THE CALCULATION OF HARMONIC CONSTANTS FROM NON-HARMONIC DATA.

by

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1. *Introduction.*

It occasionally happens that the best situation for a tide gauge is not the desired point for which tidal predictions are desired, as in the case where the gauge is up the river and the predictions are required for the bar of the river. Observations at the latter place may suffice to give tidal differences in the forms customarily found in the tide tables for secondary or subsidiary ports. One method is to predict for the site of the gauge and then to amend the predictions for the desired place, but this entails a double set of predictions being prepared, of which one only is to be printed. The adaptation of the harmonic constants so that a direct prediction can be made is a more satisfactory method. This article shows how the problem may be solved in cases where the tidal differences are not indicative of very simple changes : it is, of course, needless to use the methods described if it is obvious that all tides at the subsidiary place can be obtained by adding a constant number of minutes to the tidal predictions at the place of the gauge, for the changes required in the harmonic constants are then very simple indeed. The methods are not applicable to places where there is a large diurnal tide, but they have special value where the shallow-water tides are different at the two places. It should be understood that the principal variations in the non-harmonic constants given for a place, in the form of differences on another place, are largely due to the differing shallow-water tides.

The methods here described are of some generality and they can be applied to many other similar types of problems. For instance, they can be used to determine harmonic constants from predictions of high and low waters by expressing them in terms of non-harmonic differences on another standard prediction for which the harmonic constants are known, though there are available somewhat more accurate methods for the direct analysis of high and low waters, in one of which the formulae here given are actually used as the basis of analysis.

2. *Fundamental formulae.*

The depth of water above datum will be denoted by

$$= \Sigma \, H \cos (\sigma t + V - g) = \Sigma \, A \cos \sigma t + B \sin \sigma t \qquad \ldots\ldots(1)$$

and we shall use the following notation.

$$\left.\begin{array}{ll} t: & \text{the time, in mean solar hours} \\ \sigma: & \text{the speed of a tidal constituent, in radians per mean solar hour} \\ T: & \text{the time of high or low water, in mean solar hours} \\ V: & \text{the phase of the corresponding equilibrium constituent, at } t = 0 \\ g: & \text{the lag of phase of the tidal constituent} \\ H: & \text{the amplitude of the tidal constituent} \\ Z: & \text{the height of high or low water} \end{array}\right\} \quad \ldots\ldots (2)$$

together with

$$\left.\begin{array}{ll} \sigma_0: & \text{the value of } \sigma \text{ for } M_2 = 0.50587 \text{ radians per mean solar hour} \\ n: & \text{the value of } \sigma/\sigma_0 = 1.9768\,\sigma, \text{ a pure number} \\ \theta: & \text{the value of } \sigma_0\,T = 0.5059\,T \text{ radians} \\ T: & \text{equal to } \theta/\sigma_0 = 1.9768\ \theta \text{ hours.} \end{array}\right\} \quad \ldots\ldots (3)$$

The times of high or low water are given when the gradient is zero, so that we have

$$\Sigma\ \sigma\ A \sin \sigma\ T - \Sigma\ \sigma\ B \cos \sigma\ T = 0 \qquad \ldots\ldots (4)$$

and if the mean lunar constituent is supposed to be dominant, then we may use

$$\Sigma\ n\ A \sin n\ \theta - \Sigma\ n\ B \cos n\ \theta = 0 \qquad \ldots\ldots (5)$$

$$Z = \Sigma\ A \cos n\ \theta + \Sigma\ B \sin n\ \theta \qquad \ldots\ldots (6)$$

It will also be supposed that the origin of time is close to high water or to low water as the case may be, so that we can regard θ as small, and on expanding $\cos n\,\theta$ and $\sin n\,\theta$ in terms of $n\,\theta$ we obtain

$$\theta\,\Sigma\,n^2\,A = \Sigma\,n\,B - \frac{1}{2}\,\theta^2\Sigma\,n^3\,B + \frac{1}{6}\,\theta^3\Sigma\,n^4\,A - \cdots\cdots \qquad \ldots\ldots (7)$$

$$Z = \Sigma\,A + \theta\,\Sigma\,nB - \frac{1}{2}\,\theta^2\Sigma\,n^2\,A - \frac{1}{6}\,\theta^3\,\Sigma\,n^3\,B + \frac{1}{24}\,\theta^4\Sigma\,n^4 A + \cdots\cdots \qquad (8)$$

By elimination, either of $\frac{1}{2}\,\theta^2\,\Sigma\,n^2\,A$ or $\theta\,\Sigma\,nB$, using (7), we obtain the alternative expressions

$$Z = \Sigma\,A + \frac{1}{2}\,\theta\,\Sigma\,nB + \frac{1}{12}\,\theta^3\,\Sigma\,n^3\,B - \frac{1}{24}\,\theta^4\,\Sigma\,n^4\,A \qquad \ldots\ldots (9)$$

$$Z = \Sigma\,A + \frac{1}{2}\,\theta^2\,\Sigma\,n^2\,A + \frac{1}{3}\,\theta^3\,\Sigma\,n^3\,B - \frac{1}{8}\,\theta^4\,\Sigma\,n^4\,A \qquad \ldots\ldots (10)$$

3. *Types of data.*

The data which will be utilised may be regarded as referring to average conditions in the synodical month from new moon to new moon. Those constituents which depend upon lunar or solar distance or declination will thus be left out of consideration, for in the average month their effects will be annulled. Thus the only constituents which need to be considered are those whose arguments are intimately related to the arguments of M_2 and S_2, the principal lunar and solar semidiurnal constituents ; that is, we shall take those constituents for which V is a linear combination of v and v', the values of V for M_2 and S_2, respectively. These are as follows :

Constituent	V	*Constituent*	V
M_2	v	M_4	2 v
S_2	v'	MS_4	v + v'
		S_4	2 v'
$2\ SM_2$	2 v' − v	M_6	3 v
μ_2	2 v − v'	$2\ MS_6$	2 v + v'
		$2\ SM_6$	v + 2 v'

.... (11)

We shall choose conditions so that we have four sets of observations at intervals of 180° in v or in v', and for precision we shall define four standard phases as follows :

Phase number	V of M_2	V of S_2
0	v	v'
1	v	v' + 180°
2	v + 180°	v'
3	v + 180°	v' + 180°

.... (12)

where v and v' are now taken to be special values of V for M_2 and S_2 respectively.

One special case in which the phases are related approximately in the manner just specified is that of springs and neaps. There are many variations in the definition of springs and neaps, but for the purposes of these investigations it will be assumed that for average springs we can take

$$v = g \text{ of } M_2 \quad \text{and} \quad v' = g \text{ of } S_2 \qquad \ldots .\ (13)$$

The sequence of phases is then

HWS, HWN, LWS, and LWN.

Another special case arises when data are given for Full and Change of the Moon, and also at the Quadratures. The appropriate conditions are then specified by

$$v = g \text{ of } M_2 \quad \text{and} \quad V = nv \qquad \ldots .\ (14)$$

The sequence of phases is then

HWF, HWQ, LWF, and LWQ,

where the symbol F is used for Full and Change, and the symbol Q for Quadrature.

For convenience, we may use C, D, φ in place of A, B, θ for this case.

4. *Combinations of constituents.*

The analytical processes use values of $A = H \cos (g - V)$ and $B = H \sin (g - V)$ as defined in (1), but a consideration of the values of V given in (11) and (12) shew that certain constituents cannot be separated by the direct processes of analysis. For instance, A_0, M_4 and S_4 have the same phases for all the four special cases of (12).

Consequently we define special values of A and B as used in the analytical processes, as follows :

$$\left.\begin{array}{lll} A_0 & = & \text{the value of mean sea level} \\ A_2 & = & \text{the sum of the values of A for } M_2 \text{ and } 2\,SM_2 \\ A'_2 & = & \text{the sum of the values of A for } S_2 \text{ and } \mu_2 \\ A_4 & = & \text{the sum of the values of A for } M_4 \text{ and } S_4 \\ A_6 & = & \text{the sum of the values of A for } M_6 \text{ and } 2\,SM_6 \\ A'_4 & = & \text{the value of A for } MS_4 \\ A'_6 & = & \text{the value of A for } 2\,MS_6 \end{array}\right\} \quad \ldots\ldots (15)$$

with similar definitions for the corresponding values of B, C, D.

It is assumed that these values are given for the special conditions in (12) associated with phase-number O. For other phases the values will be reversed in sign or will take the same sign as above.

It is necessary, however, to obtain as accurately as possible by inference the values of A and B for the constituents

$$2\,SM_2,\ \mu_2,\ M_4,\ S_4,\ M_6,\ 2\,MS_6 \text{ and } 2\,SM_6 \qquad \ldots\ldots (16\,a)$$

As all these depend upon shallow-water effects it is not possible to use the equilibrium relations, and the inference must depend upon regional relations. If there are sufficient ports on the vicinity for which the constants are known then it is a simple matter to infer the required constants for the place under consideration. If only one place is known then some judgment may have to be exercised as to the possible changes between the known place and the place for which constants are being sought. It may be that the constants for $2\,SM_2$ and μ_2 are not known at the standard place so that they will have to be ignored. It will be possible, however, in most cases to obtain good approximations to M_4 and M_6 by inference, from which we may derive amplitudes and phases of S_4, $2\,MS_6$ and $2\,SM_6$ as follows :

$$\left.\begin{array}{rcll} S_4 / M_4 & = & S_2^2 / M_2^2\ , & S_4^\circ - M_4^\circ = 2\,(S_2^\circ - M_2^\circ) \\ 2\,MS_6 / M_6 & = & 3\,S_2 / M_2\ , & 2\,MS_6^\circ - M_6^\circ = (S_2^\circ - M_2^\circ) \\ 2\,SM_6 / M_6 & = & 3\,S_2^2 / M_2^2\ , & 2\,SM_6^\circ - M_6^\circ = 2\,(S_2^\circ - M_2^\circ) \end{array}\right\} \ldots (16\,b)$$

The analytical processes will ultimately give values for MS_4 and with these it may be possible to improve the approximations as will be shewn in due course.

5. *Z and T at standard port.*

We shall first consider the problem when all the data are given for springs and neaps. It might be thought that values of Z would be given at once from the known values of MHWS, MHWN, MLWS, and MLWN at the standard port, but these values are affected by constituents such as N_2 as may be seen from the expansions given in (7) and (8), for if we eliminate θ then we obtain to a first approximation

$$Z = \Sigma A + 1/2 \, (\Sigma \, nB)^2 \, / \, \Sigma \, n^2A$$

The value of the denominator in the last term is positive for all high waters and the contribution to the numerator for any constituent is also positive so that the value of Z is always too large for high waters on account of such constituents as N_2. For the purpose of analysis, therefore, we must be able to interpret the differences between the standard port and the subsidiary port in terms of known quantities. It may be taken that the mean differences used are practically independent of N_2 and other constituents not connected with M_2 and S_2.

The method now to be described is based on the assumption that T is small so that we can use the expansions given in paragraph 2. The calculation of the values of A and B defined in (15) is a straightforward process as no inference is needed if all constants are known for the standard port. The formulae (7) and (9) require values of powers of n which may be taken as in the following table :

	n	n_2	n^3	n^4
with A_0	0.000	0.000	0.000	0.000
with A_2, B_2	1.000	1.000	1.000	1.000
with A'_2, B'_2	1.035	1.071	1.108	1.147
with A_4, B_4	2.000	4.000	8.00	16.00
with A'_4, B'_4	2.035	4.142	8.43	17.16
with A_6, B_6	3.000	9.000	27.00	81.0
with A'_6, B'_6	3.035	9.212	27.96	84.8

.... (17)

With the help of this table we form the values of

$$n^2A, \quad nB, \quad \text{and} \quad n^3B$$

and then we apply the multipliers + 1 and — 1 as given in the following table and sum the products. As an example, the value of ΣX for HWN is given by

$$X_0 + X_2 - X'_2 + X_4 - X'_4 + X_6 - X'_6.$$

(We denote by X in this table any of the quantities A, n^2A, nB...)

Multipliers for

	HWS	HWN	LWS	LWN
X_0	1	1	1	1
X_2	1	1	—1	—1
X'_2	1	—1	—1	1
X_4	1	1	1	1
X'_4	1	—1	1	—1
X_6	1	1	—1	—1
X'_6	1	—1	—1	1

.... (18)

The same table is used if the data are given for HWF, HWQ, LWF, LWQ.

From the results for each phase we have to use the formula (7) to obtain values of θ. When θ is small enough, as may be taken to be usually the case, we can obtain a close approximation by solving the quadratic equation given by the first three terms of (7), and the solution is given by

$$\theta_1 = \frac{2b}{1+d} \qquad \ldots. \ (19)$$

where

$$b = \Sigma\, nB \,/\, \Sigma\, n^2A, \quad c = \Sigma\, n^4A \,/\, \Sigma\, n^2A, \quad d^2 = 1 + 2\,bc \qquad \ldots (20\,a)$$

With this approximate value of θ we can then compute

$$e = 1/6\ \theta^3\ \Sigma\ n^4A \ /\ \Sigma\ n^2A \qquad \ldots (20\,b)$$

and thence obtain a closer approximation

$$\theta = \theta_1 + e \qquad \ldots (20\,c)$$

With the values of θ for the four phases the corresponding values of T and Z can be obtained from the formulae (3) and (9), and an example of the whole process is given in Example 1, which shews examples of the computations for springs and neaps and for lunar phases. If all the data are for springs and neaps it is unnecessary to compute T and Z for the lunar phases.

6. *Z, T, v, and v' at subsidiary port, data for springs and neaps.*

For the subsidiary port we obtain the values of Z by simply adding to the values of Z at the standard port the differences in heights at the subsidiary port on the standard port. For T, we can similarly add the time-differences to the values of T for the standard port, but as it is possible or even likely that the values of the times so computed will be larger than is advisable with the series (7) and (10), we subtract any convenient value, denoted by c, this being approximately the average of the previously computed times T. It is the residual, however, which we shall now denote by T, and θ is derived by the use of formula (3).

Hence,

$$\left.\begin{array}{lll} Z & = & \text{(Z at standard port), plus the height-difference} \\ T & = & \text{(T at standard port), plus the time-difference, less} \\ & & \quad \text{a constant time (c).} \\ \theta & = & 0.5059\,T. \end{array}\right\} \ldots. \ (21)$$

This deduction of a mean time c corresponds to a shift in the time origin. At the standard port we took

$$v = M_2^\circ\ , \quad v' = S_2^\circ$$

and therefore at the subsidiary port we take

$$v = M_2^\circ + 29^\circ c\ , \quad v' = S_2^\circ + 30^\circ c \qquad \ldots. \ (22)$$

where c is in mean solar hours, and M_2° and S_2° are the values of g at the standard port.

7. *Analysis of data for subsidiary port, springs and neaps.*

The analysis proceeds by successive approximations to the solution of (7) and (10). We shall require from (21) the values of Z, θ, θ^2, and θ^3, for each of the four phases, and we shall also require the values of A and B for each of the constituents in (16*a*), by inference from regional considerations.

Much use is made of combinations which can be described by the following table, in which P and Q are any assigned variables :

$$\left.\begin{array}{ccccc} & P_0 & P_1 & P_2 & P_3 \\ Q_0 & 1 & 1 & -1 & -1 \\ Q_1 & 1 & -1 & -1 & 1 \\ Q_2 & 1 & 1 & 1 & 1 \\ Q_3 & 1 & -1 & 1 & -1 \end{array}\right\} \quad \ldots\ldots \quad (23)$$

The table will be used in both directions, with appropriate variables so that if the values of P are given we can determine the corresponding values of Q by combining the four values of P. Similarly if Q is given we can determine values of P ; for example, we may have

$$Q_1 = P_0 - P_1 - P_2 + P_3$$
$$\text{or} \quad P_1 = Q_0 - Q_1 + Q_2 - Q_3.$$

For the *first approximation* we can obtain crude values of A by writing Z for the four values of P, and applying the formulae so that Q_0, Q_1, Q_3, omitting Q_2, are respectively replaced by $4\,A_2$, $4\,A'_2$, and $4\,A'_4$. Then apply the table vertically by replacing Q by A_2, A'_2, $4\,A_4$ (the inferred value), and $4\,A'_4$, when we get values $\Sigma\, n^2A$ in place of P.

For the *second approximation*, we use the values of $\Sigma\, n_2A$ just obtained and compute

$$0.5\,\theta^2\,\Sigma\, n^2A \qquad \text{and} \qquad \theta\,\Sigma\, n^2A$$

and from these we obtain from (7) and (10), taking only the first terms, the values of

$$\Sigma\, A \qquad \text{and} \qquad \Sigma\, nB$$

The application of the formulae (23) to these, taken as P, and dividing the results by 4, gives values of $A_2 + A_6$, $A'_2 + A'_6$, $A_0 + A_4$, and A'_4 in one case, and values of $B^2 + 3\,B_6$, $1.035\ B'_2 + 3\,B'_6$, $2\,B_4$, and $2.035\ B'_4$ in the other.

From these we correct for the inferred values of A and B and so obtain first approximations to

$$A_0,\ A_2,\ A'_2,\ A_4,\ A'_4,\ A_6,\ A'_6$$

and the corresponding values of B.

For the *third approximation* we use (17) and (18) to obtain values of A and B mutiplied by powers of n, and we proceed as for the second approximation. The values of A and B thus obtained should then be accurate enough but if there is any doubt as to this then a further approximation can be made.

The final values of A and B can then be corrected for the perturbations shewn in (15) so as to give the values for the main constituents.

The derivation of H and g from A and B requires no special instructions as they follow the usual procedure. The values of V follow from (11) and (22).

The final results should be scrutinised to see if they conform to regional relationships as regards ratios of amplitudes and differences of phase-lags. A special case is that of the relation between M_4 and MS_4 for the former was inferred but the latter has been deduced by analysis. If the phase relations obtained are not in conformity with the regional values or if the ratio of amplitudes disagrees with regional values, it is necessary to start afresh with new inferred values for the quarter-diurnal constituents. Normally we expect the relations

$$M_4 \,/\, MS_4 = 1/2\; M_2 \,/\, S_2 \quad \text{and} \quad M_4^\circ - MS_4^\circ = M_2^\circ - S_2^\circ$$

to be in reasonable agreement with the derived values.

8. *Times for lunar phases, heights for springs and neaps.*

In British waters it has been customary to give time differences for the lunar phases of Full and Change and also Quadrature, while height differences have been given for springs and neaps. The deduction of harmonic constants from such data is a much more difficult problem than that already discussed. It will be assumed that the calculations for data at full and change of the moon and at quadratures will be expressed in terms of C, D, and φ where these variables are defined exactly as A, B, and θ, which will be used only for springs and neaps. In the former case the values of V will be taken as equal to nv, where n and v have the same meaning as previously, and for springs and neaps the values of V will be defined as in (11). The values of v and v' will follow from the same formula as has been used previously, (22), and V will be now restricted to have this meaning.

If we define

$$\psi = V - nv$$

then since

$$\left.\begin{array}{ll} A = H\cos(g - V) & C = H\cos(g - nv) \\ B = H\sin(g - V) & D = H\sin(g - nv) \end{array}\right\} \quad \ldots\ldots (24)$$

therefore

$$\left.\begin{array}{ll} A = C\cos\psi + D\sin\psi & C = A\cos\psi - B\sin\psi \\ B = -C\sin\psi + D\cos\psi & D = A\sin\psi + B\cos\psi \end{array}\right\} \quad \ldots\ldots (25)$$

with

$$A = C\ ,\quad B = D \quad \text{for} \quad M_2\ ,\ M_4 \ \text{and}\ M_6 \qquad \ldots\ldots (26)$$

The process of analysis is complicated by the fact that the heights of high and low waters are the principal data for finding A and C, while the times are the principal data for finding B and D, if the data are complete, but in the case considered the data are incomplete because we have no facts given as to the times at springs and neaps or the heights at lunar phases. We are able to obtain by direct methods approximate values of A and D but the approximate values of B and C have to be obtained from the formulae (25) which may be written in the forms

$$\left.\begin{aligned} B &= D \sec \psi - A \tan \psi \\ C &= A \sec \psi - D \tan \psi \end{aligned}\right\} \quad \dots \quad (27)$$

As a beginning in the analysis it is permissible to take C = A for the first approximation in the calculation of D. Example 3 illustrates the procedure. As in the previous example the differences in times and heights are tabulated together with the values of T and Z at the standard port. The mean time difference c is subtracted as before and so we obtain the values of T, Z, v and v' for the subsidiary port. From the values of T we derive the corresponding values of φ for the lunar phases. The inferred values of the minor constituents are tabulated and combined as in Example 2.

For the first approximations we assume that we can replace

$$\Sigma\, nD = \varphi\, \Sigma\, n^2C$$

$$\text{by} \quad \Sigma\, nD = \varphi\, \Sigma\, n^2A$$

and the processes for finding A and D are exactly the same as in Example 1. We then write down the values of A and D for smaller constituents so that we can deal with the values of A and D for the four major constituents M_2, S_2, M_2, and MS_2. Then we use formulae (26) and (27) to obtain B and C for these constituents, and on using values of B and C for the smaller inferred constituents we readily obtain values of the combined constituents, denoted by B_2, B'_2, etc., and C_2, C'_2, etc., as in (15).

The derivation of $\Sigma\, nB$, $\Sigma\, n^2A$, $\Sigma\, n^2C$, $\Sigma\, n^3D$ for use with the second approximation follows the familiar lines ; and the analysis for springs and neaps, and for the lunar phases, follows as in Example 1. The derivation of A and D enables values of B and C to be obtained as in the first approximation. A comparison of the values obtained by the two approximations will indicate whether a third approximation is needed, using more terms in the equations, such as are used in Example 1.

Finally the values of H and g can be obtained from A and B, or from C and D using the values of V given above.

The values of the harmonic constants should be considered in relation to the regional characteristics. It is evident that the inferred shallow-water constituents have a large influence on the results, and it is worth while considering whether a better inference can be made in the light of the constants deduced in the analysis. For example, the relationship between M_4 and MS_4 should certainly be examined, as pointed out in the previous section, for the latter constituent and one phase of the former one are given by the analytical results.

9. *Inference of other harmonic constants.*

When the analysis is completed the results can be used for predictions provided we can infer the harmonic constants for other important constituents whose speeds are not simply related to the speeds of the principal lunar and solar constituents. Where the subsidiary port is close to the standard port it may be sufficient to reduce the amplitudes proportionally to the reductions in M_2, M_4, M_6. More generally, it may be desirable to use regional variations in N_2/M_2, $N_2^\circ - M_2^\circ$ to obtain the best values for the important constituent N_2, and similarly for K_1 and O_1. For other constituents the relationships at neighbouring ports may be used, or the following formulae (based partly upon equilibrium relationships) may be used :

	H	g		H	g
K_2	0.272 S_2	$S_2^\circ + 0.08\,(S_2^\circ - M_2^\circ)$	P_1	0.331 K_1	$K_1^\circ - 0.08\,(K_1^\circ - O_1^\circ)$
T_2	0.059 S_2	$S_2^\circ - 0.04\,(S_2^\circ - M_2^\circ)$	J_1	0.056 K_1	$K_1^\circ + 0.50\,(K_1^\circ - O_1^\circ)$
ν_2	0.194 N_2	$N_2^\circ - 0.13\,(N_2^\circ - M_2^\circ)$	Q_1	0.194 O_1	$O_1^\circ - 0.50\,(K_1^\circ - O_1^\circ)$
2 N_2	0.133 N_2	$N_2^\circ + 1.00\,(N_2^\circ - M_2^\circ)$			
MN_4	2 $(N_2/M_2)\,M_4$	$M_4^\circ + (N_2^\circ - M_2^\circ)$	$2MN_6$	3 $(N_2/M_2)\,M_6$	$M_6^\circ + (N_2^\circ - M_2^\circ)$
MK_4	0.272 MS_4	$MS_4^\circ + (K_2^\circ - S_2^\circ)$			

The above relationships can only be used if the phase differences are all less than, say, 60°. It is always better to use regional relationships if these are well established.

(See examples on the following pages)

EXAMPLE I : T AND Z AT STANDARD PORT

	H	g	SPRINGS AND NEAPS V	V	g−V	A	B	FULL AND QUADRATURE nv	$g-nv$	C	D
A_0	10.00	...	...	...	...	10.00	...	...	...	10.00	...
M_2	7.00	330°	v	330°	000°	7.00	0.00	330.0°	000.0°	7.00	0.00
S_2	3.00	020	v'	020	000	3.00	0.00	341.6	038.4	2.35	1.86
$2SM_2$	0.20	050	$2v'-v$	070	−020	0.19	−0.07	353.1	056.9	0.11	0.17
μ_2	0.20	300	$2v-v'$	280	020	0.19	0.07	318.8	−018.8	0.19	−0.06
M_4	0.50	200	$2v$	300	260	−0.09	−0.49	300.0	260.0	−0.09	−0.49
MS_4	0.40	250	$v+v'$	350	260	−0.07	−0.39	311.6	−61.6	0.19	−0.36
S_4	0.10	300	$2v'$	400	260	−0.02	−0.10	323.1	−23.1	0.09	−0.04
M_6	0.10	100	$3v$	270	190	−0.10	−0.02	270.0	190.0	−0.10	−0.02
$2MS_6$	0.10	150	$2v+v'$	320	190	−0.10	−0.02	281.6	228.4	−0.07	−0.08
$2SM_6$	0.04	200	$v+2v'$	370	190	−0.04	−0.01	293.1	266.9	0.00	−0.04

COMBINATIONS (15) — DERIVATIVES (17)

COMBINATIONS (15)				n^2A	n^4A	nB	n^3B
A_0	10.00						
A_2	7.19	B_2	−0.07	7.19	7.19	−0.07	−0.07
A'_2	3.19	B'_2	0.07	3.42	3.66	0.07	0.08
A_4	−0.11	B_4	−0.59	−0.44	−1.76	−1.18	−4.72
A'_4	−0.07	B'_4	−0.39	−0.29	−1.20	−0.79	−3.28
A_6	−0.14	B_6	−0.03	−1.22	−10.98	−0.09	−0.81
A'_6	−0.10	B'_6	−0.02	−0.92	−8.48	−0.06	−0.56

COMBINATIONS (15)				n^2C	n^4C	nD	n^3D
C_0	10.00						
C_2	7.11	D_2	0.17	7.11	7.11	0.17	0.17
C'_2	2.54	D'_2	1.80	2.72	2.91	1.86	1.99
C_4	0.00	D_4	−0.53	0.00	0.00	−1.06	−4.24
C'_4	0.19	D'_4	−0.36	0.79	3.26	−0.73	−3.04
C_6	−0.10	D_6	−0.06	−0.90	−8.10	−0.18	−1.62
C'_6	−0.07	D'_6	−0.08	−0.64	−5.93	−0.24	−2.23

DATA FOR SPRINGS AND NEAPS

	HWS	HWN	LWS	LWN
ΣA	19.96	13.92	−0.32	6.00
Σn^2A	7.74	3.32	−9.20	−3.62
Σn^4A	−11.57	0.47	5.65	−1.59
ΣnB	−2.12	−0.56	−1.82	−0.22
Σn^3B	−9.36	−1.84	−6.64	−1.04
b	−0.274	−0.169	0.198	0.061
c	−1.209	−0.554	0.722	0.287
d^2	1.662	1.187	1.286	1.035
d	1.289	1.090	1.138	1.017
θ_1	−0.239	−0.162	0.185	0.060
e	0.003	0.000	−0.001	0.000
θ	−0.236	−0.162	0.184	0.060
T	−0.47	−0.32	0.36	0.12
ΣA	19.96	13.92	−0.32	6.00
$\frac{1}{2}\theta\Sigma nB$	0.250	0.045	−0.167	−0.007
$\frac{1}{12}\theta^3\Sigma n^3B$	0.010	0.000	−0.003	0.000
$-\frac{1}{24}\theta^4\Sigma n^4A$	0.001	0.000	0.000	0.000
Z	20.22	13.97	−0.49	5.99

DATA FOR FULL AND QUADRATURE

	HWF	HWQ	LWF	LWQ
ΣC	19.67	14.35	0.71	5.27
Σn^2C	9.08	3.34	−7.50	−4.92
Σn^4C	−0.75	−1.23	7.27	−5.29
ΣnD	−0.18	−1.96	−3.40	1.30
Σn^3D	−8.97	−2.41	−5.59	0.01
b	−0.020	−0.585	0.454	−0.264
c	−0.988	−0.722	0.745	−0.002
d^2	1.039	1.846	1.676	1.001
d	1.019	1.358	1.295	1.000
θ_1	−0.020	−0.496	0.396	−0.264
e	0.000	0.008	−0.010	−0.003
θ	−0.020	−0.488	0.386	−0.267
T	−0.04	−0.96	0.76	−0.53
ΣC	19.67	14.35	0.71	5.27
$\frac{1}{2}\theta\Sigma nD$	0.002	0.478	−0.656	−0.174
$\frac{1}{12}\theta^3\Sigma n^3D$	0.000	0.023	−0.027	0.000
$-\frac{1}{24}\theta^4\Sigma n^4C$	0.000	0.002	−0.007	0.001
Z	19.67	14.85	0.02	5.10

EXAMPLE 2 : ANALYSIS FOR SUBSIDIARY PORT : ALL DATA AT SPRINGS AND NEAPS

	HWS	HWN	LWS	LWN	
DIFFERENCES	1·10	1·22	0·42	0·88	HRS
T, STANDARD PORT	−0·47	−0·32	0·36	0·12	"
SUM	0·63	0·90	0·78	1·00	"
$c = 0{\cdot}83$ T	−0·20	0·07	−0·05	0·17	

	HWS	HWN	LWS	LWN
DIFFERENCES	1·62	1·33	−1·69	0·13
Z, STANDARD PORT	20·22	13·97	−0·49	5·99
SUM Z	21·84	15·30	−2·18	6·12

$v = 330^\circ{\cdot}0 + 29c = 354^\circ{\cdot}1$ $\qquad v' = 020^\circ{\cdot}0 + 30c = 045^\circ{\cdot}1$

INFERRED :-

	$2SM_2$	μ_2	M_4	S_4	M_6	$2MS_6$	$2SM_6$
A	0·20	0·24	−0·59	−0·15	0·06	0·08	0·04
B	−0·15	0·07	−0·12	0·03	0·14	0·12	0·04

$A_6 = 0{\cdot}10$ $\quad A_6' = 0{\cdot}08$ $\quad A_4 = -0{\cdot}74$

$B_6 = 0{\cdot}18$ $\quad B_6' = 0{\cdot}12$

FIRST APPROXIMATION :- $A_2 = 8{\cdot}50$ $\quad A_2' = 3{\cdot}71$ $\quad 4A_4 = -2{\cdot}96$ $\quad 4A_4' = -1{\cdot}76$

SECOND APPROXIMATION :-

PHASE	0	1	2	3
θ	−0·101	0·038	−0·024	0·086
θ^2	0·0102	0·0014	0·0006	0·0074
$\Sigma n^2 A$	7·29	3·39	−16·73	−5·79
Z	21·84	15·30	−2·18	6·12
$-\frac{1}{2}\theta^2 \Sigma n^2 A$	−0·04	0·00	0·00	0·02
sum $= \Sigma A$	21·80	15·30	−2·18	6·14
$\theta \Sigma n^2 A = \Sigma n B$	−0·74	0·13	0·40	−0·49

$A_2 + A_6 = 8{\cdot}28$ $\qquad B_2 + 3B_6 = -0{\cdot}13$

$A_2' + A_6' = 3{\cdot}70$ $\qquad 1{\cdot}035 B_2' + 3B_6' = -0{\cdot}44$

$A_0 + A_4 = 10{\cdot}26$ $\qquad 2B_4 = -0{\cdot}18$

$A_4' = -0{\cdot}45$ $\qquad 2{\cdot}035 B_4' = 0{\cdot}00$

$A_0 = 11{\cdot}00$

$A_2 = 8{\cdot}18$ $\qquad B_2 = -0{\cdot}67$

$A_2' = 3{\cdot}62$ $\qquad B_2' = -0{\cdot}77$

$A_4 = -0{\cdot}74$ $\qquad B_4 = -0{\cdot}09$

$A_4' = -0{\cdot}45$ $\qquad B_4' = 0{\cdot}00$

$A_6 = 0{\cdot}10$ $\qquad B_6 = 0{\cdot}18$

$A_6' = 0{\cdot}08$ $\qquad B_6' = 0{\cdot}12$

n^2A	n^3B	n^4A
8·18	−0·67	8·18
3·88	−0·85	4·16
−2·96	−0·72	−11·84
−1·86	0·00	−7·70
0·90	4·86	8·10
0·74	3·36	6·79

THIRD APPROXIMATION :-

PHASE	0	1	2	3
θ	−0·101	0·038	−0·024	0·086
θ^2	0·0102	0·0014	0·0006	0·0074
θ^3	−0·0010	0·0000	0·0000	0·0006
$\Sigma n^2 A$	8·88	3·36	−18·52	−5·56
$\Sigma n^3 B$	5·98	0·96	−7·42	−2·40
$\Sigma n^4 A$	7·69	1·19	−46·77	−9·47
Z	21·84	15·30	−2·18	6·12
$-\frac{1}{2}\theta^2 \Sigma n^2 A$	−0·04	0·00	0·00	0·02
$-\frac{1}{3}\theta^3 \Sigma n^3 B$	0·00	0·00	0·00	0·00
sum $= \Sigma A$	21·80	15·30	−2·18	6·14
$\theta \Sigma n^2 A$	−0·89	0·13	0·44	−0·48
$\frac{1}{2}\theta^2 \Sigma n^3 B$	0·03	0·00	0·00	−0·01
$-\frac{1}{6}\theta^3 \Sigma n^4 A$	0·00	0·00	0·00	0·00
sum $= \Sigma n B$	−0·86	0·13	0·44	−0·49

$A_2 + A_6 = 8{\cdot}28$ $\qquad B_2 + 3B_6 = -0{\cdot}17$

$A_2' + A_6' = 3{\cdot}70$ $\qquad 1{\cdot}035 B_2' + 3B_6' = -0{\cdot}48$

$A_0 + A_4 = 10{\cdot}26$ $\qquad 2B_4 = -0{\cdot}20$

$A_4' = -0{\cdot}45$ $\qquad 2{\cdot}035 B_4' = -0{\cdot}02$

$A_0 = 11{\cdot}00$

$A_2 = 8{\cdot}18$ $\qquad B_2 = -0{\cdot}71$

$A_2' = 3{\cdot}62$ $\qquad B_2' = -0{\cdot}81$

$A_4 = -0{\cdot}74$ $\qquad B_4 = -0{\cdot}10$

$A_4' = -0{\cdot}45$ $\qquad B_4' = -0{\cdot}01$

$A_6 = 0{\cdot}10$ $\qquad B_6 = 0{\cdot}18$

$A_6' = 0{\cdot}08$ $\qquad B_6' = 0{\cdot}12$

EXAMPLE 3 :- ANALYSIS FOR SUBSIDIARY PORT

DATA :- TIMES AT LUNAR PHASES F, Q
HEIGHTS AT SPRINGS AND NEAPS

	HWS	HWN	LWS	LWN
DIFFERENCES	1·77	1·30	−1·76	0·25
Z, STANDARD PORT	20·22	13·97	−0·49	5·99
SUM Z	21·99	15·27	−2·25	6·24

	HWF	HWQ	LWF	LWQ
DIFFERENCES	1·09	1·02	0·41	0·78 HR.
T, STANDARD PORT	−0·04	−0·96	0·76	−0·53
SUM	1·05	0·06	1·17	0·25
$c = 0·63$ T	0·42	−0·57	0·54	−0·38
φ	0·212	−0·288	0·273	−0·192

$v = 330° + 29°c = 348°·3$ $\quad v' = 020° + 30°c = 038°·9$

ψ FOR S_2 (AND MS_4) $= v' - nv = 038°·9 - 360°·5 = 38°·4$

INFERRED :-	$2SM_2$	μ_2	M_4	S_4	M_6	$2MS_6$	$2SM_6$			
A	0·25	0·25	−0·57	−0·09	0·03	0·03	0·01	$A_6 = 0·04$	$A_6' = 0·03$	$A_4 = -0·66$
B	−0·04	0·04	−0·20	−0·03	0·15	0·15	0·06	$B_6 = 0·21$	$B_6' = 0·15$	
C	0·18	0·24	−0·57	−0·02	0·08	−0·05	−0·04	$C_6 = -0·01$	$C_6' = -0·05$	$C_4 = -0·59$
D	0·18	−0·08	−0·20	−0·10	0·15	0·14	0·04	$D_6 = 0·19$	$D_6' = 0·14$	

FIRST APPROXIMATIONS

$\Sigma n^2 A =$	8·49	3·51	−17·41	−5·15
$\varphi =$	0·212	−0·288	0·273	−0·192
$\Sigma n D =$	1·80	−1·01	−4·75	0·99

$A_2 + A_6 = 8·32$
$A_2' + A_6' = 3·80$
$A_0 + A_4 = 10·32$
$A_4' = -0·44$

		n^2A
A_0	10·48	...
A_2	8·28	8·28
A_2'	3·77	4·04
A_4	−0·66	−2·64
A_4'	−0·44	−1·82
A_6	0·04	0·36
A_6'	0·03	0·27

$D_2 + 3D_6 = 1·14$
$1·035 D_2' + 3D_6' = 2·14$
$2D_4 = -0·74$
$2·035 D_4' = -0·73$

		n^3D
D_2	0·57	0·57
D_2'	1·66	1·83
D_4	−0·37	−2·96
D_4'	−0·36	−3·03
D_6	0·19	5·13
D_6'	0·14	3·91

	A	D
$2SM_2$	0·25	0·18
μ_2	0·25	−0·08
S_4	−0·09	−0·10

	A	D	SEC ψ	TAN ψ	B	C
M_2	8·03	0·39	1·000	0·000	0·39	8·03
S_2	3·52	1·74	1·276	0·793	−0·57	3·11
M_4	−0·57	−0·27	1·000	0·000	−0·27	−0·57
MS_4	−0·44	−0·36	1·276	0·798	−0·11	−0·28

	B	$\dot{C}$
$2SM_2$	−0·04	0·18
μ_2	0·04	0·24
S_4	−0·03	−0·02

		nB
$B_2 =$	0·35	0·35
$B_2' =$	−0·53	−0·55
$B_4 =$	−0·30	−0·60
$B_4' =$	−0·11	−0·23
$B_6 =$	0·21	0·63
$B_6' =$	0·15	0·46

		n^2C
$C_2 =$	8·21	8·21
$C_2' =$	8·35	8·59
$C_4 =$	−0·59	−2·36
$C_4' =$	−0·28	−1·16
$C_6 =$	−0·01	−0·09
$C_6' =$	−0·05	−0·46

SECOND APPROXIMATIONS

	HWS	HWN	LWS	LWN
ΣnB	0·06	0·70	−1·72	−1·44
Σn^2A	8·49	3·51	−17·41	−5·15
$\Sigma nB / \Sigma n^2A = \theta$	0·007	−0·200	0·099	0·280
θ^2	0·000	0·040	0·010	0·078
$-\frac{1}{2}\theta^2 \Sigma n^2A$	0·00	−0·07	0·09	0·20
Z	21·99	15·27	−2·25	6·24
sum = ΣA	21·99	15·20	−2·16	6·44

	HWF	HWQ	LWF	LWQ
Σn^2C	7·73	3·79	−14·77	−6·19
Σn^3D	5·45	0·03	−17·43	0·11
φ	0·212	−0·288	0·273	−0·192
φ^2	0·045	0·083	0·075	0·037
$\varphi \Sigma n^2C$	1·64	−1·09	−4·03	1·19
$\frac{1}{2}\varphi^2 \Sigma n^3D$	0·12	0·00	−0·65	0·00
sum = ΣnD	1·76	−1·09	−4·68	1·19

THE REMAINING PROCESSES ARE AS ILLUSTRATED ABOVE. THE VALUES OF ΣA AND ΣnD FOR THE TWO APPROXIMATIONS ARE IN CLOSE AGREEMENT, SO THAT A THIRD APPROXIMATION IS HARDLY NECESSARY