SIMPLIFYING ASTRONOMICAL FIX IN POLAR AREAS

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Contemporary exigencies have emphasized the strategic and commercial advantages of transpolar travel owing to the shorter distances from one continent to the next. In view of the quasi-permanent character of ice packs, these routes are of particular interest in air navigation, where a swift and easy method of position-fixing is essential. Pending the establishment of some suitable radio-electrical substructure, dependence is primarily upon the rapid obtaining of a position line through astronomical sights, as the experience of the " Aries " showed in 1945. Instruments adapted to this type of navigation were tested during the " Aries " flight, and include the stabilized gyromagnetic compass, automatic astro-compass, averaging and integrating sextant and dead reckoning automatic recorder. It is however believed that the type of nautical document enabling immediate utilization of observations—whether consisting of a diagram or table—is still in need of accurate determination.

The following method makes it possible to plot a position line merely by reading off a triple-entry card for 10° of co-latitude. It is more particularly intended for use in air navigation because it is fast, but involves no loss in nautical accuracy nor intercepts over 30 miles in length.

Present methods.—The so-called " Hinks " method used by Ellsworth in his polar expeditions consisted in taking the pole as an auxiliary position. Here, as one knows, altitude is equal to declination and azimuth to the hour angle. Without referring to a table, the following position line data can therefore be obtained directly :—

$$Hv - He = Hv - D \text{ and } Z = \pi - P$$

if P is the polar angle, but the intercept range of distance may reach considerable values, equal to those of the minutes of co-latitude, and the influence of curvature must largely be taken into account.

Let us designate the co-latitude as C, declination as D and the polar angle of the heavenly body as g. The altitude circle equation then becomes :

$$\text{Sin } he = \text{Cos } C \sin D + \cos D \sin C \cos g$$

A limited expansion of the dead-reckoned altitude h_e can be obtained for a co-latitude point C near the pole, that is an expansion of He in the neighbourhood of C = 0. We then have

$$He = D + C \cos g - \frac{1}{2} C^2 \sin^2 g \tan D + \dots$$

the term of the second order reaching appreciable proportions at 10° from the pole.

Expansion of azimuth is analogous :

$$Z \text{ or } \pi - Z = C \sin g \tan D + \dots$$

One might consider giving Z and He in a triple-entry table with arguments D, C and g. This table, limited to 10° of co-latitude, would only be the equivalent of the more general Tables H. O. 214 for high latitudes. Mr. D. H. Sadler, super-intendent of the " Almanac Office ", supplied a specimen of a triple-entry table in the Institute of Navigation Journal, where, with arguments remaining at D. C and g, the data are the difference C = He — D and azimuth. The only necessary operation in order to apply the Marcq St. Hilaire method consists in subtracting ε read off from the H table from the difference Hv — D established from observation.

Thanks to this simple arithmetical operation, considerable reduction of the triple-entry table is obtained since the influence of D is only effective with regard to terms of the second and third orders. The intercepts, however, in case reckoning is poor, may still be large, and the author justifiably is thus led to consider correction for curvature or the use of a curved template as in the case of the navigators of the " Aries ".

Neither in this latter table nor in the proposed table are the differences $\varepsilon = H - D$ computed by approximate expressions of the expansions, but, as shown further on, by precise equations. It can already clearly be seen, however, that if g is established, as well as $\varepsilon = H - D$ and either D or H that are close enough for terms of the second order, values of g will be obtained with no great difference between them for large intervals in H, whence a double reduction of the table, first owing to the fact that ε is limited to co-latitude value, and secondly because H is only operative at the second order. These are the advantages that have led to the following procedure.

Residual method.—Where : $\varepsilon = Hv - D$ or $Hv = D + \varepsilon$, ε being obtained from observation as regards Hv and ephemerides with reference to D, the applying of Neperian analogies to the position triangle gives :

$$\tan\left(45 - \frac{\varphi}{2}\right) \tan\left(\frac{H+D}{2}\right) = \frac{\cos\frac{Z+g}{2}}{\cos\frac{Z-g}{2}}$$

$$\cotan\left(45 - \frac{\varphi}{2}\right) \tan\left(\frac{H-D}{2}\right) = \frac{\sin\frac{Z-g}{2}}{\sin\frac{Z+g}{2}}$$

i. e., using the previous expressions :

(1) $$\tan\frac{C}{2}.\ \tan\left(H - \frac{\varepsilon}{2}\right) = \tan A$$

(2) $$\cotan\frac{C}{2}.\ \tan\frac{\varepsilon}{2} = \tan B$$

Where : $$\tan A = \frac{\cos\frac{Z+g}{2}}{\cos\frac{Z-g}{2}} \quad \text{and} \quad \tan B = \frac{\sin\frac{Z-g}{2}}{\sin\frac{Z+g}{2}} \qquad (3)$$

Then : $$\tan(45 - A) = \tan\frac{Z}{2}.\ \tan\frac{g}{2}$$

$$\tan(45 - B) = \frac{\tan\frac{g}{2}}{\tan\frac{Z}{2}}$$

are easily obtained.

Hence :

(4) $$\tan(45 - A)\ \tan(45 - B) = \tan^2\frac{g}{2}$$

(5) $$\frac{\tan(45 - A)}{\tan(45 - B)} = \tan^2\frac{Z}{2}$$

Formulae (1), (2), (3), (4) and (5), which are well-known, lead to precise calculation in terms of tangents of the point on the altitude circle lying on the dead reckoned latitude or auxiliary latitude parallel $90° - C$, for an altitude H and a given difference $H - D$.

Meaning of auxiliary angles.—Let us assume an altitude circle of the first species, not containing the pole, i. e. where $Hv > D$ or $\varepsilon > 0$ (Fig. 1). Let Z be the point of the circle defined by its co-ordinates C and g (the polar angle). Let PMAN be the meridian of the heavenly body, α the half-gore containing the circle tangent to the limiting meridian at point L of co-latitude C_I or latitude φ_I. α and φ_I are known to be parameters of this circle connected with H and D by the relationships :

$$\sin\varphi_I = \frac{\sin D}{\sin H}$$

$$\sin\alpha = \frac{\cos H}{\cos D}$$

The figure furthermore shows :

$$\tan \frac{\varepsilon}{2} = \tan \frac{PM}{2}$$

$$\tan (H - \frac{\varepsilon}{2}) = \cotan \frac{PN}{2}$$

It is a known fact that when expressing the power of the pole with reference to the altitude circle,

$$\tan \frac{PM}{2} \cdot \tan \frac{PN}{2} = \tan^2 (45 - \frac{\varphi_1}{2}) \text{ and } \tan \frac{PM}{2} \cotan \frac{PN}{2} = \tan^2 (45 - \frac{\alpha}{2})$$
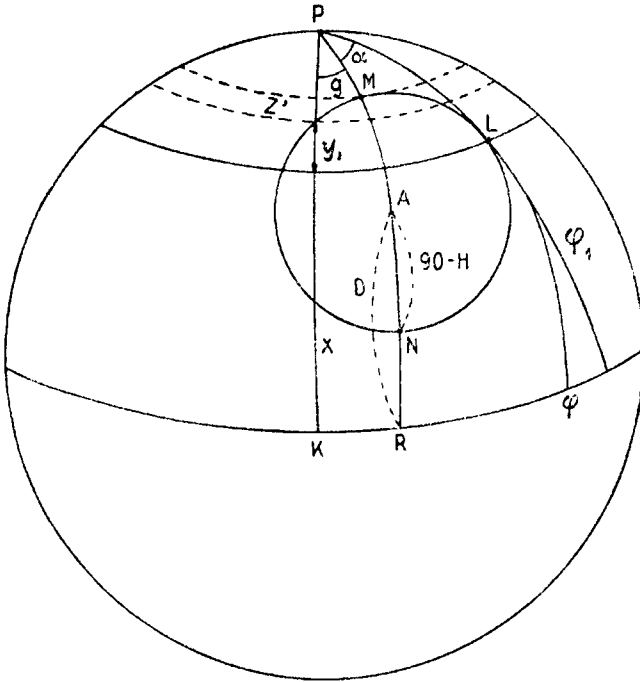
are obtained.

Fig. 1

This power is constant for all circles inscribed within the same gore, and azimuths along a same meridian remain constant in view of the fact that :

(6) $$\sin g = \sin \alpha \cdot \sin Z$$

Therefore :

(7) $$\tan A \tan B = \tan^2 (45 - \frac{\alpha}{2})$$

and by means of equations (3) and (6) :

(8) $$\frac{\tan A}{\tan B} = \frac{\tan^2 \frac{G}{2}}{\tan^2 (45 - \frac{\varphi_1}{2})}$$

The relationship of the second part of the equation (8) is none other than :

$$\cotan^2 (45 - \frac{y_1}{2}) = \frac{\tan^2 \frac{G}{2}}{\tan^2 (45 - \frac{\varphi_1}{2})}$$

y_I being the arc of meridian the meridional part of which is the difference of the meridional parts of points Z' and L of co-latitude C and C_I : $\lambda y_I = \lambda C_I - \lambda C$, it can easily be established that :

$$(9) \qquad \cos y_I = \frac{\cos \alpha}{\cos g}$$

which is a simplified form of the altitude circle equation.

Finally, the significance of auxiliary angles A and B is given by relations (10) and (11) :

$$(10) \qquad \tan A = \frac{\tan (45 - \frac{\alpha}{2})}{\tan (45 - \frac{y_1}{2})}$$

$$(11) \qquad \tan B = \tan (45 - \frac{\alpha}{2}) \, \tan (45 - \frac{y_1}{2})$$

With reference to altitude circles inscribed within the same gore, the relations :

$$(12) \qquad \tan \frac{\varepsilon}{2} = \tan (45 - \frac{\alpha}{2}) \, \tan (45 - \frac{\varphi_1}{2})$$

$$\tan (H - \frac{\varepsilon}{2}) = \frac{\tan (45 - \frac{\alpha}{2})}{\tan (45 - \frac{\varphi_1}{2})}$$

are obtained, that is, on fig. 1 : $\tan \frac{PM}{2} \, \cotan \frac{PL}{2} = \text{constant}$.

Diagrammatic interpretation (fig. 2). — On the stereographic projection polar chart, the determining of point Z' on the dead reckoned parallel of co-latitude C amounts to that of a circle of which points M and N having horizontal tangents are known. If C and g are geographic co-ordinates of Z', the meridians being straight lines, the following is obtained :

$$\overline{Z'K^2} = \overline{MK} \times \overline{KN}$$

with :

$$Z'K = PZ' \sin g = \tan \frac{C}{2} \sin g$$

$$MK = PK - PM = (\tan \frac{C}{2} \cos g - \tan \frac{\varepsilon}{2})$$

$$KN = PN - PK = \cotan (H - \frac{\varepsilon}{2}) - \tan \frac{C}{2} \cos g.$$

Whence the equation :

$$\tan^2 \frac{C}{2} \sin^2 g = (\tan \frac{C}{2} \cos g - \tan \frac{\varepsilon}{2}) \, [\cotan (H - \frac{\varepsilon}{2}) - \tan \frac{C}{2} \cos g]$$

Working this out and simplifying, and substituting p for $\tan \frac{C}{2}$, the equation of the altitude circle in polar co-ordinates is of course obtained :-

$$\tan^2 \frac{C}{2} = \tan \frac{C}{2} \cotan (H - \frac{\varepsilon}{2}) \cos g - \tan \frac{\varepsilon}{2} \cotan (H - \frac{\varepsilon}{2}) + \tan \frac{C}{2} \tan \varepsilon \cos g$$

$$(13) \qquad \cos g = \frac{p^2 + \tan \frac{\varepsilon}{2} \cotan (H - \frac{\varepsilon}{2})}{p \, [\cotan (H - \frac{\varepsilon}{2}) + \tan \frac{\varepsilon}{2}]}$$

On the figure, if O is the centre of the altitude circle and I its projection on the meridians PZ'Z", it can readily be proved that :

$$\cos g = \frac{\overline{PZ'^2} + \overline{PL'^2}}{2 \, PO. \, PZ'} = \frac{\overline{PZ'^2} + \overline{PL'^2}}{(PM + PN) \, PZ'}$$
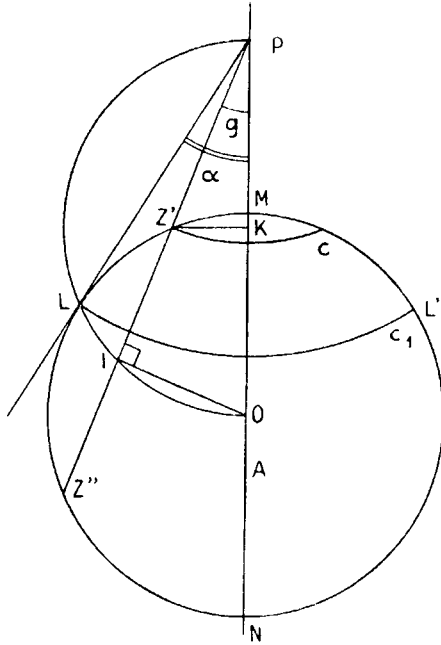
Fig. 2

if L' is the point of contact of the tangent from P :

$$\overline{PL'}^2 = PZ' \times PZ''.$$

Finally :

$$\cos g = \frac{PZ' + PZ''}{PM + PN} = \frac{PI}{PO}$$

Since I is by construction the centre of Z'Z'', if equation (13) is arranged in regular order with respect to p, we get :

(14) $\quad p^2 - p \cos g \ [\cotan (H - \frac{\varepsilon}{2}) + \tan \frac{\varepsilon}{2}] + \tan \frac{\varepsilon}{2} \cotan (H - \frac{\varepsilon}{2}) = 0$

There are two roots in p whose product is :

$$\tan^2 (45 - \frac{\varphi_1}{2}) = \tan \frac{\varepsilon}{2} \cotan (H - \frac{\varepsilon}{2})$$

For practical purposes, it is preferable to fix p and to read off $\pm$ g from a table having as arguments C or p, ε and H.

The values of auxiliary angles A and B can be obtained from equation (13), by replacing tan $\frac{\varepsilon}{2}$ and tan $(H - \frac{\varepsilon}{2})$ by their values in terms of A and B defined by equations (1) and (2). Then :

$$\cos g = \frac{\tan A + \tan B}{1 + \tan A \tan B} = \frac{\sin (A + B)}{\cos (A - B)} = \frac{\tan (45 - \frac{\alpha}{2}) \ (1 + \tan^2 (45 - \frac{y_1}{2}))}{\tan (45 - \frac{y_1}{2}) \ (1 + \tan^2 (45 - \frac{\alpha}{2}))}$$

that is :

(9) $$\cos g = \frac{\cos \alpha}{\cos y_I}$$

This form (9), to which one is led, is the simplest way of expressing the altitude circle equation on a sphere and on a chart, in terms of the parameter α (semi-angle of gore containing the circle) and taking : $y_I = C_I - C$, C_I being the co-latitude of contact parallel LL'.

Utilization.—The method uses formulae (1) and (2) which through (4) and (5) lead up to the values of g and Z, making it possible to plot the position line by means of an auxiliary or dead reckoned parallel of co-latitude C, but in the case of polar areas and stereographic projection, it is interesting to note that :

1. C and ε are small angles having a 10° limit;

2. ε is known by the difference Hv — D immediately obtained with the help of sextant and ephemeris;

3. On the stereopolar chart, the expressions $\tan \dfrac{C}{2}$ and $\tan \dfrac{\varepsilon}{2}$ represent the actual distances from the pole to the apex of the altitude circle and to the point of the circle under consideration.

In principle the table consists of ten cards each referring to 1° of co-latitude, but near the pole, say at less than 5°, the following tables can be included on a single page.

The horizontal argument is the difference ε = Hv — D expressed in an even number of degrees. It will therefore only take up 10 lines at latitude 80°, 5 lines at latitude 85°, etc.

The vertical argument is the approximate true altitude Hv, derived from observation, expressed in even degrees and with very large intervals as accurate interpolation takes place on ε = Hv — D by means of the intercept itself, declination being fixed.

Altitude does not in fact directly appreciably affect the elements sought for, as will be seen later on.

Preparations for use may therefore be made by reading off g and Z from the card beforehand with reference to approximate altitude to within less than 5°. Only the intercept, equivalent to the residual ε = Hv — D, changes.

Tabular intervals.—Co-latitude intervals amount to 1 degree, suitable for accurate air navigation. The difference ε = H — D intervals are also one degree apart, the remainder properly constituting the intercept since it only affects the altitude.

Finally, altitude intervals are very large, longitude g showing only a variation of 6° between altitudes of 10° and 70° at latitude 85°. We have already mentioned that this factor affected longitude only within the limits of the 2nd order Hv — D expansion in relation to C. If this limited expansion is considered :
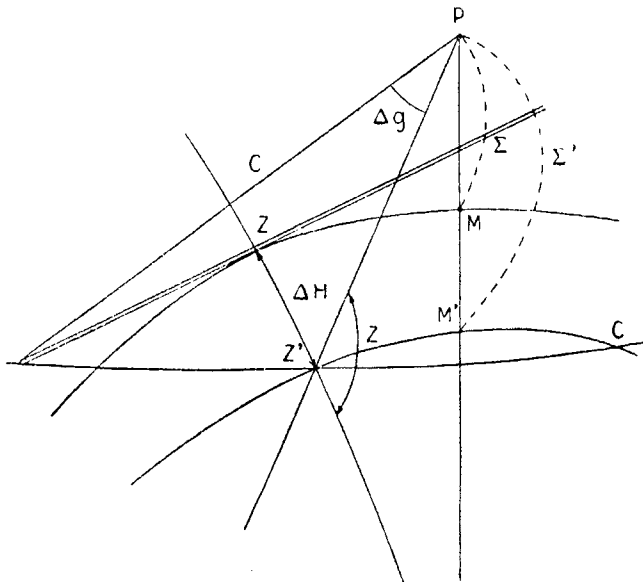

Fig. 3

$$Hv - D = \varepsilon' = C \cos g - \frac{C^2}{2} \sin^2 g \tan H \text{ arc } 1'$$

which can be written as :

$$\varepsilon + \frac{C^2}{2} \tan H \text{ arc } 1' = C \cos g + \frac{C^2}{2} \cos^2 g \tan H \text{ arc } 1'$$

Whence by dividing : with $\frac{d\,\varepsilon}{d\,H} = 0$, ε being taken as constant :-

$$\frac{dg}{d\,H} = \frac{\dfrac{C^2}{2} \dfrac{\text{arc } 1'}{\cos^2 H} (1 - \cos^2 g)}{C \sin g \left(\dfrac{C}{2} \cos g \tan H \text{ arc } 1' - 1\right)}$$

The principal part of this difference is :

$$\frac{dg}{d\,H} = \frac{C}{2} \frac{\text{arc } 1' \sin g}{\cos^2 H}.$$ Now at co-latitude $10°$: $\frac{C}{2}$ arc $1'$ is approximately $\frac{1}{12}$.

A change in g on the order of that of H could therefore only occur in the case of very considerable altitudes impracticable at the pole.

On the other hand a difference in longitude due to a difference affecting $Hv - D = \varepsilon$ would be very appreciable, but as regards utilization we again repeat that it is not to be considered, as from the chosen point read off from the table, the difference $\varepsilon' - \varepsilon = Hv - D - (He - D)$ will be carried over onto the intercept. It could moreover be verified that $\Delta g = \dfrac{\Delta H}{\sin Z, \sin C}$.

At co-latitude $10°$: $\Delta g = \dfrac{\Delta H \times 100}{8 \times \sin Z}$ approximately. Which means that Δg is considerable (fig. 3) when azimuth is slight. But the value ΔH alone interests us since it constitutes the actual intercept. This intercept does not exceed 30' given the reading interval of ε. It should moreover be noted that on co-latitude parallel c, the distance along the parallel is only $\Delta g \sin c$, or $\dfrac{\Delta H}{\sin Z}$.

Negative (H — D) differences.—The tables appear to be concerned with positive $Hv - D$ differences only, that is for $D < H$ in connection with so-called altitude circles of the first species not containing the pole.

It is of course realized that in the case of altitude circles containing the pole, that is, for $D > H$, complete equivalence of factors exists provided g and Z are substituted for one another. It will therefore suffice to take g in the azimuth column and, conversely, Z in the longitude column opposite the same arguments.

The tables therefore cover every conceivable case, equivalence of the altitude circles being determined by the equivalence of maximum azimuth of the circle containing the pole with angle x of the gore containing the circle of the first species.

Meridian transits.—Another form of solution equivalent to equations (4) and (5) is given by the expression :

$$\tan^2 \frac{g}{2} = \frac{\sin \dfrac{C - \varepsilon}{2}}{\sin \dfrac{C + \varepsilon}{2}} \times \frac{\cos \left(\dfrac{C}{2} + (H - \dfrac{\varepsilon}{2})\right)}{\cos \left(\dfrac{C}{2} - (H - \dfrac{\varepsilon}{2})\right)}$$

Therefore $g = 0$ or $180°$ in the following cases :

$$c = \varepsilon \text{ and } \frac{C}{2} + (H - \frac{\varepsilon}{2}) = 90°$$

Which give: $H = 180° - (D + c)$ circle of 2nd species $H < D$; $H = C + D$ circle of 1st species $H > D$.

Specimen tables.—The appended specimen tables are more especially designed for use with the British Special Polar Air Chart in stereographic projection on the scale of $\dfrac{1}{4 \times 10^6}$. The data shown have therefore been computed

with sufficient approximation only for such use. It may be pointed out, however, that the formulae used (in terms of tangents) are particularly accurate and capable of supplying a value of longitude as well as an intercept to the nearest half minute of arc.

The intercepts are moreover invariably under 30' since the H — D difference is given to the nearest degree. And there is no interpolation apart from the carry-over of the difference (Hv — ε) in minutes as an intercept on the vertical of the heavenly body. The tables have been computed for altitudes up to 60° only, these alone being easily used in polar air navigation.

Method of use and examples.— The first operation consists in obtaining HAao and D from the Almanac according to the general rule. The second consists in selecting an auxiliary latitude C° in the vicinity of the probable latitude; knowing that the heavenly body under consideration is at a very roughly approximate altitude Ho, the heavenly body is sighted at an altitude Hv; then Hv — D is obtained and rounded off to an even number of degrees ε. g_0 and Z are read off in the columns for the even values of ε and Ho. The longitude of the computed point is then : G = HAao — g.

The computed point is thus Z' of co-ordinates G and C. From Z' we project in azimuth Z the remaining fraction of ε in minutes towards the heavenly body, i.e. (Hv — D) — ε if an excessive difference ε was taken.

If successive sights are taken, there is no need to reenter the table and it will suffice to carry over for each observation the difference between true altitude and that used in obtaining ε ; the discrepancy constitutes the intercept in the case of each observation. If H is found to be smaller than D, ε is negative, and tabular columns g and Z should be interchanged.
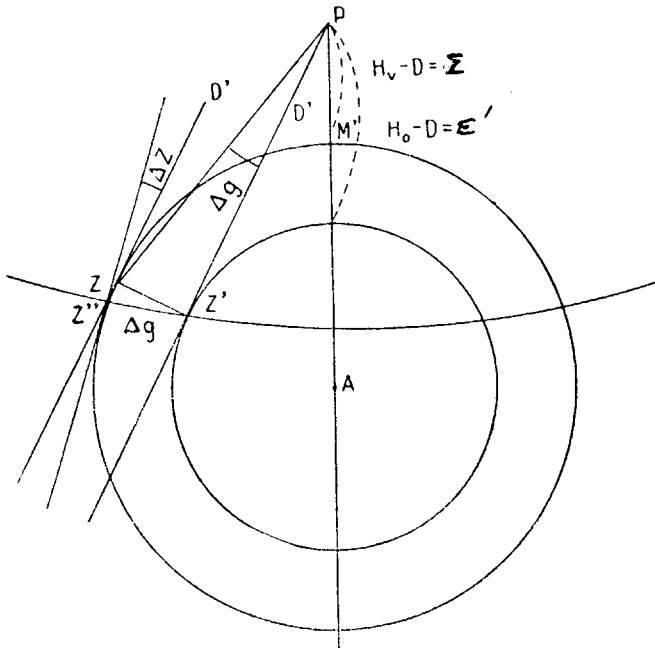

Fig. 4

Examples (Appended diagram).—On the stereographic polar chart, it will be assumed for purposes of simplification in these examples that the heavenly body is on the meridian of Greenwich, i.e. HAao = O and G = g reckoned from 0 to 180° East or West.

Example No. 1.—At a point having as co-ordinates C = 10°, g = 54°, we have a heavenly body of 20° N. declination located on the meridian of Greenwich at true altitude Hv 25°38',5 Zv azimuth = 122°,5.

On the 10° co-latitude card, for Ho = 20° and a difference Hv — D = ε = 5° (remainder 38°5'), the following is read off : g_0 = 58°40' Zo = 118,6.

From point Z' with co-ordinates C = 10°, g_0 = 58°40' W. the intercept is laid out to a length of 38',5 in the direction of Z since Hv is greater than He.

Example No. 2.—At a point having as co-ordinates C = 5°, g = 23° W., we have a heavenly body of D = 59° N. declination on the meridian of Greenwich at true altitude Hv = 63°32',2 and azimuth Zv = 153°,2.

We therefore have Hv — D = ε = 4°32',2. The card shows for ε = 4° and Ho = 60° : g_0 = 34°38' W., Zo = 140°,5. The remainder is Δε = 32',2 which is plotted as an intercept.

Remarks.—I) As the heavenly body is not usually found on the meridian of Greenwich, the value read off from the card represents the polar angle, which, as one is aware, is g = HAag if the heavenly body is in the West; 24 — g = HAag
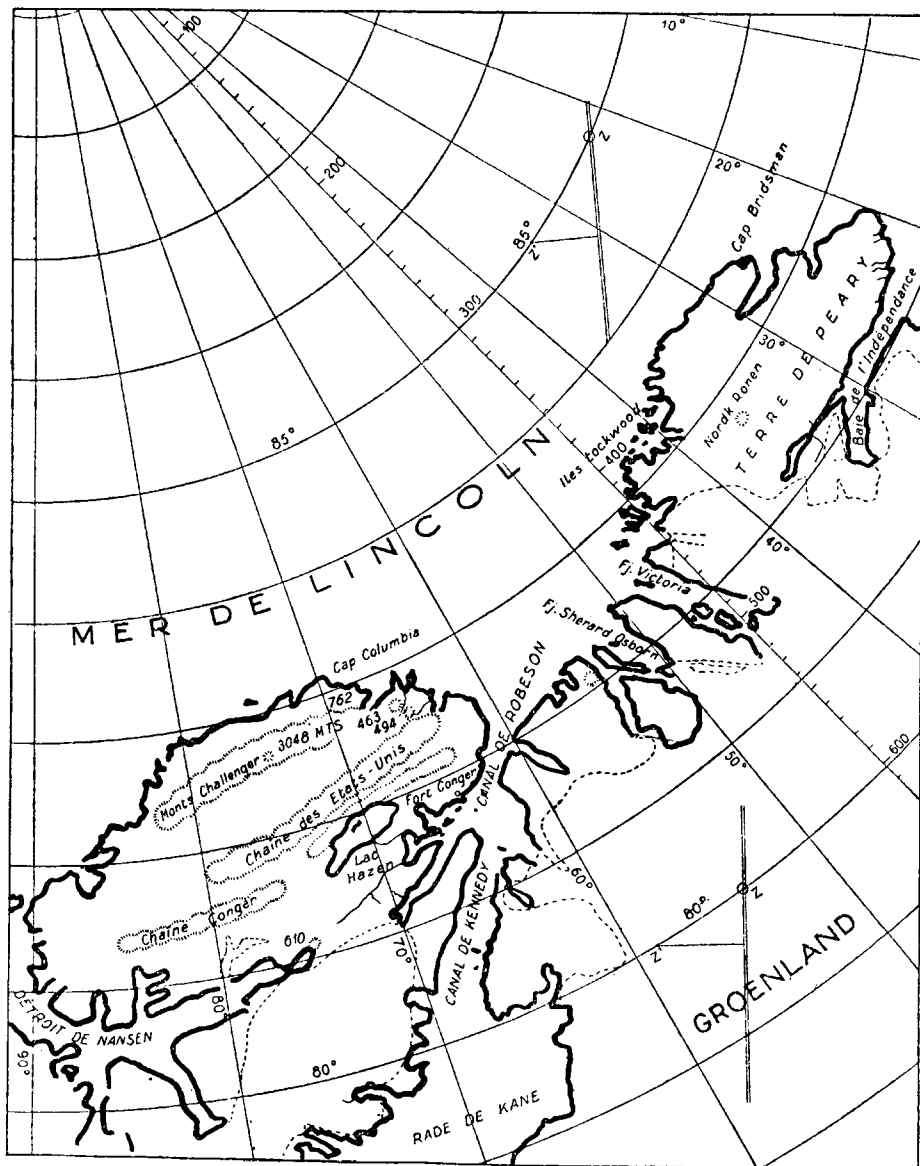

Fig. 5

APPENDIX
SPECIMEN TABLES FOR POLAR NAVIGATION

APPENDICE
SPECIMEN DES TABLES DE NAVIGATION POLAIRE

CO-LATITUDE 5° / COLATITUDE 5°

Approximate altitude H ; Hauteur approchée H : Difference / Différence ε = H — D (1)	0° g	0° Z	10° g	10° Z	20° g	20° Z	30° g	30° Z	40° g	40° Z	50° g	50° Z	60° g	60° Z
1	78°29',4	101°33',0	78°03',6	101°07',2	77°36',2	100°39',6	77°05',2	100°08',1	76°27',4	99°30',0	75°37',2	98°38',0	74°20',0	97°20',0
2	66°28',4	113°36',3	66°04',4	113°12',0	65°38',8	112°46',2	65°10',2	112°17',4	64°35',6	111°41',6	63°49',4	110°54',0	62°40',2	109°41',2
3	53°12',0	126°54',2	52°51',0	126°33',2	52°28',6	126°10',8	52°04',0	125°45',4	51°34',2	125°14',6	50°55',0	124°33',4	49°56',8	123°30',6
4	36°56',4	143°00',8	36°40',6	142°54',0	36°24',2	142°37',4	36°05',8	142°18',4	35°43',8	141°55',4	35°15',0	141°24',6	34°32',8	140°38',2
5	°	180	°	180	°	180	°	180	°	180	°	180	°	180

CO-LATITUDE 10° / COLATITUDE 10°

Approximate altitude H ; Hauteur approchée H : Difference / Différence ε = H — D (1)	0° g	0° Z	10° g	10° Z	20° g	20° Z	30° g	30° Z	40° g	40° Z	50° g	50° Z	60° g	60° Z
1	84°19',0	95°46',0	83°26',4	94°53',4	82°30',6	—	81°27',6	93°57',4	80°10',6	92°53',8	75°50',4	91°35',8	75°37',4	87°09',8
2	78°34',6	101°35',6	77°43',0	100°43',8	76°48',4	—	75°47',0	99°48',6	74°32',4	98°46',4	70°30',0	97°30',0	70°24',2	93°11',0
3	72°42',6	107°02',6	71°52',2	106°42',0	70°59',4	—	70°00',2	105°48',6	68°49',0	104°48',0	64°55',0	103°34',2	64°55',0	99°24',6
4	66°38',2	113°41',0	65°50',0	112°52',6	64°59',6	—	64°03',4	112°01',4	62°56',0	111°03',4	59°17',8	109°52',8	59°17',8	105°55',8
5	60°15',0	120°07',6	59°29',6	119°21',8	58°42',2	—	57°49',6	118°33',4	56°47',0	117°39',0	53°26',8	116°32',6	53°26',8	112°51',0
6	53°24',6	127°00',8	52°42',4	126°18',4	51°59',0	—	51°10',8	125°33',6	50°13',8	124°43',6	47°14',2	123°42',8	47°14',2	120°20',2
7	45°51',6	134°34',4	45°14',2	133°56',8	44°35',4	—	43°53',0	133°17',2	43°02',8	132°32',4	40°27',2	131°38',4	40°27',2	128°39',5
8	37°09',0	143°16',4	36°37',4	142°44',6	36°05',0	—	35°29',4	142°11',4	34°48',2	141°34',0	32°40',6	140°48',8	32°40',6	138°20',0
9	26°04',2	154°16',4	25°41',4	153°53',3	25°18',0	—	24°52',4	153°29',2	24°22',8	153°02',2	22°52',6	152°29',6	22°52',6	150°42',4
10	°	180	°	180	°	180	°	180	°	180	°	180	°	180

(1) N. B.—Whenever ε = H — D is negative or D > H, interchange columns g and Z.

(1) N. B. — Lorsque ε = H — D est négatif ou D > H : permuter les colonnes g et Z.

if the heavenly body is in the East.   P is reckoned from 0 to 180° eastwards and westwards as longitude.

II) It will be noticed that azimuth at the computed point Z' (C, g_0) often considerably differs (13° in the latter case) from azimuth at the accurate point. This is due to the fact that the former azimuth is that of a tangent to an altitude circle whose radius may differ by as much as 30' from that of a true altitude circle even though on the same parallel.   While taking into account the fact that determination of the meridian is inaccurate near the pole on a stereopolar chart, it is nevertheless possible to obtain a suitable azimuth value—generally to less than the nearest degree—by interpolating the azimuth vertically by inspection for remainders of ε. As variation of Z is not linear, a value so derived is merely approximate especially when ε is in the neighbourhood of co-latitude, and azimuth changes rapidly. It is, however, likely that in these extreme cases the bearing of the heavenly body will be obtained directly through use of the astro-compass.   It may furthermore be pointed out that in polar areas, if point Z' is considered as having been obtained as the computed point of Z'D' of inaccurate bearing, the error in azimuth to within the second order approximates Δg, the difference in longitude measured on the chart between Z" and Z' and having a value of $\dfrac{\Delta H}{\sin Z}$ or $\dfrac{\Delta \varepsilon}{\sin Z}$.   In the majority of cases this will not require consideration.

Generalizing of method.— ε and H are the terms that completely determine the position of an altitude circle in all cases.   The particular case involving small co-latitudes limits on the one hand argumentation in C and $\varepsilon = H - D$ and on the other hand enables the problem to be entered with a very rough value of H. But accuracy is retained (insofar as the dead reckoned parallel method is favourable) all over the globe.   A triple-entry table generalizing the method for all latitudes would be considerably smaller than H. O. 214.   For, C would be restricted to between 0 and 90°, ε to the value of C, and H limited from 0 to 90°.   One-degree intervals would be suitable for the three arguments and intercepts would have a maximum length of 30 miles.   So that a table having three arguments and giving at a single opening the computed point related to an intercept under thirty miles long would not consist of more than 250 pages.