GEOMETRIC PROPERTIES OF POSITION LINES IN HYPERBOLIC NAVIGATION AND THEIR LAYOUT ON THE REFERENCE ELLIPSOID

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The following paper consists of two separate parts, each completing the other. In Part I, which deals with hyperbolae on the sphere as a general case, equations of spherical hyperbolae in orthogonal and polar spherical co-ordinates are established, as well as the relationship between the geographical co-ordinates, and the equation of transformation of hyperbolae to the Cassini-Soldner projection system. These equations enable construction of the curves on the sphere.

A study of these expressions reveals certain noteworthy properties of the curves mentioned, particularly one which gives the two branches of the hyperbolae characteristics of an ellipse.

A very simple condition was also discovered, which the parameters of both position lines of the hyperbolic net must satisfy in order to intersect, at the same time supplying a method for determining the co-ordinates of their point of intersection.

Part II of this paper relates to simple expressions on the reference ellipsoid when deduced from the sphere.

The method used consists mainly of an appropriate conformal representation of the ellipsoid on a suitably selected sphere, a projection which makes changes in the linear reduction modulus as small as possible. Actually, in the relationship under discussion, this modulus has a value of one along the suitably selected parallel, while its first and second derivatives, in relation to the are of the meridian, are nil. Absolute error when transferring the corresponding linear elements of the ellipsoid to the sphere is on the order of 2 m. per 1,000-m arc, reckoned from the selected parallel.

GEOMETRIC PROPERTIES AND EQUATIONS OF HYPERBOLAE ON THE SPHERE

1. Equations in orthogonal spherical coordinates of hyperbolae on the sphere.

1.—In view of the importance of radio methods of navigation, a discussion of some of the questions relative thereto and a statement as to the inherent properties of spherical hyperbolae representing in these methods lines of position on the sphere, should be of interest.

Let us therefore consider on a sphere of unit radius two points A and B (fig. 1) which we will designate as foci, and let us determine the equation of the locus of points P on the sphere, the difference in spherical distance of which to the two foci has a constant value n ; let us relate this locus, which we shall term a spherical hyperbola, to the system of orthogonal spherical coordinates defined by the great circle arc passing through A and B and through origin O, the midpoint of the smaller of the two arcs bounded by these two points.

By definition, between the great circle arcs : BP = σ, AP = σ_1, we should have the relationship :

$$\sigma - \sigma_1 = n \qquad (1)$$

and consequently, in the spherical triangle ABP the inequality :

$$\sigma - \sigma_1 \leqslant m$$

in which m designates arc AB ; we also have :

$$n \leqslant m \qquad (2)$$

Let Q be the point where the basic circle of the reference system meets the great circle at right angles to it and passing through P ; as a result, the orthogonal spherical coordinates of a standard point of the curve are the following :

$$x = \text{arc OQ} \; ; \; y = \text{arc QP}.$$

Now in the spherical triangles BPQ, APQ, which are right-angled at Q by construction, and taking into account equation (1), we have the relations :

$$\cos \sigma = \cos \left(\frac{m}{2} + x\right) \cos y \qquad \cos (\sigma - n) = \cos \left(\frac{m}{2} - x\right) \cos y \; ;$$

consequently, in order to obtain the desired equation in terms of x and y, we need only eliminate parameter σ in these equations.

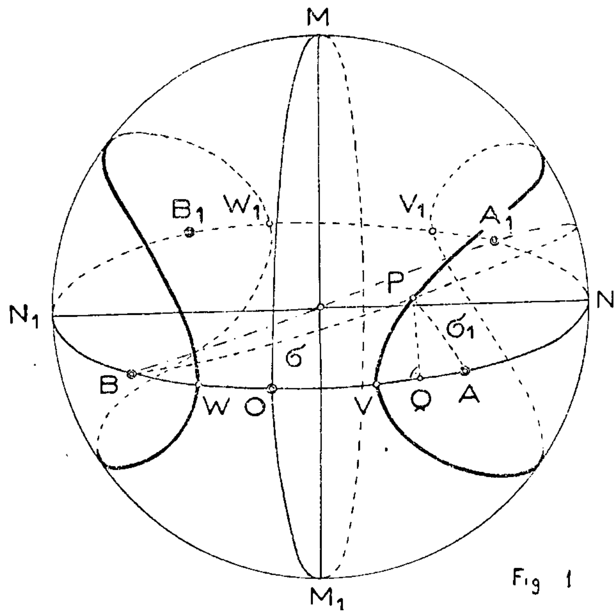

Fig 1

Let us develop the first member of the second equation and substitute, after squaring it, the value of cos σ supplied by the first ; we successively obtain :

$$\sqrt{1 - \cos^2 \sigma} \sin n = \cos \left(\frac{m}{2} - x\right) \cos y - \cos \sigma \cos n$$

$$\cos^2 y \left[\cos^2 \left(\frac{m}{2} + x\right) + \cos^2 \left(\frac{m}{2} - x\right) - 2 \cos \left(\frac{m}{2} + x\right) \cos \left(\frac{m}{2} - x\right) \cos n \right] = \sin^2 n$$

whence we of course arrive at :

$$\cos^2 y \left[\cos^2 \frac{m}{2} \sin^2 \frac{n}{2} + \left(\cos^2 \frac{n}{2} - \cos^2 \frac{m}{2}\right) \sin^2 x \right] = \sin^2 n \cos^2 n \qquad (3)$$

which connects the orthogonal spherical co-ordinates of the points on the spherical hyperbola.

2.—By examining this relationship, we can immediately deduce the symmetry of the locus with reference to the ordinates and to the abscissae, i.e., with reference to basic circle NON_1 and the great circle perpendicular to it passing through origin O.

All terms of equation (3) are essentially positive; it can therefore be deduced from inequality (2) that :

$$\cos^2 \frac{n}{2} - \cos^2 \frac{m}{2} \geqslant 0$$

Then, since

$$\cos^2 y \leqslant 1$$

the equation is satisfied for all values of x verifying the inequality :

$$\frac{\sin^2 \frac{n}{2} \cos^2 \frac{n}{2}}{\cos^2 \frac{m}{2} \sin^2 \frac{n}{2} + \left(\cos^2 \frac{n}{2} - \cos^2 \frac{m}{2}\right) \sin^2 x} \leqslant 1$$

whence :

$$\sin^2 x \geqslant \sin^2 \frac{n}{2} \qquad (4)$$

is obtained and consequently :

$$\frac{n}{2} \leqslant /x/ \leqslant \pi - \frac{n}{2} \qquad (4')$$

From (3) we get $y = O$ where $/x/ = \frac{n}{2}$, $\pi - \frac{n}{2}$; the curve is therefore composed of two branches; the first where x varies from $\frac{n}{2}$ to $\pi - \frac{n}{2}$; in the second x varies between $-\frac{n}{2}$ and $-(\pi - \frac{n}{2})$; furthermore, both branches meet the basic circle, the first at points $V(\frac{n}{2})$, $V_I(\pi - \frac{n}{2})$, and the second at points $W(\frac{n}{2})$, $W_I\left(-(\pi - \frac{n}{2})\right)$.

As in the equation found, the same results are obtained where $x = \frac{\pi}{2} - \varepsilon$ and $x = \frac{\pi}{2} + \varepsilon$; both branches are therefore symmetrical with respect to the circle of ordinates whose abscissae are respectively equal to $\frac{\pi}{2}$ and $-\frac{\pi}{2}$.

When the two points $A_1\ (\pi\ \frac{m}{2})$ $B_1\left(-(\pi - \frac{m}{2})\right)$ diametrically opposite on the circle to points A and B, are taken as foci, keeping without variation constant n, the resulting hyperbola coincides with that having A and B as foci.

$$\text{arc } B_I P - \text{arc } A_I P = n \qquad (5)$$

As equation (3) does not change, we get :

$$\cos^2 (\pi - \frac{m}{2}) = \cos^2 \frac{m}{2}.$$

In the particular case where $n = O$, the equation of the curve can be reduced to the expression :

$$\cos^2 y \sin^2 x = O$$

which analytically represents the great circle passing through O perpendicular to the base.

Finally, if the two foci are diametrically opposite, $m = \pi$, and from equation (3), the following

$$\cos^2 y \sin^2 x = \sin^2 \frac{n}{2}$$

is obtained, which gives a simultaneous representation of the two circles with a spherical radius of $\frac{n}{2}$ with centres at the two foci, to which the hyperbola is reduced.

3—.Let us furthermore show that the two branches of the curve have the property of ellipses on the sphere ; specifically, the points of the branch, for instance, that contain points A and A_I have the property that the sum of their spherical distances to A and A_I is always equal to $\pi - n$.

It is true that owing to the property of the hyperbola that :

$$\text{arc BP} - \text{arc AP} = n \qquad (1')$$

but on the other hand :

$$\text{arc BP} = \pi - \text{arc } A_I P$$

therefore, by construction, the great circle passing through B and P contains point A_I diametrically opposite B.

Substituting the first inequality for the second, we have the relationship :

$$\text{arc AP} + \text{arc } A_I P = \pi - n \qquad (5)$$

proving our assertion.

The ellipse equation, when the base remains fixed and the origin is chosen at the mid-point of arc AA_1, is obtained by developing in a manner entirely analogous to that used in obtaining equation (3), or more simply by taking : $x = \frac{\pi}{2} + x'$ in the latter, in the form :

$$\cos^2 y \left[\cos^2 \frac{m}{2} \sin^2 \frac{n}{2} + (\cos^2 \frac{n}{2} - \cos^2 \frac{m}{2}) \cos^2 x\right] = \sin^2 \frac{n}{2} \cdot \cos^2 \frac{n}{2} \qquad (3')$$

The length of the semi-axis of the ellipse corresponds to a value of y obtained in taking $x = 0$, and is

$$y = \text{arc} \cos \left(\frac{\sin \frac{n}{2}}{\sin \frac{m}{2}} \right).$$

The above-mentioned branch may furthermore be regarded as a spherical ellipse with a major axis $\pi + n$ and foci B, B_1, which immediately results when (1') et (5) are added together member by member :

$$\text{arc } BP + \text{arc } B_1 P = 2n + (\text{arc } AP + \text{arc } A_1 P)$$

and taking (5) into account :

$$\text{arc } BP + \text{arc } B_1 P = \pi + n \qquad (5')$$

4.—It is therefore obvious that with regard to the particular system of co-ordinates considered on the sphere, (3) represents the equation of the transformed spherical hyperbola in the Cassini-Soldner projection system when the great circle containing foci A and B is chosen as the basic meridian of the representation, and midpoint O of arc AB is taken as origin.

Figure 2 shows the transformed curves of a branch of two homofocal hyperbolae drawn on a sphere of unit radius having as parameter n values of 10° and 15° respectively, with $m = 20°$.

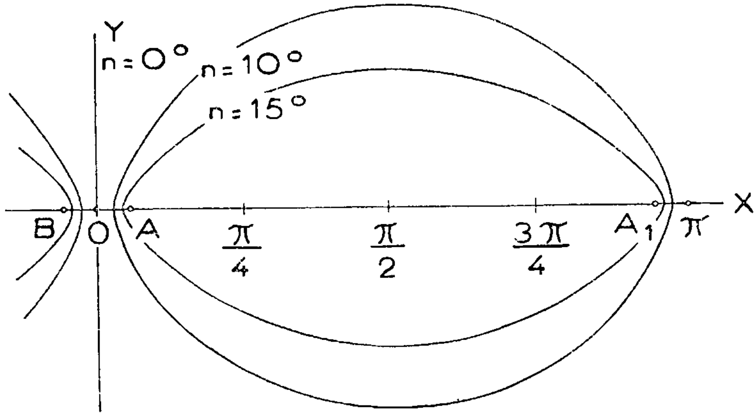

Fig 2

2. Relations between geographical coordinates of points of a spherical hyperbola and its representation in the Cassini-Soldner system of projection. Equations of transformed curve.

1.—On a sphere of unit radius, let us take as original meridian of longitudes Λ, that which passes through focus A of latitude Φ_0, and let Λ_1, Φ_1 be the geographical co-ordinates of the other focus B (fig. 3). If point P (Λ, Φ) is on the hyperbola of parameter n_1, and arcs AP, BP are designated respectively as σ and σ_1, then we should have :

$$\sigma_1 = \sigma \pm n_1$$

according as the point under consideration is or is not on the branch whose focus is A.

With regard to spherical triangles APQ, BPQ, having a common side from P to pole Q of the reference system, we get :

$$\cos \sigma = \sin \Phi_0 \sin \Phi + \cos \Phi_0 \cos \Phi \cos \Lambda$$

$$\cos (\sigma \pm n_1) = \sin \Phi_1 \sin \Phi + \cos \Phi_1 \cos \Phi \cos (\Lambda_1 - \Lambda)$$

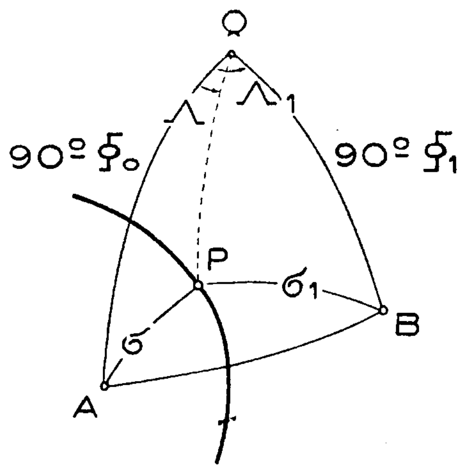

Fig. 3

from which, by eliminating σ, the following expression is easily obtained, connecting the geographical co-ordinates of the standard point of the spherical hyperbola of parameter n_1 :

$$\cos^2\Phi\Big[\cos^2\Phi_0\cos^2\Lambda+\cos^2\Phi_1\cos^2(\Lambda-\Lambda_1)-2\cos\Phi_0\cos\Phi_1\cos n_1\cos(\Lambda-\Lambda_1)\cos\Lambda$$
$$+2\sin\Phi_0\sin\Phi_1\cos n_1-\sin^2\Phi_0-\sin^2\Phi_1\Big]+2\sin\Phi\cos\Phi\Big[\sin\Phi_0\cos\Phi_0\cos\Lambda$$
$$+\sin\Phi_1\cos\Phi_1\cos(\Lambda-\Lambda_1)-\sin\Phi_1\cos\Phi_0\cos n_1\cos\Lambda-\sin\Phi_0\cos\Phi_1\cos n_1\cos(\Lambda-\Lambda_1)\Big]$$
$$=\sin^2 n_1-\sin^2\Phi_0-\sin^2\Phi_1+2\sin\Phi_0\sin\Phi_1\cos n_1 \tag{6}$$

This relation may be expressed in the following form, which is more convenient for purposes of computation :

$$\Big[A_1\sin^2\Lambda+A^2\sin 2\Lambda+A_3\Big]\cos^2\Phi+\Big[B_1\sin\Lambda+B_2\cos\Lambda\Big]\sin 2\Phi+C=0 \tag{6'}$$

by taking :

(6'') $A_1=\alpha-2\beta\cos\Lambda_1$ $\quad A_2=\beta\sin\Lambda_1$ $\quad A_3=\beta\cos\Lambda_1+\gamma\cos\Phi_0$ $\quad \delta\sin\Phi_0-\varepsilon\sin\Phi_1$

$\beta_1=\zeta\sin\Lambda_1$ $\quad B_2=\delta\cos\Phi_0+\zeta\cos\Lambda_1$ $\quad C=\delta\sin\Phi_0+\varepsilon\sin\Phi_1-\sin^2 n_1$

whence :

(7') $\alpha=\cos^2\Phi_1-\cos^2\Phi_0$ $\qquad \beta=\cos\Phi_1(\cos\Phi_1\cos\Lambda_1-\cos\Phi_0\cos n_1)$

$\gamma=\cos\Phi_0-\cos\Phi_1\cos\Lambda_1\cos n_1$ $\qquad \delta=\sin\Phi_0-\sin\Phi_1\cos n_1$

$\varepsilon=\sin\Phi_1-\sin\Phi_0\cos n_1$ $\qquad \zeta=\varepsilon\cos\Phi_1$

2.—The basic meridian of Cassini-Soldner projection system is the one that passes through focus A of the hyperbola to be represented, a focus furthermore chosen as origin in the system. If Q be the pole of the basic meridian (fig. 4), then the orthogonal spherical co-ordinates of standard point P on a sphere of unit radius are the following :

$$x = \text{arc AD} = \text{angle AQD} \qquad y = \text{arc DP}$$

and consequently, the equation of the transformation of the hyperbola will be obtained from (6) by substituting in this equation, with regard to the position of the basic circles relating to the two systems, x and y respectively for Λ and Φ, and by taking $\Phi_0 = 0$.

The equation sought for accordingly takes the following form, where the orthogonal co-ordinates of the other focus of the hyperbola of parameter n_1 are designated by x_1 and y_1 :

$$\cos^2 y\Big[\Big[2\cos x_1\cos y_1(\cos n_1-\cos x_1\cos y_1)-\sin^2 y_1\Big]\sin^2 x-\Big[\sin x_1\cos y_1(\cos n_1-\cos x_1\cos y_1)\Big]\sin 2x+\Big[\sin^2 x_1\cos^2 y_1-2\cos x_1\cos y_1(\cos n_1-\cos x_1\cos y_1)\Big]\Big]+\sin 2y\Big[\Big[\sin x_1\sin y_1\cos y_1\Big]\sin x-\Big[\sin y_1(\cos n_1-\cos x_1\cos y_1)\Big]\cos x\Big]+\Big[\sin^2 y_1-\sin^2 n_1\Big]=0. \tag{8}$$

By referring on the other hand the position of focus B to the other location at A by means of the polar angle α_1 and polar distance m_1, we then get with regard to spherical triangle ABC, right-angled at C, the relations :

$$\cos x_1 \cos y_1 = \cos m_1 \qquad \sin y_1 = \sin m_1 \sin \alpha_1$$
$$\sin x_1 \cos y_1 = \sin m_1 \cos \alpha_1$$

From the preceding equation (8), the following is obtained, giving the equation of the transformation :

$$\cos^2 y \Big[\big[2\cos m_1 (\cos n_1 - \cos m_1) - \sin^2 m_1 \sin^2 \alpha_1 \big] \sin^2 x -$$
$$\big[\sin m_1 \cos \alpha_1 (\cos n_1 - \cos m_1) \big] \sin 2x + \big[\sin^2 m_1 \cos^2 \alpha_1 - 2\cos m_1 (\cos n_1 - \cos m_1) \big] \Big]$$
$$+ \sin 2y \Big[\big[\sin^2 m_1 \sin \alpha_1 \cos \alpha_1 \big] \sin x - \big[\sin m_1 \sin \alpha_1 (\cos n_1 - \cos m_1) \big] \cos x \Big]$$
$$+ \big[\sin^2 m_1 \sin^2 \alpha_1 - \sin^2 n_1 \big] = 0. \qquad (8')$$

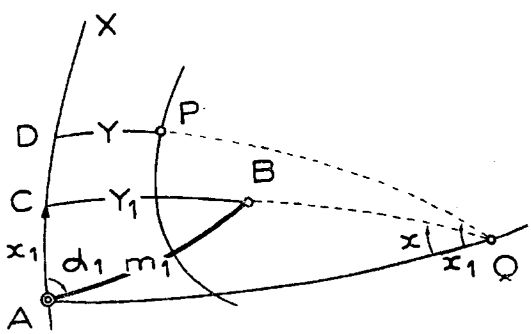

Fig. 4

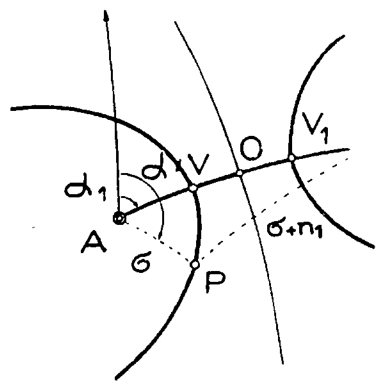

Fig 5

3. **Conditions which the parameters of the two position lines of a hyperbolic net must satisfy in order to meet.**

1.—In order to deal with this problem, the hyperbolae should be referred to a system of polar co-ordinates on the sphere of unit radius. The pole of the system coincides with focus A ; let α_1 and m_1 be the co-ordinates of the other focus, B (fig. 5). Let σ and α designate the co-ordinates of standard point P of the branch of the hyperbola containing the pole, and let us take :

$$BP - AP = n_1$$

In spherical triangle ABP, we get :

$$\cos (\sigma + n_1) = \cos m_1 \cos \sigma + \sin m_1 \sin \sigma \cos (\alpha - \alpha_1)$$

and by means of the simple development :

$$\operatorname{cotg} \sigma = \frac{\sin m_1}{\cos n_1 - \cos m_1} \cos (\alpha - \alpha_1) + \frac{\sin n_1}{\cos n_1 - \cos m_1} \qquad (9')$$

in which the coefficients :

$$\frac{\sin m_1}{\cos n_1 - \cos m_1} = M_1 \qquad \frac{\sin n_1}{\cos n_1 - \cos m_1} = N_1 \qquad (10)$$

are, owing to equation (2), essentially positive quantities, the first being larger than the second.

Similarly, for the other branch we get :

$$\operatorname{cotg} \sigma = \frac{\sin m_1}{\cos n_1 - \cos m_1} \cos (\alpha - \alpha_1) - \frac{\sin n_1}{\cos n_1 - \cos m_1} \qquad (9'')$$

Briefly :

$$\operatorname{cotg} \sigma = M_1 \cos (\alpha - \alpha_1) \pm N_1 \qquad (9)$$

consequently represents the equation in polar spherical co-ordinates of the hyperbola having a focus at the pole, a positive or negative sign being chosen according as the branch under consideration is the one containing the pole or the other one.

2.—Let $\sigma_\alpha = AP_I$ designate the polar distance corresponding to the branch outside the pole, and $\sigma_{\pi + \alpha} = AP$ that of the other branch, the first referring to value α and the second to value $\pi + \alpha$ of the polar angle.

From (9) we have :

$$\text{cotg}\ \sigma_\alpha + \text{cotg}\ \sigma_{\pi + \alpha} = 0$$

i.e. :

$$\sigma_\alpha + \sigma_{\pi + \alpha} = \pi.$$

The rendering in ordinary terms of this relation between arcs AP and AP_I of the same great circle with common extremities at focus A is facilitated by examination of figure 6.

3.—By effecting variations of n_I in the interval (O, m_I), we get curves of a family of homofocal hyperbolae ; as regards $n_I = O$ in particular we get :

$$\text{cotg}\ \sigma = \text{cotg}\ \frac{m_1}{2} \cos (\alpha - \alpha_1)$$

which is the equation of the great circle cutting orthogonally arc AB at its midpoint.

4.—Let us then consider a second family of homofocal hyperbolae with their focus at pole A, and let m_2, α_2 be the polar co-ordinates of the other focus (fig. 7).

At the meeting points of the hyperbola of parameter n_I of the first family with that of parameter n_2 of the second, the following equality will of course be obtained :

$$M_1 \cos (\alpha - \alpha_1) \pm N_1 = M_2 \cos (\alpha - \alpha_2) \pm N_2 \qquad (11)$$

where :

$$M_2 = \frac{\sin m_2}{\cos n_2 - \cos m_2} \qquad N_2 = \frac{\sin n_2}{\cos n_2 - \cos m_2} \qquad (10')$$

which, when solved with regard to α , will give the polar angles, and next, by using one of the formulae (9), the polar distances σ of the points under consideration.

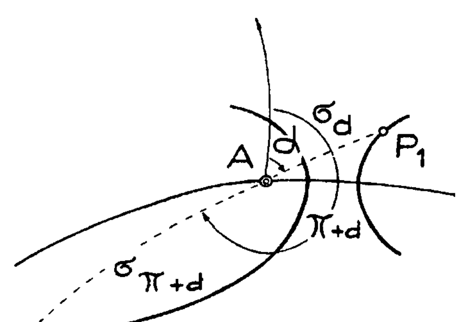

Fig 6

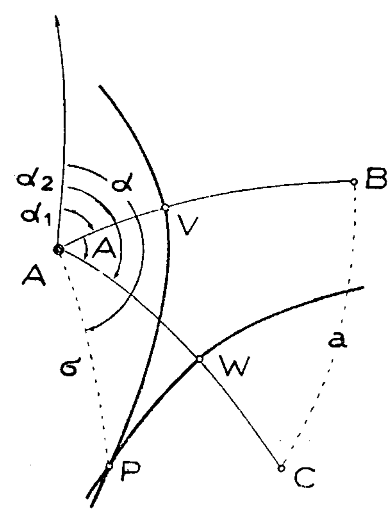

Fig 7

To solve formula (11), the following form should be used :

$$(M_1 \sin \alpha_1 - M_2 \sin \alpha_2) \sin \alpha + (M_1 \cos \alpha_1 - M_2 \cos \alpha_2) \cos \alpha = \pm N_2 \mp N_1$$

and by putting :

$$\text{tg}\ \varepsilon = \frac{M_1 \cos \alpha_1 - M_2 \cos \alpha_2}{M_1 \sin \alpha_1 - M_2 \sin \alpha_2} \qquad (12)$$

we get :

$$(M_1 \cos \alpha_1 - M_2 \cos \alpha_2) \text{cotg}\ \varepsilon \sin \alpha + (M_1 \cos \alpha_1 - M_2 \cos \alpha_2) \cos \alpha = \pm N_2 \mp N_1$$

and consequently :

$$\sin(\alpha + \varepsilon) = \frac{\pm N_2 \mp N_1}{M_1 \cos \alpha_1 - M_2 \cos \alpha_2} \sin \varepsilon \qquad (13)$$

a relation that solves the problem for general cases.

In order to effect this equation, we must have :

$$\left(\frac{\pm N_2 \mp N_1}{M_1 \cos \alpha_1 - M_2 \cos \alpha_2} \right)^2 \sin^2 \varepsilon \leqslant 1$$

or :

$$\sin^2 \varepsilon \leqslant \left(\frac{M_1 \cos \alpha_1 - M_2 \cos \alpha_2}{N_2 \pm N_1} \right)^2 .$$

which amounts to the same thing.

From formula (12) we take :

$$\sin^2 \varepsilon = \frac{tg^2 \varepsilon}{1 + tg^2 \varepsilon} = \frac{(M_1 \cos \alpha_1 - M_2 \cos \alpha_2)^2}{M_1^2 + M_2^2 - 2 M_1 M_2 \cos(\alpha_2 - \alpha_1)}$$

and by substitution in the preceding equality, designating as $A = \alpha_2 - \alpha_1$ the angle made at A by the directions of foci B and C, we get :

$$N_1^2 \pm N_2^2 + 2 N_1 N_2 \leqslant M_1^2 + M_2^2 - 2 M_1 M_2 \cos A$$

whence, taking into account equations (10) and (10'), the following relation is easily obtained :

$$(\cos n_1 + \cos m_1)(\cos n_2 - \cos m_2) + (\cos n_1 - \cos m_1)(\cos n_2 + \cos m_2) \geqslant 2 \sin m_1 \sin m_2 \cos A - 2 \sin n_1 \sin n_2$$

from which we take :

$$\cos(n_1 - n_2) \geqslant \cos m_1 \cos m_2 + \sin m_1 \sin m_2 \cos A$$

But in triangle ABC, whose side opposite angle A is designated as a, we get, following the cosin theorem :

$$\cos a = \cos m_1 \cos m_2 + \sin m_1 \sin m_2 \cos A.$$

From this the following inequalities result, which parameters n_1 and n_2 of the curves of both families must satisfy in order to meet :

$$| n_1 - n_2 | \leqslant a \qquad (14)$$

$$n_1 + n_2 \leqslant a. \qquad (14')$$

The first refers to both systems of branches each having pole A on the inside or outside, and the other is valid for other cases.

LAYOUT OF POSITION LINES ON REFERENCE ELLIPSOID IN HYPERBOLIC NAVIGATION AND RELATED PROBLEMS

1. Correspondence in representation of the sphere on the ellipsoid.

1.—By using an appropriate representation of the sphere on the ellipsoid, we can transfer the data discovered with regard to the sphere onto the ellipsoid.

This method, devised by Gauss as an approximate substitution of spherical trigonometry for spheroidal trigonometry, consists in essence of the following. Let us suppose that we have to represent a surface strip of no great width on another surface ; we then have to set up a correspondence between the two surfaces in such a way that there will be no angular distortions, and that the modulus of linear reduction n will be equal to 1 along a certain line so selected that the strip will be appropriately divided lengthwise, and will vary as slowly as possible within the strip.

In the theory of conformal representations, proof is given in particular that if the two surfaces in question are surfaces of revolution, as in the present instance, and if the strip itself is in the direction of the parallels, fluctuations in the value of modulus n can be kept at a low level with regard to unity, at most of the third order, when the width of the strip measured along the meridians is taken as an infinitesimal of the first order. Or, more accurately, it is always possible to have first and second derivatives of n, with respect to the arc of the meridian, equal to zero along the parallel of the area in which the modulus has the value of 1.

It might be well to mention how this possibility is limited to the first derivative at most in the representation of a non-developable surface on the plane, a possibility which was made use of, in the case of the ellipsoid, in Lambert's isogonic conical projection.

2.—Let us designate as ρ and N the main radii of curvature : radius of curvature of the meridian and major normal, at a standard point P having as geographical co-ordinates φ, λ of the ellipsoid of revolution of parameters a, semi major axis, and e, numerical value of the eccentricity :

$$\rho = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \varphi)^{3/2}} \qquad N = \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi}}$$

and let us write down the value of the linear element ds of the latter in the isometrical form

$$ds^2 = N^2 \cos^2 \varphi \,(dv^2 + d\lambda^2) \tag{15}$$

designating as v the meridional part :

$$v = \int_0^{\varphi} \frac{\rho}{N \cos \varphi} d\varphi = \log \text{nep} \operatorname{tg} \left(\frac{\pi}{4} + \frac{\varphi}{2}\right) \left[\frac{1 - e \sin \varphi}{1 + e \sin \varphi}\right]^{\frac{e}{2}} \tag{16}$$

Let us recall the analogue expression for the linear element dS of the point of latitude Φ and longitude Λ located on the sphere of radius R

$$dS^2 = R^2 \cos^2 \Phi \,(dV^2 + d\Lambda^2) \tag{15'}$$

or :

$$V = \int_0^{\Phi} \frac{d\Phi}{\cos \Phi} = \log \text{nep} \operatorname{tg} \left(\frac{\pi}{4} + \frac{\Phi}{2}\right) \tag{16'}$$

The most usual autogonal representation of the ellipsoid on the sphere, with corresponding parallels, will be obtained by the following equality :

$$V + i\Lambda = K_1 (v + i\lambda) + \log \text{nep} K_2 \tag{17}$$

where i represents the imaginary unit $\sqrt{-1}$, and K_1 and K_2 are two real constants.

In order to obtain the desired correspondences from the latter, the real and imaginary parts should be separated ; we then get :

$$V = K_1 v + \log \text{nep} K_2 \qquad \Lambda = K_1 \lambda \tag{18}$$

of which the former expression, taking into account (16) and (16'), can also be written :

$$\operatorname{tg}\left(\frac{\pi}{4} + \frac{\Phi}{2}\right) = K_2 \left[\operatorname{tg}\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) \left[\frac{1 - e \sin \varphi}{1 + e \sin \varphi}\right]^{\frac{e}{2}}\right]^{K_1} \tag{19}$$

2. **Determining of constants.**

1.—Since representation is conformal, modulus of linear reduction n is simply obtained by means of the relationship between the arc of the parallel on the sphere and the corresponding one on the ellipsoid, and taking the second equation (18) into account as well as the relation occuring between radius r of the parallel on the ellipsoid and the major normal,

$$n = \frac{dS}{ds} = \frac{K_1 R \cos \Phi}{r} = \frac{K_1 R \cos \Phi}{N \cos \varphi} \tag{20}$$

It can be seen that modulus n is a function of latitude only, and therefore remains constant throughout the length of each parallel.

Therefore, letting n_0 be the value of the modulus on parallel φ_0 of the ellipsoid, to which parallel Φ_0 corresponds on the sphere of radius R_0

$$n_0 = \frac{K_1 R_0 \cos \Phi_0}{N_0 \cos \varphi_0} \tag{20'}$$

At latitude φ, the following modulus, through Taylor's development, will be obtained :

$$n = n(\varphi) = n(\varphi_0 + (\varphi - \varphi_0)) = n_0 + (\varphi - \varphi_0)\left(\frac{d\,n}{d\,\varphi}\right)_{\varphi_0} + \frac{(\varphi - \varphi_0)^2}{2}\left(\frac{d^2 n}{d\varphi^2}\right)_{\varphi_0} + \frac{(\varphi - \varphi_0)^3}{3!}\left(\frac{d^3 n}{d\varphi^3}\right)_{\varphi_0} + \ldots\ldots \qquad (21)$$

If representation is desired of an ellipsoidal zone with a minimum amount of alterations, constants K_1, K_2 and radius R of the sphere will have to be selected in such a way that $n_0 = 1$ will be obtained on parallel φ_0, called the normal parallel, and the first and second derivatives of the modulus with respect to latitude will cancel out, as then, as shown by equation (21), fluctuations around the unit of linear reduction will only be dependent upon powers of difference ($\varphi - \varphi_0$) greater than the second power.

For $n_0 = 1$, from formula (20') we can readily deduce

$$R_0 = \frac{N_0 \cos\varphi_0}{K_1 \cos\Phi_0} \qquad (22)$$

2.—In order to satisfy both other conditions, the first and second derivatives of formula (20) with respect to φ will first have to be determined. Therefore :

$$\frac{d}{d\,\varphi}\,n = -\frac{K_1 R_0}{r^2}\left[r \sin\Phi\,\frac{d\,\Phi}{d\,\varphi} + \cos\Phi\,\frac{d\,r}{d\,\varphi}\right] \qquad (23)$$

In order to get :

$$\frac{d\,\Phi}{d\,\varphi} = \frac{d\,\Phi}{d\,v}\cdot\frac{d\,v}{d\,\varphi}$$

from formula (19) expressed as

$$\operatorname{tg}\left(\frac{\pi}{4} + \frac{\Phi}{2}\right) = K_2\,\varepsilon^{K_1 v} \qquad (19')$$

designating as ε the natural logarithm base, we get by differentiating :

$$\frac{1}{2\cos^2\left(\frac{\pi}{4} + \frac{\Phi}{2}\right)}\,d\,\Phi = K_1 K_2\,\varepsilon^{K_1 v}\,d\,v$$

whence :

$$\frac{d\,\Phi}{d\,v} = 2 K_1 K_2\,\varepsilon^{K_1 v}\cos^2\left(\frac{\pi}{4} + \frac{\Phi}{2}\right)$$

and, taking (19') into account

$$\frac{d\,\Phi}{d\,v} = 2 K_1 \sin\left(\frac{\pi}{4} + \frac{\Phi}{2}\right)\cos\left(\frac{\pi}{4} + \frac{\Phi}{2}\right) = K_1 \cos\Phi$$

on the other hand, from (16) :

$$\frac{d\,v}{d\,\varphi} = \frac{\rho}{N\cos\varphi} = \frac{\rho}{r}$$

Therefore :

$$\frac{d\,\Phi}{d\,\varphi} = K_1\,\frac{\rho}{r}\cos\Phi \qquad (24)$$

The following is readily arrived at :

$$\frac{d\,r}{d\,\varphi} = \frac{d}{d\,\varphi}\,\frac{a\cos\varphi}{\sqrt{1 - e^2\sin^2\varphi}} = -\,\rho\sin\varphi \qquad (24')$$

and by substituting in formula (23) and consideration being given to (20), we finally get :

$$\frac{d}{d\,\varphi}\,n = n\,\frac{\rho}{r}\left[\sin\varphi - K_1\sin\Phi\right] \qquad (23')$$

By deriving again with respect to φ, we easily obtain :

$$\frac{d^2}{d\varphi^2}\,n = \frac{1}{n}\left(\frac{d\,n}{d\,\varphi}\right)^2 + \frac{r}{\rho}\left(\frac{d\,n}{d\,\varphi}\right)\frac{d}{d\,\varphi}\left(\frac{\rho}{r}\right) + n\,\frac{\rho}{r}\left(\cos\varphi - K_1^2\,\frac{\rho}{r}\cos^2\Phi\right) \qquad (25)$$

Therefore, since we have the two equalities :

$$\left(\frac{d}{d\,\varphi}\,n\right)\varphi_0 = 0 \qquad \left(\frac{d^2}{d\,\varphi^2}\,n\right)\varphi_0 = 0 \qquad (26)$$

we should have, in accordance with the foregoing, the relations :

$$K_1 = \frac{\sin \varphi_0}{\cos \Phi_0} \tag{27}$$

$$\cos \varphi_0 = K_1^2 \frac{\rho_0}{r_0} \cos^2 \Phi_0 \tag{28}$$

From the latter, we get :

$$K_1 \cos \Phi_0 = \sqrt{\frac{N_0}{\rho_0}} \cos \varphi_0$$

and by substituting in formulas (22) and (27), we get respectively the radius of the sphere

$$R_0 = \sqrt{\rho_0 N_0} \tag{29}$$

and the value of Φ_0, corresponding to φ_0

$$\operatorname{tg} \Phi_0 = \sqrt{\frac{\rho_0}{N_0}} \operatorname{tg} \varphi_0 \tag{30}$$

The latter, account being taken of (28), can be expressed as :

$$R_0 \operatorname{cotg} \Phi_0 = N_0 \operatorname{cotg} \varphi_0 \tag{30'}$$

whence it follows that the geodetic curvatures of the two parallels corresponding to φ_0 on the ellipsoid and Φ_0 on the sphere are equal ; since we know that the geodetic curvature of a parallel of a surface of revolution is measured by the inverse of the side of the cone circumscribed on the surface along the same parallel.

Knowing the value of Φ_0, obtained by means of formula (30), we can deduce K_1 from formula (26) ; moreover the first formula of (18) expressed as

$$\log \text{nep } K_2 = \log \text{nep } \operatorname{tg} \left(\frac{\pi}{4} + \frac{\Phi}{2} \right) - K_1 v$$

will give the value of the second constant K_2 where Φ_0 is substituted for Φ and v_0 for the value of meridional part v relating to geographical latitude φ_0 :

$$\log K_2 = \log \operatorname{tg} \left(\frac{\pi}{4} + \frac{\Phi_0}{2} \right) - M K_1 v_0 \tag{31}$$

denoting by M (= 0.434 2944 819 … …) the modulus of Briggian logarithms designated by log.

3.—The values of K_1, K_2 and Φ_0 being known, the meridional part and the longitude of point P (v, λ) of the ellipsoid corresponding to point P' (Φ, Λ) of the sphere of radius $R_0 = \sqrt{\rho_0 N_0}$ are obtained by means of expressions (18) :

$$v = \frac{1}{M K_1} \left[\log \operatorname{tg} \left(\frac{\pi}{4} + \frac{\Phi}{2} \right) - \log K_2 \right] \qquad \lambda = \frac{1}{K_1} \Lambda \tag{18'}$$

Since it is desired, in order to compute the former, to make use of the numerical tables in (18) and (31) which in terms of φ and μ supply values v and V of meridional parts on the ellipsoid and sphere expressed in minutes of arc, it will be necessary to write :

$$V' = K_1 v' + \frac{1}{M \text{ arc } 1'} \log K_2 \qquad v' = \frac{1}{K_4} V' - \frac{1}{K_1 \text{ M arc } 1'} \log K_2 \tag{18''}$$

$$\log K_2 = M \text{ arc } 1'. (V'_0 - K_1 v'_0) \tag{31'}$$

3. **Evaluation of the order of approximation obtained.**

1.—The modulus of linear reduction in the correspondence for latitude φ is obtained, through choice of constants, from the development :

$$n = \frac{d S}{d s} = 1 + \frac{(\varphi - \varphi_0)^3}{3!} \left(\frac{d^3}{d \varphi^3} n \right)_{\varphi_0} + \dots \dots \dots \tag{21'}$$

Passing from infinitesimal to finite elements, we get by integrating :

$$S - s = \frac{1}{6} \left(\frac{d^3}{d \varphi_3} n \right)_{\varphi_0} \int_0^s (\varphi - \varphi_0)^3 \, d s + \dots \dots \dots \tag{32}$$

representing by S the arc of the sphere corresponding to arc s on the ellipsoid.

In order to get some idea as to the order of approximation obtained when substituting the arc of the sphere for the corresponding arc of the ellipsoid, the magnitude of the first term, which is the preponderant one, of the development of the second member of the preceding equality will have to be obtained.

Since it is only a question of establishing an order of magnitude, substitution of the following approximate expression for difference ($\varphi - \varphi_0$) will suffice :

$$\varphi - \varphi_0 = \frac{s}{\rho_0} \cos \alpha_0$$

where α_0 designates the azimuth, on parallel φ_0, of the arc being considered, and we have :

$$S - s = \frac{s^4}{24 \rho_0^3} \left(\frac{d^3}{d\varphi^3} n\right)_{\varphi_0} \cos^3 \alpha_0 + \dots\dots\dots \qquad (32')$$

With regard to the value of the derivative computed on the normal parallel, we get from formula (25), derived with respect to φ

$$\frac{d^3}{d\varphi^3} n = \frac{d^2 n}{d\varphi^2}\left[\frac{2}{n}\frac{dn}{d\varphi} + \frac{r}{\rho}\frac{d}{d\varphi}\left(\frac{\rho}{r}\right)\right] + \frac{dn}{d\varphi}\left[-\frac{1}{n^2}\left(\frac{dn}{d\varphi}\right)^2 + \frac{d}{d\varphi}\left(\frac{r}{\rho}\right)\frac{d}{d\varphi}\left(\frac{\rho}{r}\right) + \frac{r}{\rho}\frac{d^2}{d\varphi^2}\left(\frac{\rho}{r}\right)\right.$$
$$\left. + \frac{\rho}{r}\left(\cos\varphi - K_1^2 \frac{\rho}{r}\cos^2\Phi\right)\right] + n\left[\cos\varphi - K_1^2\frac{\rho}{r}\cos^2\Phi\right]\frac{d}{d\varphi}\left(\frac{\rho}{r}\right) - n\frac{\rho}{r}\left[\sin\varphi + \right.$$
$$\left. + K_1^2\cos^2\Phi\frac{d}{d\varphi}\left(\frac{\rho}{r}\right) - 2K_1^3\left(\frac{\rho}{r}\right)^2 \sin\Phi\cos^2\Phi\right]$$

whence the following value sought for is obtained from (26) and (28), bearing in mind that n^0 is equal to unity :

$$\left(\frac{d^3}{d\varphi^3} n\right)_{\varphi_0} = -\frac{\rho_0}{r_0}\left[\sin\varphi_0 + K_1^2\cos^2\Phi_0\left(\frac{d}{d\varphi}\frac{\rho}{r}\right)_{\varphi_0} - 2K_1^3\left(\frac{\rho_0}{r_0}\right)^2\sin\Phi_0\cos^2\Phi_0\right]$$

Expressing the entire quantity in terms of φ_0, using formulae (27) and (28), and observing that :

$$\frac{d}{d\varphi}\frac{\rho}{r} = \frac{\rho}{r}\left[\operatorname{tg}\varphi + \frac{e^2 \sin 2\varphi}{1 - e^2\sin^2\varphi}\right]$$

by simple deduction we get :

$$\left(\frac{d^3}{d\varphi^3} n\right)_{\varphi_0} = -2\frac{\rho_0}{r_0}\left[\sin\varphi_0 + \frac{e^2\sin\varphi_0\cos^2\varphi_0}{1 - e^2\sin^2\varphi_0} - \frac{\rho_0}{r_0}\sin\varphi_0\cos\varphi_0\right]$$

But :

$$\frac{\rho}{r} = \frac{1 - e^2}{(1 - e^2\sin^2\varphi_0)\cos\varphi_0}$$

and therefore, by means of simple developments, we arrive at :

$$\left(\frac{d^3}{d\varphi^3} n\right)_{\varphi_0} = -\frac{2e^2(1 - e^2)\sin 2\varphi_0}{(1 - e^2\sin^2\varphi_0)^2} \qquad (33)$$

which may also be more briefly expressed as :

$$\left(\frac{d^3}{d\varphi^3} n\right)_{\varphi_0} = -2\frac{\rho_0 N_0}{a^2} e^2 \sin 2\varphi_0 \qquad (33')$$

Formula (32') consequently becomes :

$$S - s = -\frac{1}{12} e^2 s^4 \frac{N_0}{a^2 \rho_0^2} \sin 2\varphi_0 \cos^3\alpha_0 + \dots\dots\dots$$

and since we have :

$$\frac{N}{\rho^2} = \frac{(1 - e^2\sin^2\varphi)^{5/2}}{a(1 - e^2)^2}$$

we finally get :

$$S - s = -\frac{1}{12} e^2 \frac{s^4}{a^3}\frac{(1 - e^2\sin^2\varphi_0)^{5/2}}{(1 - e^2)^2}\sin 2\varphi_0\cos^3\alpha_0 + \dots\dots\dots \qquad (34)$$

We can then assume as maximum value of this difference :

$$S - s = -\frac{1}{12}\frac{e^2}{a^3(1 - e^2)^2} s^4 \qquad (34')$$

Which means that by substituting the arc of the corresponding ellipsoid for the arc of the sphere, a relative error is made that is not in excess of :

$$\frac{S - s}{s} = \frac{1}{12} \frac{e^2}{(1 - e^2)^2} \left(\frac{s}{a}\right)^3 \qquad (34")$$

2.—The following table gives the amounts of absolute and relative errors corresponding to a few values of arc :

arcs in kilometres	*Absolute error in metres*	*Relative error*
50	0.000 014	1 : 370 0000 000
100	0.000 22	1 : 46 0000 000
200	0.003 5	1 : 5 7000 000
500	0.14	1 : 3700 000
800	0.90	1 : 890 000
1 000	2.2	1 : 460 000
1 500	11	1 : 140 000
2 000	35	1 : 57 000
2 500	85	1 : 29 000
3 000	177	1 : 17 000

3. *Example* 1.—Determine the autogonal correspondence between points on the ellipsoid of Hayford and those of a sphere, with minimum alteration of corresponding arcs within a zone of mean latitude $\varphi_0 = 45°$.

Radius R_0 of the sphere is obtained from formula (29) : $R_0 = 6\,378\,352$ m., while formula (30) supplies a latitude of $\Phi_0 = 44° 57' 05''.79$ for the parallel of the sphere corresponding to the $\varphi_0 = 45°$ latitude parallel on the ellipsoid.

With regard to constants, formula (27) gives $K_1 = 1.000\,84566$ and (31) $\log K_2 = 0.001\,22630$.

Values in metres of magnitudes : ρ_0 , N_0 , $\sqrt{\rho_0 N_0}$, appearing in the foregoing expressions, can easily be obtained in terms of argument φ_0 from *Tables for the International Ellipsoid of Reference* published by the International Geodetical and Geophysical Union, Special Publication No. 2, Paris 1928, or from the *Table of Meridional Parts* of the International Hydrographic Bureau, Special Publication No. 21, Monaco, 1928, which supply values of v_0 and V_0 expressed in minutes in terms of φ_0 and Φ_0 respectively.

Applying formula (18), equations of the correspondence consequently become :

$$v' = 0.999\,1551\ V' - 9'.6989 \qquad \lambda = 0.999\,1551\ \Lambda \qquad (a)$$

when going from the sphere to the ellipsoid, and

$$V' = 1.000\ 8457\ v' + 9'.7071 \qquad \Lambda = 1.000\ 8457\ \lambda \qquad (a')$$

when transferring from the ellipsoid to the sphere.

The sole purpose in applying the considerable degree of accuracy used in the numerical computation of this example and of the following ones is in order to show the practical side or convergency of the formulae.

4. **Conversion of meridional parts into geographical latitudes.**

1.—Given value v of the meridional part in relation to the ellipsoid, in order to convert to geographical latitude, we shall beging by putting :

$$v = \text{Nap. log tan}\left(\frac{\pi}{4} + \frac{\varphi}{2} + \Delta\right) \qquad (35)$$

Developing in a progression series according to increasing powers of Δ, we get :

$$v = \text{Nap. log tan}\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) + \frac{2}{\cos\varphi}\Delta + \frac{2}{\cos^2\varphi}\sin\varphi\,\Delta^2 + \frac{4}{3}\frac{1+\sin^2\varphi}{\cos^3\varphi}\Delta^3 +$$
$$+ \frac{2}{3}\sin\varphi\,\frac{5+\sin^2\varphi}{\cos^4\varphi}\Delta^4 + \frac{5+18\sin^2\varphi+\sin^4\varphi}{\cos^5\varphi}\Delta^5 + \ldots\ldots\ldots$$

On the other hand, by definition :

$$v = \text{Nap. log tan}\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) - \sum_{n=1} \frac{e^{2n}}{2n-1}\sin^{2n-1}\varphi$$

and by equalizing both expressions, the following development is arrived at, by means of a simple albeit laborious process, neglecting terms lower than the order of e^{12} :

$$\Delta = -\frac{e^2}{4}\sin 2\varphi.\left[1 + \frac{5}{6}e^2\sin^2\varphi + \frac{e^4}{30}(26\sin^2\varphi - 5)\sin^2\varphi + \frac{e^6}{2\,520}(2474\sin^2\varphi - 945)\sin^4\varphi + \right.$$
$$\left. + \frac{e^8}{2\,520}(2\,936\sin^4\varphi - 1\,680\sin^2\varphi + 105)\sin^4\varphi + \ldots\ldots\ldots\right] \quad (36)$$

Let auxiliary angle ψ then be introduced, connected with the meridional part by the relationship :

$$\text{Nap. log tan}\left(\frac{\pi}{4} + \frac{\psi}{2}\right) = v \quad (37)$$

which, when parallelled with equation (35), enables the following to be written :

$$\varphi = \psi - 2\Delta \quad (38)$$

On the other hand we have :

$$\sin\varphi = \sin(\psi - 2\Delta) \qquad \cos\varphi = \cos(\psi - 2\Delta)$$

By developing appropriately, and bearing in mind (36), we get the sin and cosin value of the geographical latitude in terms of angle ψ

$$\sin\varphi = \sin\psi + \frac{e^2}{2}\sin 2\psi\cos\psi\left[1 + \frac{e^2}{3}(3 - 5\sin^2\psi) + \frac{e^4}{5}(5 - 20\sin^2\psi + 16\sin^4\psi) + \ldots\ldots\ldots\right]$$

$$\cos\varphi = \cos\psi - \frac{e^2}{2}\sin 2\psi\sin\psi\left[1 + \frac{e^2}{6}(9 - 10\sin^2\psi) + \frac{e^4}{30}(60 - 155\sin^2\psi + 96\sin^4\psi) + \ldots\ldots\ldots\right]$$

and consequently :

$$\sin 2\varphi = \sin 2\psi + e^2\sin 2\psi\left[1 - 2\sin^2\psi + \frac{e^2}{6}(6 - 31\sin^2\psi + 26\sin^4\psi) + \right.$$
$$\left. + \frac{e^4}{30}(30 - 285\sin^2\psi + 546\sin^4\psi - 292\sin^6\psi) + \ldots\ldots\ldots\right]$$

Through these and formula (36), formula (38) takes the form of :

$$\varphi = \psi + \frac{e^2}{2}\sin 2\psi\left[1 + \frac{e^2}{6}(6 - 7\sin^2\psi) + \frac{e^4}{30}(30 - 85\sin^2\psi + 56\sin^4\psi) + \right.$$
$$\left. + \frac{e^6}{2\,520}(2\,520 - 12\,600\sin^2\psi + 18\,669\sin^4\psi - 8\,558\sin^6\psi) + \ldots\ldots\ldots\right]$$

Arranging according to increasing powers of sin ψ, we finally get the relationship :

$$\varphi = \psi + \frac{e^2}{2}\sin 2\psi\left[(1 + e^2 + e^4 + e^6) - \frac{1}{6}(7e^2 + 17e^4 + 30e^6)\sin^2\psi + \right.$$
$$\left. + \frac{1}{120}(224e^4 + 889e^6)\sin^4\psi - \frac{4\,279}{1\,260}e^6\sin^6\psi + \ldots\ldots\ldots\right] \quad (39)$$

which makes it possible to transform the meridional part into geographical latitude by means of auxiliary angle ψ, whose value is obtained from (37) expressed as :

$$\log\tan\left(\frac{\pi}{4} + \frac{\psi}{2}\right) = v'\,M\,\text{arc}\,1' = 0.000\,1263\,31144\,v' \quad (37')$$

2.—By taking the parameters of the International Reference Ellipsoid ($a = 6\,378\,388$ m., $e^2 = 0.006\,7226\,700$) and by expressing the values of both latitudes of formula (39) in minutes of arc, we get :

$$\varphi = \psi + 11'.555\,4186\sin 2\psi\left[1.006\,76817 - 0.007\,97268\sin^2\psi + 0.000\,08661\sin^4\psi - \right.$$
$$\left. - 0.000\,00103\sin^6\psi + \ldots\ldots\ldots\right] \quad (39')$$

How to determine this element indirectly by inverse interpolation effected by means of ordinary meridional parts tables pertaining to the International Ellipsoid, then becomes obvious.

5. **Problem of Transfer of Geographical Co-ordinates.**

1.—By means of the correspondence expressed analytically in the preceding formulae, figures are obtained on the sphere differing only slightly from those on the ellipsoid, so that by applying formulae pertaining to spherical geometry to the former, the solution to problems relating to the latter will be obtained by approximation.

As an initial application of this method, let us solve the problems inherent to the transformation of co-ordinates on the ellipsoid.

2. *Direct Problem.*—In the direct problem of transfer of co-ordinates, geographical co-ordinates φ and λ of point P, whose polar geodetic co-ordinates α_0 and s referred to point A (φ_0, λ_0) are known, are to be determined.

On the sphere of radius $R_0 = \sqrt{\rho_0 N_0}$, let us designate as Φ_0, Λ_0 the geographical co-ordinates of point A' computed by means of the preceding expressions, and as Φ and Λ those, at present unknown, of Point P', points that correspond to A and P on the ellipsoid.

If we take :

$$\sigma = \frac{s}{R_0 \text{ arc } 1''}$$

in the spherical triangle with sides $90° - \Phi_0$, $90° - \Phi$, σ, the latter two having as opposite angles α and $\Lambda - \Lambda_0$, by using logarithmic formulas we get :

$$\text{tg}(\Lambda - \Lambda_0) = \frac{\text{tg}\,\alpha_0 \sin K}{\cos(\Phi_0 + K)} \tag{40}$$

$$\text{tg}\,\Phi = \cos(\Lambda - \Lambda_0)\,\text{tg}(\Phi_0 + K)$$

where :

$$\text{tg}\,K = \cos\alpha_0\,\text{tg}\,\sigma \tag{40'}$$

values of Φ **and** Λ ; and by substituting in formulae (18), we get the geographical co-ordinates sought for relating to point P.

The equality :

$$\frac{\sin\alpha_0 \sin\sigma}{\sin(\Lambda - \Lambda_0)\cos\Phi} = 1 \tag{41}$$

makes it possible to prove the calculation.

3.—If arc s is comparatively small with relation to the radius of the sphere, it will be convenient to use a series development, progressing according to increasing powers of s, in order to get Φ and Λ.

As independent variable for determining points of the great circle passing through A' and P', let us take its arc s reckoned from A' ; V and Λ will be finite and continuous functions of s : V = V (s), $\Lambda = \Lambda$ (s), and consequently, by Mac-Laurin's development :

$$V = V_0 + s\left(\frac{dV}{ds}\right)_{A'} + \frac{s^2}{2!}\left(\frac{d^2V}{ds^2}\right)_{A'} + \dots\dots\dots$$

$$\Lambda = \Lambda_0 + s\left(\frac{d\Lambda}{ds}\right)_{A'} + \frac{s^2}{2!}\left(\frac{d^2\Lambda}{ds}\right)_{A'} + \dots\dots\dots$$

In order to obtain the values of successive derivatives of point A' appearing in these developments, it should first be observed that :

$$\frac{dV}{ds} = \frac{dV}{d\Phi}\frac{d\Phi}{ds} = \frac{1}{\cos\Phi}\frac{\cos\alpha}{R}$$

and thus successively :

$$\frac{d^2V}{ds^2} = \frac{1}{R\cos\Phi}\left[\sin\alpha\frac{d\alpha}{ds} - \cos\alpha\,\text{tg}\,\Phi\frac{d\Phi}{ds}\right]$$

Deriving Clairaut's equation

$$R\cos\Phi\sin\alpha = \text{constant}$$

we get on the other hand :

$$-R\sin\Phi\sin\alpha\frac{d\Phi}{ds} + R\cos\Phi\cos\alpha\frac{d\alpha}{ds} = 0$$

i.e. :

$$\frac{d\alpha}{ds} = \frac{\sin\alpha}{R}\,\text{tg}\,\Phi$$

and thence :

$$\frac{d^2 V}{d s^2} = \frac{1}{R^2 \cos^2 \Phi} \sin \Phi \cos 2 \alpha$$

By successive derivations, the following equalities can easily be obtained :

$$\frac{d^3 V}{d s^3} = \frac{1}{R^3 \cos^3 \Phi} \left[(1 + \sin^2 \Phi) \cos^3 \alpha - (1 + 5 \sin^2 \Phi) \sin^2 \alpha \cos \alpha\right]$$

$$\frac{d^4 V}{d s^4} = \frac{1}{R^4 \cos^4 \Phi} \sin \Phi \left[(5 + \sin^2 \Phi) \cos^4 \alpha - 18 (1 + \sin^2 \Phi) \sin^2 \alpha \cos^2 \alpha + (1 + 5 \sin^2 \Phi) \sin^4 \alpha\right]$$

We can similarly obtain the successive derivatives of longitude with reference to the arc ; from the known relation :

$$\frac{d \Lambda}{d s} = \frac{\sin \alpha}{R \cos \Phi}$$

we get :

$$\frac{d^2 \Lambda}{d s^2} = \frac{1}{R^2 \cos^2 \Phi} \sin 2 \alpha \sin \Phi$$

$$\frac{d^3 \Lambda}{d s^3} = \frac{1}{R^3 \cos^3 \Phi} 2 \sin \alpha \left[(1 + 2 \sin^2 \Phi) \cos^2 \alpha - \sin^2 \Phi \sin^2 \alpha\right]$$

$$\frac{d^4 \Lambda}{d s^4} = \frac{1}{R^4 \cos^4 \Phi} 4 \sin 2 \alpha \sin \Phi \left[(2 + \sin^2 \Phi) \cos^2 \alpha - (1 + 2 \sin^2 \Phi) \sin^2 \alpha\right]$$

Substituting values taken on by these derivatives for point A' in the preceding developments, and taking into account formulae (18), by putting

$$v'_0 = \frac{1}{K_1} V'_0 - \frac{1}{K_1 M \operatorname{arc} 1'} \log K_2 \qquad A_1 = \frac{1}{K_1 \operatorname{arc} 1' \cos \Phi_0} \qquad A_2 = \frac{1}{2 K_1 \operatorname{arc} 1'} \frac{\sin \Phi_0}{\cos^2 \Phi_0}$$

$$A_3 = \frac{1}{6 K_1 \operatorname{arc} 1'} \frac{1 + \sin^2 \Phi_0}{\cos^3 \Phi_0} \qquad A_4 = - \frac{1}{6 K_1 \operatorname{arc} 1'} \frac{1 + 5 \sin^2 \Phi_0}{\cos^3 \Phi_0} \qquad A_5 = \frac{1}{24 K_1 \operatorname{arc} 1'} (5 + \sin^2 \Phi_0) \frac{\sin \Phi_0}{\cos^4 \Phi_0}$$

$$A_6 = - \frac{1}{4 K_1 \operatorname{arc} 1'} (1 + \sin^2 \Phi_0) \frac{\sin \Phi_0}{\cos^4 \Phi_0} = - \frac{9}{2} \operatorname{tg} \Phi_0 A_3$$

$$A_7 = \frac{1}{24 K_1 \operatorname{arc} 1'} (1 + 5 \sin^2 \Phi_0) \frac{\sin \Phi_0}{\cos^4 \Phi_0} = - \frac{1}{4} \operatorname{tg} \Phi_0 A_4 \qquad (42)$$

and

$$\lambda_0 = \frac{1}{K_1} \Lambda_0 \qquad B_1 = \frac{1}{K_4 \operatorname{arc} 1'' \cos \Phi_0} \qquad B_2 = \frac{1}{K_1 \operatorname{arc} 1''} \frac{\sin \Phi_0}{\cos^2 \Phi_0} \qquad B_3 = \frac{1}{3 K_1 \operatorname{arc} 1''} \frac{1 + 2 \sin^2 \Phi_0}{\cos^3 \Phi_0}$$

$$B_4 = - \frac{1}{3 K_1 \operatorname{arc} 1''} \frac{\sin^2 \Phi_0}{\cos^3 \Phi_0} \qquad B_5 = \frac{1}{3 K_1 \operatorname{arc} 1''} (2 + \sin^2 \Phi_0) \frac{\sin \Phi_0}{\cos^4 \Phi_0}$$

$$B_6 = - \frac{1}{3 K_1 \operatorname{arc} 1''} (1 + 2 \sin^2 \Phi_0) \frac{\sin \Phi_0}{\cos^4 \Phi_0} = - B_3 \operatorname{tg} \Phi_0 \qquad (42')$$

$$a = \sigma \cos \alpha_0 \qquad b = \sigma \sin \alpha_0 \qquad \sigma = \frac{s}{R_0} \qquad (42'')$$

we get the expressions :

$$v' = v'_0 + A_1 a + A_2 (a^2 - b^2) + A_3 a^3 + A_4 a b^2 + A_5 a^4 + A_6 a^2 b^2 + A_7 b^4 + \ldots \ldots \ldots$$

$$\lambda'' = \lambda_0 + B_1 b + B_2 a b + B_3 a^2 b + B_4 b^3 + B_5 a^3 b + B_6 a b^3 + \ldots \ldots \ldots \qquad (43)$$

making it possible to compute directly the co-ordinates of the ellipsoid; the meridional part and longitude of point P, respectively expressed in minutes and seconds of arc.

It should be pointed out that with reference to our latitudes, the part relating to the combination of terms of the fourth order in the formation of both co-ordinates does not exceed one-hundredth of a second when arc s is below 70 km., and is less than a second when s does not exceed 200 km.

4. *Example* 2.—Determine the geographical co-ordinates of point P on the International Ellipsoid, the polar geographical co-ordinates of which are known : $s = 148\,715.78$ m., $\alpha_0 = 110° 49' 54''.82$, referred to point A of geographical co-ordinates : $\varphi_0 = 45°$, $\Lambda = 10°$ E.

Taking as normal parallel that passing through A, equations of the correspondence are supplied by the expressions (a) (a') which we found in the foregoing example ; in which, on the sphere of radius : $R_0 = 6\,378\,352$ m., point A' with co-ordinates : $\Phi_0 = 44°\,57'05''.79$ $\Lambda_0 = 10°00'30''.44$ corresponds to point A ; moreover : $\sigma = \frac{s}{R_0 \text{ arc } 1''} = 1°20'09''.210$.

With these elements, using formula (40') we get $K = 359°31'20''.44$ and consequently, using formulae (40) : $\Lambda = 42\,328''.75$ and $\Phi = 44°27'78\,583$ to which value $V = 2984'5930$, obtained from ordinary tables of meridional parts for the sphere, corresponds. Substituting these values obtained from formulae (a) we get : $v = 2972'.3724$, $\lambda = 42\,292''.99$, and consequently, by computing formula (39'), or by inverse interpolation in meridonal parts tables relating to the International Ellipsoid, we get the values of the co-ordinates sought : $\varphi = 44°30'38''.32$, $\lambda = 11°44'52''.99$ E.

If it is desired to apply formulae (43) for the direct computation of λ and v, the numerical values of the coefficients of these developments, corresponding to latitude Φ_0 in the present example, are the following :

$V_0 = 3013'.579\,02$	$A_1 = 4853'.502\,70$	$A_2 = 2422'.656$	$A_3 = 2421'.3$	$A_4 = -5646'.1$
	$A_5 = 2217'$	$A_6 = -1\,0877'$	$A_7 = 1409'$	
	$B_1 = 291\,210''.162$	$B_2 = 290\,718''.7$	$B_3 = 387\,298''$	
	$B_4 = -96\,743''$	$B_5 = 483\,551''$	$B_6 = -386\,644''$	

thence, having :

$$\sigma = \frac{s}{R_0}\ 0.0233\ 15707 \qquad a = -0.0082\ 91702 \qquad b = 0.0217\ 91510$$

we get : $v = 2972'.3723$, $\lambda = 11°44'52''.99$, which are in complete harmony with values found previously.

5. *Converse problem.*—Given geographical positions (φ_0, λ_0) and (φ, λ) of points A and P on the ellipsoid, determine the polar geodetic co-ordinates of point P referred to point A, or find length s of the geodetic arc connecting them and azimuth α_0 of extremity A.

Denoting as A' (Φ_0, Λ_0), P' (Φ, Λ) the points on the sphere of radius $R_0 = \sqrt{\rho_0 N_0}$ corresponding to points A and P on the ellipsoid, we get, in the same spherical triangle we examined in the direct problem and by using the same formulae, the following relationships :

$$\text{tg}\ \alpha_0 = \frac{\text{tg}\ (\Lambda - \Lambda_0) \sin K}{\cos (\Phi_0 + K)} \qquad (44)$$

$$\text{tg}\ \sigma = \frac{\text{cotg}\ (\Phi_0 + K)}{\cos \alpha_0}$$

where :

$$\text{tg}\ K = \cos (\Lambda - \Lambda_0)\ \text{cotg}\ \Phi \qquad (44')$$

whence we compute azimuth α_0, owing to isogonic correspondence, and, with sufficient approximation, arc s = σ R_0 arc 1'', which we were looking for.

Proof is likewise obtained by means of formula (41).

6. *Example* 3.—Given the geographical co-ordinates of points A ($\varphi_0 = 45°$, $\lambda = 10°$ E.), P ($\varphi = 44°30'38''.32$, $\lambda = 11°44'52''.99$ E.) ; determine length s of the geodetic arc connecting them and azimuth α_0 formed at extremity A.

The meridional parts relating to the given values of φ_0 and φ, obtained from the above-mentioned tables with parameters of the international ellipsoid, are respectively : $v_0 = 3013'.5790$, $v = 2972'.3722$. Taking then the parallel at A as normal parallel, and applying formulae (a'), we get : $V_0 = 3025'.8347$, $\Lambda_0 = 10°00'30''.44$; $V = 2984'.5930$, $\Lambda = 11°45'28''.75$.

From tables of meridional parts for the sphere, and by inverse interpolation, we get the values corresponding to V_0 and V : $\Phi_0 = 44°57'05''.79$, $\Phi = 44°27'47''.15$.

Since we also know the co-ordinates of points A' and P' which on the sphere of radius $R_1 = 6\,378\,352$ m. correspond to points A and P on the ellipsoid, through formula (44') we get : $K = 45°31'24'',767$, then from formulae (44) we get : $\alpha_0 = 110°49'54''.82$ and $\sigma = 4809''.210$, i.e. : s = σ R_0 arc 1'' = 148 715.78 m., for the polar geodetic co-ordinates sought of P referred to point A.

6. **Layout of hyperbolae on the ellipsoid.**

1.—On the reference ellipsoid, let us denote as A and B the foci of the hyperbola of parameter n_I that we desire to plot. Let the pole be focus A having as geographical co-ordinates φ_0 and λ_0, and let m_I and α_1 be the polar geodetic co-ordinates of the other focus. Then by selecting parallel φ_0 of A as the normal parallel for correspondence between the ellipsoid and the sphere of radius $R_0 = \sqrt{\rho_o N_o}$, point A' on the sphere with co-ordinates Φ_o, Λ_o will correspond to point A, while the polar spherical co-ordinates of point B, corresponding to the other focus, will again be m_I and α_1

Let us agree to take parameter n, characterizing the hyperbola of the m family of homofocal hyperbolae, negative with relation to the branch outside focus A taken as the pole — at each point of which the distance from the pole is therefore greater than the distance from the other focus — expression (9) then supplies the following relationship between the polar spherical co-ordinates of points of the hyperbola plotted on the sphere of radius R_0 :

$$\text{cotg } \sigma = M_1 \cos(\alpha - \alpha_1) + N_1 \qquad (45)$$

As m_I and n_I are in general small in relation to R_o, coefficients :

$$M_1 = \frac{\sin \frac{m_1}{R_o}}{\cos \frac{n_1}{R_o} - \cos \frac{m_1}{R_o}} \qquad N_1 = \frac{\sin \frac{n_1}{R_o}}{\cos \frac{n_1}{R_o} - \cos \frac{m_1}{R_o}} \qquad (46)$$

can be expressed as :

$$M_1 = \frac{\sin \frac{m_1}{R_0}}{2 \sin \frac{1}{2 R_o}(m_1 + n_1) \sin \frac{1}{2 R_o}(m_1 - n_1)} = \frac{1}{2}\left[\text{cotg } \frac{1}{2 R_0}(m_1 - n_1) + \text{cotg } \frac{1}{2 R_0}(m_1 + n_1)\right]$$

$$N_1 = \frac{\sin \frac{n_1}{R_0}}{2 \sin \frac{1}{2 R_o}(m_1 + n_1) \sin \frac{1}{2 R_o}(m_1 - n_1)} = \frac{1}{2}\left[\text{cotg } \frac{1}{2 R_0}(m_1 - n_1) - \text{cotg } \frac{1}{2 R_0}(m_1 + n_1)\right] \qquad (46')$$

which is more convenient in numerical calculation.

Then by developing cotangents in series, and by appropriate manipulation, we get :

$$\left.\begin{aligned} M_1 &= \frac{2 m_1}{m_1^2 - n_1^2} R_o - \frac{m_1}{6 R_o}\left[1 + \frac{1}{60 R_0^2}(m_1^2 + 3 n_1^2) + \ldots\ldots\right] \\ N_1 &= \frac{2 n_1}{m_1^2 - n_1^2} R_o + \frac{n_1}{6 R_o}\left[1 + \frac{1}{60 R_0^2}(n_1^2 + 3 m_1^2) + \ldots\ldots\right] \end{aligned}\right\} \qquad (46'')$$

where the first term is of the order of : $\frac{1}{945}\left(\frac{m_1}{R_o}\right)^5$ and therefore not greater than 1.10^{-9} when m_I is under 400 km ; however, for such value of m_I, the last term considered in the development is a contributing factor in forming the two coefficients in an amount under 4.10^{-6}.

2.—With regard to standard azimuth α, formula (40) supplies polar distance $\sigma = \frac{s}{R_o \text{ arc } 1''}$ referred to the sphere of unit radius corresponding to standard point P' on the hyperbola.

It should be noted that the polar co-ordinates, whether obtained from formulae (40) or (42), make it possible to compute values Φ and Λ which, by substitution in formulae (18), determine the geographical co-ordinates in relation to point P of the branch being considered of the hyperbola on the ellipsoid.

3 *Example* 4.—Determine by means of points on the international ellipsoid the branch of hyperbola of parameter n_I = — 14 347 m. having as foci points A (φ_0 = 45°, λ_0 = 10° E.) B (φ_1 = 45°38'43".00, λ = 11°09'31".11 E.).

By taking the parallel at A as normal parallel in the correspondence with the sphere of radius R_0 = 6 378 352 m., the polar geodetic co-ordinates of B referred to focus A will be obtained by applying the method followed in the preceding example, m_I = 115 740 m. α_1 = 51°18'00".00.

With these values, from formulae (46) we get : $M_I = 111.935\,62$, $N_I = -13.876\,16$, and then, through formula (45), the relation between the polar spherical co-ordinates of the points of the branch is given by the expression :

$$\text{cotan}\ \sigma = 111.935\,62 \cos(\alpha - 51°18'00''.00) - 13.876\,16.$$

As we desire to obtain the geographical co-ordinates of point P of the hyperbola whose polar distance AP = σ R_0 arc 1" forms azimuth $\alpha = 110°49'54''.82$ at pole A, we will first determine, through use of the preceding expression, the value of σ corresponding to α : $\sigma = 4809''.2097$, i.e. s = σ R_0 arc 1" = 148 715.78 m.

Knowing the polar co-ordinates on the sphere—which, owing to the correspondence adopted, show negligible differences when compared with the geodetic co-ordinates within a 1,000 km. area around pole A—we can finally obtain, by using the transfer of co-ordinates method, the geographical co-ordinates of point P of the hyperbola, which in the present case are $\varphi = 44°30'38''.32$, $\lambda = 11°44'52''.99$ E., the same results as in example 2.

7. **Problems inherent to the determination of the co-ordinates of a point in a net.**

1.—Let φ_0 and λ_0 be the geographical co-ordinates of the master station, i.e. of focus A common to both families of hyperbolae whose other foci are respectively at slave stations B and C, for which polar geodetic co-ordinates (m_I α_1) (m_2, α_2) referred to point A are given.

Following the convention we established with regard to the symbol attributed to parameter n, we shall denote as co-ordinates of point P in the net the values n_I and n_2 of n in relation to the branches of the hyperbolae respectively belonging to families m_I and m_2 containing it.

2. *Direct problem.*—As the first problem, let us determine the geographical co-ordinates of a point P whose n_I and n_2 co-ordinates in the net are known. The problem obviously merely consists in finding the meeting-points of the branch of parameter n_I, belonging to family m_I and the branch of parameter n_2 of the hyperbola of family m_2, a problem that we have already solved in connection with the laying-out of hyperbolae on the sphere of unit radius and which we will now transfer to the ellipsoid.

Applying the usual method, let us take parallel φ_0 at the master station as normal parallel of the correspondence. A' (Φ_0, Λ_0) will then correspond to point A on the sphere of radius $R_0 = \sqrt{\rho_0 N_0}$, while (m_I, α_1) and (m_2, α_2) will represent the spherical polar co-ordinates of the other two foci B' and C' of the hyperbolae.

From the two polar co-ordinates of the meeting-points, azimuth values are directly obtained by using formula (13) :

$$\sin(\alpha + \varepsilon) = \frac{N_2 - N_1}{M_1 \cos \alpha_1 - M_2 \cos \alpha_2} \sin \varepsilon \qquad (13)$$

where values of M and N are supplied by formulas (46) while the auxiliary quantity ε is obtained from :

$$\text{tg}\ \varepsilon = \frac{M_1 \cos \alpha_1 - M_2 \cos \alpha_2}{M_1 \sin \alpha_1 - M_2 \sin \alpha_2} \qquad (12)$$

The latter takes the logarithmic form of

$$\text{tg}\ \varepsilon = \frac{\sin(\alpha_2 - K) \cos \alpha_1}{\sin(\alpha_1 - K) \sin \alpha_2} \qquad (12')$$

when we put :

$$\text{tg}\ K = \frac{M_2 \sin \alpha_2}{M_1 \cos \alpha_1} \qquad (12'')$$

The azimuth being known, the polar distance σ is computed with reference to the unit sphere by means of formula (45) with regard to the branch of the hyperbola of parameter n_I of family m_I or of parameter n_2 of family m_2.

The polar co-ordinates being known, we go on to the geographical co-ordinates Φ, Λ on the sphere, by using formulae (40), and from these, through use of formulae (18), we finally get the required geographical co-ordinates φ and λ.

3. *Example* 5.—Master station A and the two slave stations B and C have the following geographical co-ordinates : A ($\varphi_0 = 45°$, $\lambda_0 = 10°$ E.) ; B ($\varphi_1 = 45°38'43''.00$, $\lambda_1 = 11°09'31''.11$ E.) ; C ($\varphi_2 = 43°59'56''.69$, $\lambda_2 = 10°07'34''.50$ E.). Knowing the co-ordinates of point P in the net : $n_I = -14\,347$ m., $n_2 = -7\,268$ m., determine its geographical position.

Let us assume point A as the pole of a system of polar co-ordinates, and as normal parallel of the correspondence, that passing through this same point. Applying the method followed in the converse problem of transfer of co-ordinates, we obtain the following values for the polar co-ordinates in relation to foci B and C : $m_I = 115\,740$ m., $\alpha_1 = 51°18'00''.00$; $m_2 = 111\,680$ m., $\alpha_2 = 174°47'50''.00$.

From formulae (46) and (45), we get the equations of the two branches of the hyperbolae respectively of parameters n_I and n_2 : cotan $\sigma = 111.935\,63 \cos(\alpha - 51°18'00''.00) - 13.876\,16$; cotan $\sigma = 114.70841 \cos(\alpha - 174°47'50''.00) - 7.465\,47$. Applying formulae (12), for the auxiliary angle we get the value : $\varepsilon = 67°19'49.''96$, and then, by means of formulae (13) : $\alpha = 294°30'43''.26$ and $\alpha = 110°49'54''.82$. In connection with these azimuth values, by applying any one of the previous equations we get : $\sigma = 179°06'33.63$ and $\sigma = 4809''.2097$. Leaving aside the farther point from the pole, which however satisfies the problem, we get the geographical co-ordinates sought for in relation to the second by means of the formulae found in the transfer of co-ordinates, as results of the preceding example : $\varphi = 44°30'38''.32$, $\lambda = 11°44'52''.99$ E.

4. *Converse problem.* Let us now attempt to solve by means of spherical geometry the converse of the foregoing problem, i.e., to determine co-ordinates n_I, n_2 of a point P of the net, the geographical co-ordinates being known.

With the correspondence previously established, points A' (Φ_0, Λ_0) B' (Φ_1, Λ_1), C' (Φ_2, Λ_2), P' (Φ, Λ) on the sphere of radius R_0 correspond to points A (φ_0, λ_0), B (φ_1, λ_1), C (φ_2, λ_2), P (φ, λ) on the corresponding ellipsoid. By applying formulae (44) to triangles A'P'B', A'P'C' we will obtain the lengths of sides A'P', B'P' and C'P' referred to the unit sphere, and consequently, following our convention, the differences

$$B'P' - A'P' = n_I \qquad C'P' - A'P' = n_2 \qquad (47)$$

will supply, with the symbol and with sufficient approximation, the required co-ordinates.

5. *Example* 6.—Without changing the position of points A, B, C in the previous example, compute the co-ordinate in the net of point P ($\varphi = 44°30'38''.32, \lambda = 11°44'52''.99$ E.).

Let the parallel at A be retained as normal parallel in the correspondence ; on the sphere of radius R_0 points A' ($\Phi_0 = 44°57'05''.79$) $\Lambda_0 = 10°00'30''.44$) ; B' ($\Phi_1 = 45°35'45''.00$, $\Lambda_1 = 11°\ 10'05''.08$) ; C' ($\Phi_2 = 43°57'08''.88$, $\Lambda_2 = 10°08'05''.33$); P' ($\Phi = 44°27'47''.15$, $\Lambda = 11°45'28''.75$) will correspond to the four given points on the ellipsoid.

The computing of arcs : $\dfrac{A'P'}{R_0 \text{ arc } 1''} = \sigma$, $\dfrac{B'P'}{R_0 \text{ arc } 1''} = \sigma_1$, $\dfrac{C'P'}{R_0 \text{ arc } 1''} = \sigma_2$ effected by means of formulas (44) gives : $\sigma = 4809''.210$, $\sigma_1 = 4345''.253$, $\sigma_2 = 4574''.175$, and consequently, on the sphere of radius R_0 : A'P' = 148 715.78 m., B'P' = 134 368.79 m., C'P' = 141 447.76 m. will be obtained. Whence, according to conditions set by the correspondence, the co-ordinates sought are the following :

$$n_I = 14\,346.99 \text{ m.} \qquad n_2 = -7\,268.02 \text{ m.}$$