

$$\alpha_1 = -\frac{e}{m}\sqrt{1 + \frac{r_1^2}{b^2 + e^2}},$$

$$\beta_1 = \frac{b}{m}\sqrt{1 + \frac{r_1^2}{b^2 + e^2}}.$$

The transformation is then :—

$$x_1 = \frac{by + c}{mx + p}, \qquad y_1 = \frac{ey + f}{mx + p};$$

and the circle equation will be on the second projection :—

$$\left(x + \frac{p}{m} + \frac{ec - bf}{b^2 + e^2}\sqrt{1 + \frac{r_1^2\, m^2}{b^2 + e^2}}\right)^2 + \left(y + \frac{bc + ef}{b^2 + e^2}\right)^2 = r_1^2\, m^2\frac{(ec-bp)^2}{(b^2 + e)^2}.$$

If $r_1 = 0$, this circle will also have a radius equal to nil; its equation will be:—

$$\left(x + \frac{p}{m} + \frac{ec - bf}{b^2 + e^2}\right)^2 + \left(y + \frac{bc + ef}{b^2 + e^2}\right)^2 = 0;$$

and we shall have :—

$$\alpha_1 = -\frac{e}{m}, \qquad \beta_1 = \frac{b}{m}.$$

At this point the second projection is still conformal.

It may also be verified that, for this point α_1, β_1, we have in this case :—

$$(\alpha_1 + i\,\beta_1)\left[m(\alpha_1 + \beta_1 i) + p\right] = c + if.$$

VII.—If we have $\frac{m}{n} = \frac{a}{b} = \frac{d}{e}$, without r_1 being nil, the equations (2*a*) become :—

$$m^2(\alpha_1^2 + \beta_1^2 - r_1^2) + a^2 + d^2 = 0,$$

$$m^2(\alpha_1^2 + \beta_1^2 - r_1^2) - 2m(a\,\alpha_1 + d\,\beta_1) + a^2 + d^2 = 0,$$

which can co-exist only if :—

$$a\,\alpha_1 + d\,\beta_1 = 0.$$

These conditions would require between x_1 and y_1 the relation —

$$x_1 (fn - ep) - y_1 (cn - bp) = bf + ec\ ;$$

which cannot be contemplated, whether r_1 be nil or not.

III.—CONDIONS REQUIRED FOR A CIRCLE TO BECOME A STRAIGHT LINE.

In order that the conic ($1a$) may become a straight line through projective transformation, all the terms of the second degree must have their coefficients equal to nil. We shall therefore have the following equations :—

$$(6)\quad (a - \alpha_1 m)^2 + (d - \beta_1 m)^2 = m^2 r_1^2,$$
$$(b - \alpha_1 n)^2 + (e - \beta_1 n)^2 = n^2 r_1^2,$$
$$(a - \alpha_1 m)(b - \alpha_1 n) + (d - \beta_1 m)(e - \beta_1 n) = m\, n\, r_1^2;$$

which may be put in the form :—

$$(6^a)\quad \alpha_1^2 + \beta_1^2 - r_1^2 = 2\,\frac{a\alpha_1 + d\beta_1}{m} - \frac{a^2 + d^2}{m^2} = 2\,\frac{b\alpha_1 + e\beta_1}{n} - \frac{b^2 + e^2}{n^2}$$
$$= \alpha_1 \frac{bm + an}{mn} + \beta_1 \frac{em + dn}{mn} - \frac{ab + de}{mn}.$$

By equating with the last equality half the total of the second and the third, we shall have :—

$$(7)\quad n^2 (a^2 + d^2) + m^2 (b^2 + e^2) - 2\, mn\, (ab + ed) = 0,$$

or

$$(bm - an)^2 + (em - dn)^2 = 0,$$

a relation which can only exist if m and n be different from zero :—

$$\frac{a}{b} = \frac{d}{e} = \frac{m}{n}$$

The transformation will then be :—

$$x_1 = \frac{a}{m} + \frac{c - a\frac{p}{m}}{mx + ny + p}, \qquad y_1 = \frac{d}{m} + \frac{f - d\frac{p}{m}}{mx + ny + p}.$$

This would require the point x_1, y_1 to lie on an equation straight line :—

$$x_1 \left(f - d\frac{p}{m}\right) - y_1 \left(c - a\frac{p}{m}\right) + \frac{dc - af}{m} = 0\ ;$$

which cannot be assumed. Consequently, we may see that the conic ($1a$) will never become a straight line through projective transformation.

If m or n where nil, the transformation would comprise only y or x, which could not be contemplated.

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