THE ANALYSIS OF HIGH AND LOW WATERS.

(Part I : Semidiurnal Tides Predominant).

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INTRODUCTION

Non-harmonic tidal constants have always been obtained from observations of high and low waters, but very little success has been achieved by methods designed to give harmonic constants. The formidable nature of the problem is revealed in a method designed by Sir George Darwin (1890), and the complexity of the method and its limitation to diurnal and semidiurnal tides have combined to rule it out as a practical method. There is, however, a need for a method which is not difficult for computers and which does not involve any abnormal processes, for there is a great deal of tidal data which are at present unusable because of the lack of a method of analysis which will give harmonic constants. It is conceded that no analysis of high and low water data can be so exact as one which uses hourly heights, partly because of the fewer data per day (8 observed quantities against 24), partly because so few observations per day do not suffice to give all species of tide, partly because of the measure of uncertainty with which times of high and low water can be observed, and partly because of the intrinsic difficulties of the analysis. Methods of this kind, therefore, can only be recommended when hourly heights are not available.

The remark that the fewness of the observations per day limits the analysis needs elaboration. To determine a tide of one species requires two observations or two analytical quantities per day. Eight observations will give tides of species diurnal, semi-diurnal, and third-diurnal, but will not be sufficient to discriminate between long-period and quarter-diurnal tides. The sixth-diurnal tides also cannot be separated from the semi-diurnal tides, nor the eighth-diurnal from the quarter-diurnals, (though it is possible, by very elaborate methods, to analyse for, say, certain constituents of the sixth-diurnal species whose increments of phase per day are larger than the increments of phase of any semi-diurnal constituents). It is necessary to make certain assumptions with regard to the quarter-diurnal tides in order to obtain the maximum information from the analysis.

The constants derived from the analysis of high and low waters will therefore differ, perhaps appreciably, from those derived from hourly heights if the shallow-water tides are important. One interesting possibility emerges, however, in that, as these special constants represent the data from which they are derived, and that all the data are fully used, the special constants should give direct on the machine the predictions which normally would require harmonic shallow-water corrections to the machine predictions derived from the standard constituents, but this possibility has not yet been fully examined.

The method of analysis expounded in Part I of this paper is based upon certain series expansions which are only valid when the diurnal and shallow-water tides are small compared with the semi-diurnal tides, but as these conditions are very commonly satisfied the method is applicable for most places in the world. It requires observations of tides over 32 solar days.

There are certain principles of analysis, exemplified in this method, which are of the highest importance. The author has long taught that it is essential to separate as far as possible the several species of tide day by day, and that every possible effort should be made to ensure that this is done as completely as is possible or practical; otherwise the corrections to be made to the major processes of analysis become greatly involved. It should not be necessary to correct these major processes except for constituents all of the same species. Another fundamental principle is that all operations should be conducted, day by day if possible, around a central time origin so that all corrections for functions which involve cosines of arguments do not involve sines; that is, the operations relate together functions which are symmetrical about the time-origin and similarly those which are asymmetrical with regard to the time-origin. By this method the corrections are simplified and reduced greatly in quantity. Also, there should not normally be any need to resort to astronomical data except for the interpretation and reduction of the final constants provided by the analysis. Finally, the method of analysis should be so simple that it does not call for any special skill or understanding by the computer, provided that he is accustomed to the normal processes of tidal analysis.

INSTRUCTIONS FOR ANALYSIS

I) In columns 1 to 4 arrange the high and lower water heights Z in sequences denoted by s = 0, 1, 2, 3. A sequence of high waters consists of every alternate high water; that is, at intervals of approximately 24 hours. For sequence s = 0 choose the first high water of the day on the central day, and then write the rest in order. For sequence s = 1 write down the ensuing low water heights, about 6 hours later than the high waters of s = 0; then for sequence s = 2 write down the high waters about 12 hours later than for s = 0; and finally for s = 3 write the low waters about 18 hours later than for s = 0. These heights are written consecutively in the column, paying no attention to the elapse of a solar day when the time passes through midnight. If there are missing observations they can be filled in by estimation. There should be 31 entries in each sequence, those for days — 15 and +15 being only used in the initial processes. Normally Z will be given to 0.1 feet.

II) In columns 5 to 8 write down the solar times (in hours and decimals) of high and low water, t, which correspond to the values of Z. After passing through midnight the true sequence times should really be increased by 24 hours but this will be understood and allowed for in the calculations. It is necessary in this method of analysis to work in decimals of an hour and not in minutes, and the times should be entered to two decimals.

III) We use the notation Z(s) to indicate a value of Z according to its sequence, and we have Z(0) to Z (3) on one line. It is convenient, however, to refer to Z(3) on the previous day as Z(— 1), and to Z(2) on the previous day as Z(— 2), while Z(4) and Z(5) refer to Z(0) and Z(1) on the next day. Similarly t(s) denotes the time according to its sequence.

We compute, to two decimals of a foot,

$$\alpha(0) = \frac{1}{4}[2Z(0) + Z(2) + Z(-2)]$$

$$\alpha(1) = \frac{1}{4}[2Z(1) + Z(3) + Z(-1)]$$

$$\alpha(2) = \frac{1}{4}[2Z(2) + Z(4) + Z(0)]$$

$$\alpha(3) = \frac{1}{4}[2Z(3) + Z(5) + Z(1)]$$

These processes are very simply performed; we take for $\alpha(0)$ the mean value of Z(2) and Z(—2), which will be approximately equal to Z(0), and we take the mean of this mean with Z(0). The values of $\alpha(s)$ are given in columns 9 to 12.

IV) For the times we have a similar process, but it is rendered a little more complicated by the steady increase of time and also by its passage through 24 hours.

We define

$$t' = t \pm 12$$

and take

$$\beta(0) = \frac{1}{4}[2t(o) + t'(2) + t'(-2)]$$

with similar formulae for $\beta(1)$, $\beta(2)$, $\beta(3)$.

Whether we add or subtract 12 to t depends upon whether any sequence has passed through 24 hours relatively to another. The following examples will clarify the process:

day	s = 0	1	2	3
1	22.92	5.51	10.99	17.66
2	23.61	6.23	11.70	18.37
3	00.27	6.90	12.40	19.05

For $\beta(0)$ on day 2 we have $t(0) = 23.61$, and we need to make $t(2)$ and $t(-2)$ approximately the same; this requires us to add 12 hours to each, so that we get $\beta(0)$ on day $2 = \frac{1}{4}(47.22+23.70+22.99)$. On day 3, however, we need to subtract 12 hours from $t(2)$ and $t(-2)$ so that $\beta(0) = \frac{1}{4}(00.54 - 00.30 + 00.40)$. The principle is that $t(2)$ and $t(-2)$ should be amended by 12 hours so as to make them nearly equal to $t(0)$. Similar ideas hold with regard to $\beta(1)$; thus on day 3 we have $\beta(1) = \frac{1}{4}(13.80+6.37+7.05)$.

With a little practice the process becomes exceedingly simple. It is only necessary to remember that $\beta(s)$ is nearly equal to $t(s)$. The values of $\beta(s)$ are given in columns 13 to 16.

V) The meaning of the processes III and IV is that we have computed quantities α and β which are almost independent of the diurnal and third-diurnal tides. To obtain the diurnal tides, with the third-diurnals, we compute in columns 17-20, and 21-24, the values of

$$\alpha_1(s) = Z(s) - \alpha(s)$$
$$\beta_1(s) = t(s) - \beta(s)$$

VI) (*Commencing Sheet II*).

As the values of α_1 and β_1 are not sufficiently free from semidiurnal contributions it is necessary to combine the values so as to eliminate the undesired contributions.

We therefore compute

$$\gamma(0) = \frac{1}{4}[2\alpha_1(0) - \alpha_1(2) - \alpha_1(-2)]$$

$$\delta(0) = \frac{1}{4}[2\beta_1(0) - \beta_1(2) - \beta_1(-2)]$$

$$\gamma(1) = \frac{1}{2}[\alpha_1(1) - \alpha_1(-1)]$$

$$\delta(1) = \frac{1}{2}[\beta_1(1) - \beta_1(-1)]$$

The operations are similar to those already carried out, and the values of $\gamma(0)$, $\gamma(1)$, $\delta(0)$, $\delta(1)$ are given in columns 25, 26, 27, and 28 respectively.

VII) There are contributions to α from mean level, the semidiurnal tide, and the quarterdiurnal tide, and we therefore compute

$$\alpha_0+\alpha_4 = \frac{1}{4}\,[2\alpha(0)+\alpha(1)+\alpha(-1)]$$

$$\alpha_2 = \alpha(0) - (\alpha_0+\alpha_4)$$

The results are given in columns 29 and 30, respectively.

VIII) Similarly we compute, in columns 31 and 32,

$$\beta_2 = \frac{1}{4}\,[2\beta(0)+\beta(1)+\beta(-1)]$$

$$\beta_4 = \beta(0) - \beta_2$$

IX) The above processes do not accurately give the required diurnal, semidiurnal, and quarterdiurnal quantities and so it is necessary to correct them.

Compute $\delta^2(0)$, $\delta^2(1)$, $2\beta_4^2$ in columns 33 to 35 to 3 places of decimals, and then compute to three places of decimals, in columns 36 to 38,

$$c_1 = \frac{1}{16}\,[\delta^2(0)+\delta^2(1)+2\beta_4^2]$$

$$c_2 = \frac{1}{16}\,[\delta^2(1) - \delta^2(0)]$$

$$c_3 = \frac{1}{4}\,\delta(0)\,\delta(1)$$

X) Compute to 2 places of decimals the values of $\alpha_2 c_1$, $\alpha_2\, c_2$, and (β_4+c_3), in columns 39 to 41.

XI) In columns 42 to 44 compute

$$A = \alpha_2 - \alpha_2\, c_1$$

$$a_0+a_4 = (\alpha_0+\alpha_4)+\alpha_2\, c_2$$

$$b_4 = \frac{1}{4}\,A\,(\beta_4+c_3)$$

XII) On sheet IV of the calculations a space is reserved for the calculation of a_0 and a multiplier called λ. Referring to columns 42 and 43 underline the spring and neap values of A, the amplitude of the semidiurnal tide, and underline the corresponding values of (a_0+a_4). Transfer these marked quantities in proper order to sheet IV and compute A^2 (to one decimal). Alongside the 4 values of A^2 put multipliers 1, — 3, 3, — 1 and then form the sum of the products of these multipliers with the corresponding values of (a_0+a_4). Similarly form the sum of the products of the multipliers with A^2. In the example these are respectively

$$8.05 - (3\times 7.97) + (3\times 8.00) - 8.00 = 0.14$$

and

$$24.5 - (3\times 7.3\ ) + (3\times 45.3) - 2.6\ = 135.9$$

The ratio of these two, to 4 decimals, is 0.0010, which is called λ. Compute λA^2 and subtract from (a_0+a_4) to give a_0 at springs and neaps. The mean value of these 4 quantities gives mean sea level for the month.

XIII) Returning to column 45 compute

$$c_4 = 16\,\lambda b_4$$

and in column 46 compute

$$S = \beta_2+c_4$$

This is the time of high water of the mean semidiurnal tide for the day.

XIV) Compute S_0, the mean value of S, adding 24 hours for each value after the passage through 24. Enter this value in column 47 for the central day of the month. It is the time-origin for the whole analysis, and it is approximately equal to the value of S on the central day.

XV) Complete column 47 by adding S_0 to each of the values given in Table I, to give τ, adding or subtracting 24 to bring τ approximately equal to S. Alternatively, put S_0 on the calculating machine and add multiples of 0.8412 per day, positively and negatively from the central day. This is a lunar time scale and the values of τ are, very nearly, the times of high water of M_2.

XVI) Compute in column 48 the values of

$$\bar{\beta} = S - \tau$$

This is the time of high water of the mean semidiurnal tide for the day, relative to the lunar time scale, τ. Check that the mean value of $\bar{\beta}$ is zero.

XVII) (*Commencing Sheets III and IV*).

For the next processes of analysis it is necessary to transfer the time origin from the time of high water of the semidiurnal tide to the special lunar time scale.

In column 73 compute from $\bar{\beta}$ in column 48, to one place of decimals,

$$\varphi_2 = 29^\circ \bar{\beta}$$

In column 80 compute to nearest degree from φ_2

$$\varphi_4 = 2\ \varphi_2$$

In column 64 compute from φ_2, to one place of decimals,

$$\varphi_1 = \frac{1}{2}\ \varphi_2$$

In column 53 compute to nearest degree from φ_1

$$\varphi_3 = 3\ \varphi_1$$

XVIII) In columns 74 and 75 enter from trigonometrical tables the values of cos φ_2 and sin φ_2 to 3 decimals.

In columns 81, 82 ; 54, 55 ; 65, 66, enter $\cos\varphi$ and $\sin\varphi$ to 2 decimals for φ_4, φ_3, φ_1 respectively.

XIX) Compute to 2 decimals, in columns 76, 77, using A in column 42, the values of

$$A_2 = A\ \cos\varphi_2 \qquad\qquad B_2 = A\ \sin\varphi_2$$

XX) In column 78, compute $a_4 = \lambda A^2$ from the value of λ computed at stage (XII) and A in column 42.

Also copy the values of b_4 in column 79 from column 44.

XXI) In columns 49 and 50 compute from columns 27, 28, 42, values of

$$\varepsilon(0) = A\ \delta(0) \qquad\qquad \varepsilon(1) = A\ \delta(1)$$

XXII) In columns 51 and 52 compute from columns 25 and 26, 49 and 50, values of

$$a_3 = \frac{1}{4}\ [\gamma(0) - \varepsilon(1)] \quad , \quad b_3 = \frac{1}{4}\ [\varepsilon(0) - \gamma(1)]$$

XXIII) In columns 62 and 63 compute

$$a_1 = \gamma(0) - a_3 \quad , \quad b_1 = \gamma(1) + b_3$$

XXIV) Complete all the columns in sheets III and IV from

$$ac_4 = a_4 \cos \varphi_4 \qquad a\,s_4 = a_4 \sin \varphi_4$$
$$bc_4 = b_4 \cos \varphi_4 \qquad b\,s_4 = b_4 \sin \varphi_4$$
$$A_4 = ac_4 - b\,s_4 \qquad B_4 = bc_4 + a\,s_4$$

(with similar operations for the third-diurnal and diurnal tides).

XXV) The quantities A, B are now in a form suited to normal processes of analysis. In Table 2 are daily multipliers d_0, d_1, d_2, d_3, d_4, d_a, d_b, d_c, d_d. These columns of multipliers are each in turn placed alongside the columns of one of the functions A, B and the sum of the products is obtained, and entered on Sheet IV according to the multipliers and fonction used.

XXVI) These functions are combined by Table 3 for the diurnal and third-diurnal tides and by Table 4 for the semidiurnal and quarter-diurnal tides. The interpretation of Table 3 is that

$$A'_{10} = 1.000\ A_{10} + 0.094\ A_{11} - 0.064\ A_{12} + 0.036\ A_{13}$$

where the first suffix denotes the function A or B and the second one denotes the multiplier d.

The values of A′, B′ are entered on Sheet V in the same order as is given in Table 5.

XXVII) A further combination of the functions A′, B′ is effected by means of Table 5 to give apparent values of R cos r and R sin r for the principal constituents. The table combines at most two functions ; thus for K_1, we have

$$10^5\ R \cos r = 1665\ A'_{11} + 1746\ B'_{1a}$$
$$10^5\ R \sin r = 1665\ B'_{11} - 1746\ A'_{1a}$$

The resulting values of R cos r and R sin r are entered on Sheet V to 3 decimals, or, if the observations are not good, to 2 decimals. The values obtained as for μ_2 are not placed in the usual place in the table but at the bottom of the last column of the table, for reasons given below.

XXVIII) The calculation of H and g is effected by standard methods. We first calculate R and r for all constituents except μ_2 and enter the values on the appropriate lines in the schedule.

For μ_2 we have a special correction for $2N_2$. We infer values for this constituent from

$$R \text{ for } 2N_2 = 0.125 \times (R \text{ for } N_2)$$
$$r \text{ for } 2N_2 = 2 \times (r \text{ for } N_2) - (r \text{ for } M_2)$$

These values are subtracted from the nominal values from the analysis for μ_2, and from the residual values we compute R and r for μ_2.

Then we compute from standard tables the values of

$$V = V_0 + V'_0 + V''_0 + V'''_0$$

where

V_0 = the astronomical argument for the first day of the year
V'_0 = the increment to the first day of the month
V''_0 = the increment to the day of the month
V'''_0 = the increment to the hour of the time origin on the central day, equal to σS_0. For σ see Table 8.

If the computer has not got the full tables for all the constituents but has access only to the Admiralty Tide Tables, Part III, he can thence obtain the values of V for the principal constituents M_2, N_2, K_1, and O_1. Similarly standard methods for all constituents, or from Table 6 supplementing the Admiralty Tide Tables, Part III, will give values of f and u. From Table 6 he can then obtain the values for all the other constituents except for L_2, M_1 and M_3.

Certain constituents are perturbed by other large constituents whose speeds are nearly equal to that of the nominal constituent. Corrections for these are computed in the last column of Sheet V, using Table 7. The method of correction is the same as is used in the Admiralty Tide Tables, Part III, but the brief instructions given on the schedule should be clear enough. From these computations we obtain values of w and (1+W).

Finally we compute H to two decimals and g to one decimal from the formulae

$$H = R \div f\,(1+W)$$
$$g = r + V + u + w.$$

Remarks on Example of Analysis

The example is an artificial one which was prepared for testing the performance of a tide-predicting machine. The values of Z are given to two decimals, though the last decimal is only approximate. Normally only one decimal of a foot will be used, but this special example was intended to test the accuracy of the method, so that greater nominal accuracy is maintained in the computations than would normally be the case. No checks are necessary in most of this work as the quantities in the columns should be reasonably smooth; doubtful values may be re-checked or arbitrarily amended if the observations appear to be unsmooth. It is desirable, however, to work to two decimals of an hour throughout and to two decimals of a foot in α and α_1; this is permissible because the processes are averaging processes which tend to smooth out casual errors. The values of H have been computed to three decimals of a foot but normally two decimals will be sufficient, and the values of g have been computed to one decimal of a degree, but normally they may be given to the nearest degree.

The exact values of R cos r and R sin r used in the machine prediction are as follows:

	R cos r	R sin r	Errors			R cos r	R sin r	Errors	
	—		—			—		—	
M_2	4.000	0.063	0.003	0.023	K_1	0.479	0.165	0.005	0.002
L_2	0.159	0.082	0.002	0.004	O_1	0.156	0.481	0.009	0.011
N_2	0.794	0.411	0.014	0.001	J_1	0.080	0.041	0.004	0.000
S_2	1.588	0.823	0.001	0.015	Q_1	0.040	0.081	0.009	0.001
μ_2	0.080	0.041	0.006	0.003					
M_4	0.005	–0.160	0.019	0.003	MK_3	0.040	0.017	0.012	0.002
MN_4	0.043	–0.079	0.003	0.020	MO_3	0.015	0.040	0.010	0.004
MS_4	0.085	–0.158	0.002	0.027					

The errors for the constituents not used on the machine are the values obtained on Sheet V and it is evident that these are all small, of the order of 0.01 ft. There is a small time error of about 1 minute in the predictions, which accounts for the errors in M_2 and S_2. The principal errors are in the quarter-diurnal tides, for the obvious reason that an assumption had to be made with regard to them, but these errors are very small. The method could be used with safety for either the diurnal tides or the quarter-diurnal tides relatively greater to the semidiurnal tide but not both together.

TABLE 1.

L	τ-S_0
-14	12·22
-13	13·06
-12	13·91
-11	14·75
-10	15·59
- 9	16·43
- 8	17·27
- 7	18·11
- 6	18·95
- 5	19·79
- 4	20·64
- 3	21·48
- 2	22·32
- 1	23·16
0	0·00
1	0·84
2	1·68
3	2·52
4	3·36
5	4·21
6	5·05
7	5·89
8	6·73
9	7·57
10	8·41
11	9·25
12	10·09
13	10·94
14	11·78

TABLE 2.

L	d_0	d_1	d_2	d_3	d_4	d_a	d_b	d_c	d_d
-14	1	-2	2	-2	2	0	-1	1	-1
-13	1	-2	2	-1	1	1	-1	2	-1
-12	1	-2	1	0	-2	1	-2	2	-2
-11	1	-1	0	1	-2	1	-2	1	0
-10	1	-1	-1	2	-1	2	-2	0	1
- 9	1	-1	-2	2	0	2	-1	-1	2
- 8	1	0	-2	1	2	2	-1	-2	1
- 7	1	0	-2	-1	2	2	1	-2	0
- 6	1	1	-2	-1	1	2	1	-1	-2
- 5	1	1	-1	-2	-1	2	2	-1	-2
- 4	1	1	0	-2	-2	1	2	1	-1
- 3	1	1	1	-1	-2	1	2	2	1
- 2	1	2	1	1	0	1	1	2	2
- 1	1	2	2	2	1	1	1	1	2
0	1	2	2	2	2	0	0	0	0
1	1	2	2	2	1	-1	-1	-1	-2
2	1	2	1	1	0	-1	-1	-2	-2
3	1	1	1	-1	-2	-1	-2	-2	-1
4	1	1	0	-2	-2	-1	-2	-1	1
5	1	1	-1	-2	-1	-2	-2	1	2
6	1	1	-2	-1	1	-2	-1	1	2
7	1	0	-2	-1	2	-2	-1	2	0
8	1	0	-2	1	2	-2	1	2	-1
9	1	-1	-2	2	0	-2	1	1	-2
10	1	-1	-1	2	-1	-2	2	0	-1
11	1	-1	0	1	-2	-1	2	-1	0
12	1	-2	1	0	-2	-1	2	-2	2
13	1	-2	2	-1	1	-1	1	-2	1
14	1	-2	2	-2	2	0	1	-1	1

TABLE 3.
Combination for A'_{10}, A'_{11}, ...

	A'_{10}	A'_{11}	A'_{12}	A'_{13}
A_{10}	1·000	...	...	...
A_{11}	0·094	1·000	-0·124	-0·001
A_{12}	-0·064	0·259	1·000	-0·213
A_{13}	0·036	-0·138	0·415	1·000

	A'_{1a}	A'_{1b}	A'_{1c}
A_{1a}	1·000	-0·160	0·242
A_{1b}	0·148	1·000	-0·325
A_{1c}	-0·033	0·287	1·000

TABLE 4.
Combinations for A'_{20}, A'_{21}, ...

	A'_{20}	A'_{21}	A'_{22}	A'_{24}
A_{20}	1·000	...	...	...
A_{21}	0·081	1·000	-0·106	0·023
A_{22}	-0·009	0·078	1·000	0·037
A_{23}	...	...	...	...
A_{24}	-0·016	-0·014	-0·023	1·000

	A'_{2a}	A'_{2b}	A'_{2d}
A_{2a}	1·000	-0·194	-0·025
A_{2b}	0·042	1·000	0·044
A_{2c}	...	...	...
A_{2d}	-0·034	0·026	1·000

TABLE 5.

Combinations for $10^5 R \cos r$ and $10^5 R \sin r$.

		M_1	K_1	O_1	J_1	Q_1	OO_1	$2Q_1$
10^5 R cos r	A' 10	3448	...	...	...	...	...	...
	11	...	1665	1697	...	...	...	...
	12	...	...	...	1401	1630	...	...
	13	...	...	...	...	...	1583	1989
	B' 1a	...	1746	-1779	...	...	...	...
	1b	...	...	...	1654	-1925	...	...
	1c	...	...	...	...	...	1945	-2444
10^5 R sin r	B' 10	3448	...	...	...	...	...	...
	11	...	1665	1697	...	...	...	...
	12	...	...	...	1401	1630	...	...
	13	...	...	...	...	...	1583	1989
	A' 1a	...	-1746	1779	...	...	...	...
	1b	...	...	...	-1654	1925	...	...
	1c	...	...	...	...	...	-1945	2444

		M_3	MK_3	MO_3
10^5 R cos r	A' 30	3448	...	...
	31	...	1665	1697
	32	...	...	...
	33	...	...	...
	B' 3a	...	1746	-1779
	3b	...	...	...
	3c	...	...	...
10^5 R sin r	B' 30	3448	...	...
	31	...	1665	1697
	32	...	...	...
	33	...	...	...
	A' 3a	...	-1746	1779
	3b	...	...	...
	3c	...	...	...

		M_2	L_2	N_2	S_2	μ_2	$2SM_2$
10^5 R cos r	A' 20	3448	...	...	...	...	...
	21	...	1675	1739	...	...	...
	22	...	...	...	1538	1650	...
	24	...	...	...	...	...	1736
	B' 2a	...	1795	-1795	...	...	...
	2b	...	...	...	1731	-1731	...
	2d	...	...	...	...	...	1736
10^5 R sin r	B' 20	3448	...	...	...	...	...
	21	...	1707	1707	...	...	...
	22	...	...	...	1594	1594	...
	24	...	...	...	...	...	1730
	A' 2a	...	-1761	1829	...	...	...
	2b	...	...	...	-1670	1791	...
	2d	...	...	...	...	...	1730

		M_4	MN_4	MS_4
10^5 R cos r	A' 40	3448	...	...
	41	...	1739	...
	42	...	...	1538
	44	...	...	...
	B' 4a	...	-1795	...
	4b	...	...	1731
	4d	...	...	...
10^5 R sin r	B' 40	3448	...	...
	41	...	1707	...
	42	...	...	1594
	44	...	...	...
	A' 4a	...	1829	...
	4b	...	...	-1670
	4d	...	...	...

TABLE 6. (Supplementing A.T.T. Pt.III.)

Values of V, f, u.

	V	f	u		V	f	u
J_1*	$M_2+K_1-N_2$	as O_1	$-1{\cdot}2u(O_1)$				
Q_1	M_2-J_1	as O_1	as O_1	MK_3	M_2+K_1	$f(M_2).f(K_1)$	$u(M_2)+u(K_1)$
OO_1*	$2K_1-O_1+180°$	$3{\cdot}5f(O_1)-2{\cdot}43$	$3u(J_1)$	MO_3	M_2+O_1	$f(M_2).f(O_1)$	$u(M_2)+u(O_1)$
$2Q_1$	$2Q_1-O_1$	as O_1	as O_1				
N_2		as M_2	as M_2	M_4	$2M_2$	$2f(M_2)-1$	$2u(M_2)$
μ_2	$2M_2-S_2$	as M_2	$u(M_2)$	MN_4	M_2+N_2	$f(M_4)$	$2u(M_2)$
$2SM_2$	$2S_2-M_2$	as M_2	$-u(M_2)$	MS_4	M_2+S_2	$f(M_2)$	$u(M_2)$

* Approximately for f and u.

TABLE 7.

S_2 AND MS_4			K_1 AND MK_3			N_2 AND MN_4		
Angle	w/f	W/f	Angle	wf	Wf	Angle	w	1+W
000°	0°·7	-0·214	000°	0°·0	0·331	000°	0°·0	1·184
010	- 6·6	-0·192	010	- 2·5	0·327	010	1·6	1·182
020	-12·3	-0·131	020	- 4·9	0·316	020	3·1	1·174
030	-15·5	-0·046	030	- 7·3	0·297	030	4·6	1·163
040	-16·5	0·047	040	- 9·6	0·271	040	5·9	1·147
050	-15·6	0·134	050	-11·8	0·239	050	7·2	1·127
060	-13·4	0·207	060	-13·8	0·201	060	8·3	1·104
070	-10·3	0·258	070	-15·6	0·157	070	9·2	1·077
080	- 6·6	0·284	080	-17·1	0·107	080	9·9	1·048
090	- 2·6	0·284	090	-18·3	0·053	090	10·4	1·017
100	1·6	0·256	100	-19·1	-0·003	100	10·6	0·984
110	5·6	0·204	110	-19·3	-0·060	110	10·4	0·953
120	9·2	0·131	120	-19·0	-0·118	120	10·0	0·922
130	12·0	0·041	130	-17·8	-0·173	130	9·1	0·893
140	13·7	-0·058	140	-15·9	-0·224	140	7·8	0·867
150	13·6	-0·157	150	-13·1	-0·268	150	6·2	0·846
160	11·2	-0·245	160	- 9·3	-0·302	160	4·3	0·830
170	6·0	-0·307	170	- 4·9	-0·323	170	2·2	0·819
180	- 0·9	-0·330	180	0·0	-0·331	180	0·0	0·816
190	- 7·8	-0·308	190	4·9	-0·323	190	-2·2	0·819
200	-12·6	-0·247	200	9·3	-0·302	200	-4·3	0·830
210	-14·9	-0·163	210	13·1	-0·268	210	-6·2	0·846
220	-14·8	-0·067	220	15·9	-0·224	220	-7·8	0·867
230	-13·0	0·029	230	17·8	-0·173	230	-9·1	0·893
240	- 9·8	0·115	240	19·0	-0·118	240	-10·0	0·922
250	- 6·0	0·186	250	19·3	-0·060	250	-10·4	0·953
260	- 1·8	0·236	260	19·1	-0·003	260	-10·6	0·984
270	2·6	0·263	270	18·3	0·053	270	-10·4	1·017
280	6·9	0·265	280	17·1	0·107	280	- 9·9	1·048
290	10·8	0·241	290	15·6	0·157	290	- 9·2	1·077
300	14·1	0·192	300	13·8	0·201	300	- 8·3	1·104
310	16·5	0·124	310	11·8	0·239	310	- 7·2	1·127
320	17·5	0·039	320	9·6	0·271	320	- 5·9	1·147
330	16·8	-0·051	330	7·3	0·297	330	- 4·6	1·163
340	13·7	-0·133	340	4·9	0·316	340	- 3·1	1·174
350	8·0	-0·193	350	2·5	0·327	350	- 1·6	1·182
360	0·7	-0·214	360	0·0	0·331	360	0·0	1·184
Angle is (V+u) for K_1 f is $f(K_2)$			Angle is (2V+u) for K_1 f is $f(K_1)$			Angle is 3V for M_2 minus 2V for N_2		

TABLE 8.

σ = Increment of Phase per Mean Solar Hour
μ = Increment of Phase per Mean Lunar Day.

Constituent.		
M_1	14°·49205	0°·0000
P_1	14·95893	11·5978
S_1	15·00000	12·6180
K_1	15·04107	13·6382
O_1	13·94304	-13·6382
J_1	15·58544	27·1612
Q_1	13·39866	-27·1612
$2Q_1$	12·85429	-40·6841
OO_1	16·13910	40·9147
M_3	43·47616	0·0000
MK_3	44·02517	13·6382
MO_3	42·92714	-13·6382

Constituent.		
M_2	28°·98410	0°·0000
L_2	29·52849	13·5229
N_2	28·43973	-13·5229
2	28·51258	-11·7132
S_2	30·00000	25·2361
μ_2	27·96821	-25·2361
T_2	29·95893	24·2159
K_2	30·08214	27·2765
$2N_2$	27·89535	-27·0458
$2SM_2$	31·01590	50·4721
M_4	57·96821	0·0000
MN_4	57·42383	-13·5229
MS_4	58·98410	25·2361
MK_4	59·06624	27·2765

Constituents which are associated in the Analytical Processes are grouped in the above table.

I

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
S	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
L	·Z				t				α				β				α_1				β_1			
−15	11.84	2.95	13.36	3.79	11.57	18.12	00.06	06.51			12.69	3.31			23.86	06.43			0.67	0.48			0.20	0.08
−14	12.20	2.70	13.48	3.57	12.02	18.60	00.52	06.99	12.81	3.19	12.92	3.10	12.16	18.67	00.39	06.92	−0.61	−0.49	0.56	0.47	−0.14	−0.07	0.13	0.07
−13	12.50	2.58	13.47	3.48	12.50	19.10	01.00	07.49	12.99	3.05	13.02	3.05	12.63	19.17	00.87	07.44	−0.49	−0.47	0.45	0.43	−0.13	−0.07	0.13	0.05
−12	12.66	2.65	13.36	3.55	12.99	19.65	01.50	08.01	13.04	3.08	13.01	3.16	13.12	19.70	01.38	07.96	−0.38	−0.43	0.35	0.39	−0.13	−0.05	0.12	0.05
−11	12.65	2.90	13.07	3.76	13.53	20.18	02.00	08.60	12.93	3.28	12.81	3.44	13.64	20.24	01.91	08.55	−0.28	−0.38	0.26	0.32	−0.11	−0.06	0.09	0.05
−10	12.09	3.33	12.59	4.11	14.11	20.63	02.60	09.25	12.63	3.63	12.40	3.88	14.21	20.88	02.53	09.22	−0.19	−0.30	0.18	0.23	−0.10	−0.05	0.07	0.03
−9	12.01	3.95	11.96	4.50	14.83	21.57	03.29	10.02	12.14	4.13	11.84	4.41	14.89	21.60	03.38	10.01	−0.13	−0.18	0.12	0.09	−0.06	−0.03	0.01	0.01
−8	11.44	4.68	11.20	4.86	15.72	22.44	04.18	10.99	11.51	4.64	11.20	4.94	15.73	22.47	04.26	11.00	−0.07	0.00	0.00	−0.08	−0.01	−0.03	−0.08	−0.01
−7	10.96	5.34	10.56	5.04	16.96	23.60	05.48	12.21	10.91	5.15	10.74	5.26	16.89	23.60	05.65	12.28	0.03	0.19	−0.18	−0.22	0.07	0.00	−0.17	−0.07
−6	10.91	5.60	10.40	4.76	18.69	01.09	07.08	13.67	10.70	5.24	10.84	5.07	18.48	01.01	07.25	13.76	0.21	0.36	−0.44	−0.33	0.20	0.07	−0.17	−0.09
−5	11.64	5.19	10.88	3.64	20.16	02.64	08.40	15.01	11.14	4.74	11.54	4.30	19.95	02.49	08.55	15.12	0.50	0.45	−0.66	−0.46	0.21	0.15	−0.15	−0.11
−4	12.76	4.32	11.63	2.74	21.35	03.83	09.41	16.04	12.01	3.60	12.48	3.30	21.08	03.68	09.55	16.17	0.75	0.52	−0.85	−0.56	0.17	0.15	−0.14	−0.13
−3	13.88	3.38	12.47	1.75	22.12	04.75	10.24	16.91	12.96	2.81	13.41	2.36	21.47	04.61	10.88	17.02	0.92	0.59	−0.94	−0.61	0.15	0.14	−0.14	−0.11
−2	14.81	2.58	13.22	0.98	22.92	05.51	10.99	17.66	13.53	1.97	14.17	1.65	22.77	05.40	11.13	17.77	0.98	0.61	−0.95	−0.67	0.15	0.11	−0.14	−0.11
−1	15.44	2.05	13.77	0.60	23.61	06.23	11.70	18.37	14.47	1.42	14.66	1.26	23.48	06.12	11.82	18.47	0.97	0.63	−0.89	−0.66	0.13	0.11	−0.12	−0.10
→ 0	15.65	1.77	14.06	0.63	00.27	06.90	12.40	19.05	14.78	1.19	14.80	1.21	00.16	06.81	12.50	19.13	0.87	0.58	−0.74	−0.58	0.11	0.09	−0.10	−0.08
1	15.45	1.80	14.06	1.05	00.93	07.52	13.07	19.73	14.76	1.32	14.62	1.51	00.83	07.45	13.21	19.78	0.69	0.48	−0.56	−0.46	0.10	0.07	−0.14	−0.05
2	14.89	2.15	13.74	1.84	01.58	08.13	13.75	20.35	14.40	1.80	14.09	2.14	01.50	08.09	13.82	20.39	0.49	0.35	−0.35	−0.30	0.08	0.04	−0.07	−0.04
3	13.99	2.72	13.18	2.88	02.18	08.75	14.43	21.00	13.72	2.54	13.31	2.97	02.14	08.71	14.46	21.02	0.27	0.18	−0.13	−0.09	0.04	0.04	−0.03	−0.02
4	12.90	3.40	12.39	4.05	02.82	09.34	15.18	21.68	12.84	3.43	12.34	3.92	02.81	09.34	15.46	21.66	0.06	−0.03	0.05	0.13	0.01	0.00	0.02	0.02
5	11.70	4.19	11.50	5.24	03.45	09.95	16.01	22.41	11.82	4.42	11.32	4.91	03.53	10.00	15.91	22.36	−0.12	−0.23	0.18	0.33	−0.08	−0.05	0.10	0.05
6	10.56	4.97	10.73	6.28	04.18	10.68	17.16	23.32	10.64	5.36	10.40	5.79	04.38	10.76	16.92	23.20	−0.28	−0.39	0.33	0.49	−0.20	−0.10	0.24	0.12
7	9.56	5.64	10.33	6.98	05.18	11.50	18.68	00.68	10.04	6.14	9.80	6.41	05.55	11.75	18.31	00.44	−0.48	−0.50	0.53	0.57	−0.37	−0.25	0.37	0.24
8	8.99	6.03	10.49	6.94	06.68	12.90	20.15	02.92	9.70	6.50	9.76	6.43	07.05	13.35	19.81	02.37	−0.71	−0.47	0.73	0.58	−0.37	−0.45	0.34	0.55
9	9.09	5.80	10.97	6.34	08.28	14.73	21.18	04.09	9.91	6.22	10.16	5.92	08.47	15.12	20.99	03.69	−0.82	−0.42	0.81	0.42	−0.19	−0.39	0.19	0.40
10	9.59	5.20	11.97	5.68	09.32	15.95	21.98	04.69	10.43	5.60	10.74	5.27	09.45	16.82	21.83	04.46	−0.84	−0.40	0.83	0.41	−0.13	−0.27	0.15	0.23
11	10.24	4.50	12.13	6.04	10.04	16.62	22.68	05.18	11.04	4.93	11.36	4.61	10.17	16.78	22.49	05.04	−0.80	−0.43	0.77	0.43	−0.13	−0.16	0.14	0.18
12	10.94	3.65	12.67	4.38	10.66	17.18	23.46	05.67	11.67	4.28	11.97	3.96	10.77	17.30	23.04	05.57	−0.73	−0.43	0.70	0.42	−0.11	−0.12	0.12	0.10
13	11.61	3.25	13.12	3.80	11.17	17.74	23.67	06.12	12.25	3.67	12.52	3.41	11.29	17.82	23.55	06.06	−0.64	−0.42	0.60	0.39	−0.12	−0.08	0.12	0.06
14	12.23	2.78	13.45	3.33	11.68	18.26	00.15	06.65	12.76	3.17	12.96	2.98	11.79	18.32	00.04	06.59	−0.53	−0.39	0.49	0.35	−0.11	−0.06	0.11	0.06
15	12.72	2.48	13.66	3.04	12.18	18.42	00.65	07.16	13.18	2.88			12.29	18.66			−0.41	−0.35			−0.11	−0.04		

II

	25	26	27	28	29	30	31	32	33	34	35
L	$\gamma(0)$	$\gamma(1)$	$\delta(0)$	$\delta(1)$	a_0+a_4	α_2	β_2	β_4	$\delta^2(0)$	$\delta^2(1)$	$2\beta_4^2$
									(3 decs)		
-14	-0·61	-0·48	-0·15	-0·08	8·03	4·78	12·36	-0·20	0·022	0·006	0·080
-13	-0·50	-0·47	-0·13	-0·07	8·03	4·96	12·84	-0·21	0·017	0·005	0·088
-12	-0·39	-0·43	-0·13	-0·05	8·05	4·99	13·35	-0·23	0·017	0·002	0·105
-11	-0·29	-0·38	-0·11	-0·06	8·08	4·85	13·87	-0·23	0·012	0·004	0·105
-10	-0·20	-0·31	-0·09	-0·05	8·07	4·56	14·46	-0·25	0·008	0·002	0·130
-9	-0·14	-0·20	-0·05	-0·03	8·07	4·07	15·15	-0·26	0·002	0·001	0·135
-8	-0·06	-0·04	0·02	-0·02	8·03	3·48	15·99	-0·26	0·000	0·000	0·135
-7	0·06	0·14	0·10	0·00	7·98	2·93	17·09	-0·20	0·010	0·000	0·080
-6	0·26	0·29	0·18	0·07	7·98	2·72	18·56	-0·08	0·032	0·005	0·012
-5	0·52	0·39	0·18	0·12	8·02	3·12	20·04	-0·09	0·032	0·014	0·016
-4	0·75	0·49	0·16	0·13	8·03	3·98	21·24	-0·16	0·026	0·017	0·053
-3	0·91	0·56	0·14	0·14	8·01	4·95	22·18	-0·21	0·020	0·020	0·088
-2	0·96	0·61	0·14	0·11	8·00	5·83	22·99	-0·22	0·020	0·012	0·097
-1	0·94	0·65	0·13	0·11	8·00	6·47	23·71	-0·23	0·017	0·012	0·106
→ 0	0·84	0·62	0·11	0·10	8·00	6·78	00·40	-0·24	0·012	0·010	0·115
1	0·67	0·53	0·11	0·08	8·01	6·75	01·06	-0·23	0·012	0·006	0·106
2	0·47	0·40	0·09	0·04	8·03	6·37	01·72	-0·22	0·008	0·002	0·097
3	0·26	0·24	0·04	0·04	8·03	5·69	02·34	-0·20	0·002	0·002	0·080
4	0·05	0·03	0·01	0·01	8·02	4·82	02·99	-0·18	0·000	0·000	0·065
5	-0·12	-0·18	-0·07	-0·04	8·00	3·82	03·68	-0·15	0·005	0·002	0·045
6	-0·27	-0·36	-0·18	-0·07	7·99	2·85	04·47	-0·09	0·032	0·005	0·016
7	-0·46	-0·50	-0·35	-0·18	8·00	2·04	05·51	0·04	0·121	0·032	0·002
8	-0·67	-0·52	-0·36	-0·34	8·01	1·62	06·97	0·08	0·130	0·116	0·013
9	-0·80	-0·46	-0·23	-0·47	8·02	1·79	08·61	-0·14	0·053	0·220	0·039
10	-0·83	-0·41	-0·15	-0·34	8·10	2·33	09·68	-0·23	0·022	0·116	0·106
11	-0·80	-0·32	-0·14	-0·20	8·07	2·97	10·39	-0·22	0·020	0·040	0·097
12	-0·73	-0·43	-0·12	-0·13	8·08	3·61	10·97	-0·20	0·014	0·017	0·080
13	-0·64	-0·42	-0·12	-0·09	8·03	4·22	11·49	-0·20	0·014	0·008	0·080
14	-0·54	-0·39	-0·11	-0·06	8·02	4·74	11·98	-0·19	0·012	0·004	0·072

	36	37	38	39	40	41	42	43	44	45	46	47	48
L	c_1	c_2	c_3	$\alpha_2 c_1$	$\alpha_2 c_2$	β_4+c_3	A	a_0+a_4	b_{24}	c_4	β_2+c_4 = S	γ	$\bar{\beta}$
	(3 decs)			(2 decs)									
-14	0·007	-0·001	0·003	0·03	0·00	-0·20	4·75	8·03	-0·24	..	12·36	12·19	0·17
-13	7	-1	2	3	0	-0·21	4·43	8·03	-0·26	..	12·84	13·03	-0·19
-12	8	-1	2	4	0	-0·23	4·95	8·05	-0·29	..	13·35	13·88	-0·53
-11	8	0	2	4	0	-0·23	4·81	8·08	-0·28	..	13·87	14·72	-0·85
-10	9	0	1	4	0	-0·25	4·52	8·07	-0·28	..	14·46	15·56	-1·10
-9	9	0	0	4	0	-0·26	4·03	8·07	-0·26	..	15·15	16·40	-1·25
-8	8	0	0	3	0	-0·26	3·45	8·03	-0·22	..	15·99	17·24	-1·25
-7	6	-1	0	2	0	-0·20	2·91	7·98	-0·15	..	17·09	18·08	-0·99
-6	3	-2	3	1	-1	-0·08	2·71	7·97	-0·05	..	18·56	18·92	-0·36
-5	4	-1	5	1	0	-0·08	3·11	8·02	-0·06	..	20·04	19·76	0·28
-4	6	-1	5	2	0	-0·15	3·96	8·03	-0·15	..	21·24	20·61	0·63
-3	8	0	5	4	0	-0·20	4·91	8·01	-0·25	..	22·18	21·45	0·73
-2	8	-1	4	5	-1	-0·22	5·78	7·99	-0·32	-0·01	22·98	22·29	0·69
-1	8	0	4	5	0	-0·23	6·42	8·10	-0·37	-0·01	23·70	23·13	0·57
0	9	0	3	6	0	-0·24	6·72	8·00	-0·40	-0·01	00·39	-00·03	0·42
1	8	0	2	5	0	-0·23	6·70	8·01	-0·39	-0·01	01·05	00·81	0·24
2	7	0	1	4	0	-0·22	6·33	8·03	-0·35	-0·01	01·71	01·65	0·06
3	5	0	0	3	0	-0·20	5·66	8·03	-0·28	..	02·34	02·49	-0·15
4	4	0	0	2	0	-0·18	4·80	8·02	-0·22	..	02·99	03·33	-0·34
5	3	0	1	1	0	-0·15	3·81	8·00	-0·14	..	03·68	04·18	-0·50
6	3	-2	3	1	-1	-0·09	2·84	7·98	-0·06	..	04·47	05·02	-0·55
7	10	-6	16	2	-1	0·06	2·02	7·99	0·03	..	05·51	05·86	-0·35
8	16	-1	30	2	0	0·11	1·60	8·00	0·04	..	06·97	06·70	0·27
9	19	10	27	3	2	-0·11	1·76	8·04	-0·05	..	08·61	07·54	1·07
10	15	6	13	4	1	-0·22	2·29	8·11	-0·13	..	09·68	08·38	1·30
11	9	1	7	3	0	-0·21	2·94	8·07	-0·15	..	10·39	09·22	1·17
12	7	0	4	2	0	-0·20	3·59	8·06	-0·18	..	10·97	10·06	0·91
13	6	0	3	3	0	-0·20	4·19	8·03	-0·28	..	11·49	10·91	0·58
14	5	0	2	2	0	-0·19	4·72	8·02	-0·26	..	11·98	11·74	0·24

III

	49	50	51	52	53	54	55	56	57	58	59
L	$\epsilon(0)$	$\epsilon(1)$	a_3	b_3	ϕ_3	$\cos\phi_3$	$\sin\phi_3$	ac_3	as_3	bc_3	bs_3
−14	−0.71	−0.38	−0.06	−0.06	7°	0.99	0.12	−0.06	−0.01	−0.06	−0.01
−13	−0.64	−0.34	−0.04	−0.04	−8	0.99	−0.14	−0.04	0.01	−0.04	0.01
−12	−0.65	−0.25	−0.04	−0.06	−23	0.92	−0.39	−0.04	0.02	−0.05	0.02
−11	−0.53	−0.29	0.00	−0.04	−37	0.80	−0.60	0.00	0.00	−0.03	0.02
−10	−0.41	−0.23	0.01	−0.02	−48	0.67	−0.74	0.01	−0.01	−0.01	0.01
−9	−0.20	−0.12	0.00	0.00	−54	0.59	−0.81	0.00	0.00	0.00	0.00
−8	0.07	−0.07	0.00	0.03	−54	0.59	−0.81	0.00	0.00	0.02	−0.02
−7	0.29	0.00	0.02	0.04	−43	0.73	−0.68	0.01	−0.02	0.03	−0.03
−6	0.49	0.19	0.02	0.05	−16	0.96	−0.28	0.02	−0.01	0.05	−0.01
−5	0.56	0.37	0.04	0.04	12	0.98	0.21	0.04	0.01	0.04	0.01
−4	0.63	0.51	0.06	0.04	28	0.88	0.47	0.05	0.03	0.04	0.02
−3	0.69	0.64	0.06	0.03	32	0.85	0.53	0.05	0.03	0.03	0.02
−2	0.81	0.64	0.06	0.05	30	0.87	0.50	0.05	0.03	0.04	0.02
−1	0.83	0.71	0.06	0.04	25	0.91	0.42	0.05	0.03	0.04	0.02
0	0.74	0.67	0.04	0.03	16	0.95	0.31	0.04	0.01	0.03	0.01
1	0.74	0.54	0.03	0.05	10	0.98	0.17	0.03	0.01	0.05	0.01
2	0.57	0.25	0.03	0.04	2	1.00	0.03	0.03	0.00	0.04	0.00
3	0.23	0.23	0.01	0.00	−7	0.99	−0.12	0.01	0.00	0.00	0.00
4	0.05	0.05	0.00	0.00	−15	0.97	−0.26	0.00	0.00	0.00	0.00
5	−0.27	−0.15	0.01	−0.02	−22	0.93	−0.37	0.01	0.00	−0.02	0.01
6	−0.51	−0.20	−0.02	−0.04	−24	0.91	−0.41	−0.02	0.01	−0.04	0.02
7	−0.71	−0.36	−0.02	−0.05	−15	0.97	−0.26	−0.02	0.01	−0.05	0.01
8	−0.57	−0.54	−0.03	−0.01	12	0.98	0.21	−0.03	−0.01	−0.01	0.00
9	−0.40	−0.53	0.01	0.01	47	0.68	0.73	0.01	0.01	0.01	0.01
10	−0.34	−0.78	−0.01	0.02	56	0.56	0.83	−0.01	−0.01	0.01	0.02
11	−0.41	−0.59	−0.05	0.00	51	0.63	0.78	−0.03	−0.04	0.00	0.00
12	−0.43	−0.47	−0.06	0.00	40	0.77	0.64	−0.05	−0.04	0.00	0.00
13	−0.50	−0.35	−0.06	−0.02	25	0.91	0.42	−0.05	−0.03	−0.02	−0.01
14	[illegible]	−0.25	−0.06	−0.03	10	0.98	0.17	−0.06	−0.01	−0.03	0.00

	60	61	62	63	64	65	66	67	68	69	70	71	72
L	A_3	B_3	a_1	b_1	ϕ_1	$\cos\phi_1$	$\sin\phi_1$	ac_1	as_1	bc_1	bs_1	A_1	B_1
−14	−0.05	−0.07	−0.55	−0.54	2°.4	1.00	0.04	−0.55	−0.02	−0.54	−0.02	−0.53	−0.56
−13	−0.05	−0.03	−0.46	−0.51	−2.8	1.00	−0.05	−0.46	0.02	−0.51	0.03	−0.49	−0.49
−12	−0.06	−0.02	−0.35	−0.49	−7.7	0.99	−0.13	−0.35	0.04	−0.49	0.06	−0.41	−0.45
−11	−0.02	0.00	−0.29	−0.42	−12.3	0.98	−0.21	−0.28	0.06	−0.41	0.08	−0.36	−0.35
−10	0.00	0.00	−0.21	−0.35	−16.0	0.96	−0.28	−0.20	0.06	−0.33	0.10	−0.30	−0.27
−9	0.00	0.00	−0.14	−0.20	−18.1	0.95	−0.31	−0.13	0.04	−0.19	0.06	−0.19	−0.15
−8	0.02	0.00	−0.06	−0.01	−18.1	0.95	−0.31	−0.06	0.02	−0.01	0.00	−0.06	0.01
−7	0.04	−0.01	0.04	0.18	−14.4	0.97	−0.25	0.04	−0.01	0.17	−0.04	0.08	0.16
−6	0.03	0.01	0.24	0.34	−5.2	1.00	−0.09	0.24	−0.02	0.34	−0.03	0.27	0.32
−5	0.03	0.05	0.48	0.43	4.0	1.00	0.07	0.48	0.03	0.43	0.03	0.45	0.46
−4	0.03	0.08	0.69	0.53	9.2	0.99	0.16	0.68	0.11	0.52	0.09	0.59	0.63
−3	0.03	0.08	0.85	0.59	10.6	0.98	0.18	0.83	0.15	0.58	0.11	0.72	0.73
−2	0.03	0.08	0.90	0.66	10.0	0.98	0.17	0.88	0.15	0.65	0.11	0.77	0.80
−1	0.03	0.08	0.88	0.69	8.2	0.99	0.14	0.87	0.13	0.68	0.10	0.77	0.81
0	0.03	0.05	0.80	0.65	6.1	0.99	0.11	0.79	0.09	0.64	0.07	0.72	0.73
1	0.02	0.04	0.64	0.58	3.5	1.00	0.06	0.64	0.04	0.58	0.04	0.60	0.62
2	0.03	0.03	0.44	0.44	0.8	1.00	0.01	0.44	0.00	0.44	0.00	0.44	0.44
3	0.01	0.01	0.25	0.24	−2.2	1.00	−0.04	0.25	−0.01	0.24	−0.01	0.26	0.23
4	0.00	0.00	0.05	0.03	−5.0	1.00	−0.09	0.05	0.00	0.03	0.00	0.05	0.03
5	0.00	0.01	−0.13	−0.20	−7.2	0.99	−0.13	−0.13	0.02	−0.20	0.03	−0.16	−0.18
6	−0.04	−0.01	−0.25	−0.40	−8.0	0.99	−0.14	−0.25	0.04	−0.40	0.06	−0.31	−0.36
7	−0.03	−0.01	−0.44	−0.55	−5.0	1.00	−0.09	−0.44	0.04	−0.55	0.05	−0.49	−0.51
8	−0.03	−0.04	−0.64	−0.53	3.9	1.00	0.07	−0.64	−0.04	−0.53	−0.04	−0.60	−0.57
9	0.00	0.02	−0.81	−0.45	15.5	0.96	0.27	−0.78	−0.22	−0.43	−0.12	−0.66	−0.67
10	−0.03	−0.02	−0.82	−0.39	18.8	0.95	0.32	−0.78	−0.26	−0.37	−0.13	−0.65	−0.63
11	−0.03	−0.07	−0.75	−0.42	17.0	0.96	0.29	−0.72	−0.22	−0.40	−0.12	−0.60	−0.62
12	−0.05	−0.09	−0.67	−0.43	13.2	0.97	0.23	−0.65	−0.15	−0.42	−0.10	−0.55	−0.57
13	−0.04	−0.08	−0.58	−0.44	8.4	0.99	0.15	−0.57	−0.09	−0.44	−0.07	−0.50	−0.53
14	−0.06	−0.07	−0.48	−0.42	3.5	1.00	0.08	−0.48	−0.04	−0.42	−0.03	−0.45	−0.46

IV

	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88
L	q_2	$\cos q_2$	$\sin q_2$	A_2	B_2	a_4	b_4	q_4	$\cos q_4$	$\sin q_4$	ac_4	as_4	bc_4	bs_4	A_4	B_4
-14	4.9°	0.996	0.085	4.75	0.40	0.02	-0.24	9°	0.99	0.16	0.02	0.00	-0.24	-0.04	0.06	-0.24
-13	-5.5	0.995	-0.096	4.91	-0.47	0.02	-0.26	-11	0.98	-0.19	0.02	0.00	-0.25	0.05	-0.03	-0.25
-12	-15.4	0.964	-0.266	4.77	-1.32	0.02	-0.29	-31	0.86	-0.52	0.02	-0.01	-0.25	0.15	-0.13	-0.26
-11	-24.6	0.909	-0.416	4.37	-2.00	0.02	-0.28	-49	0.66	-0.75	0.01	-0.02	-0.18	0.21	-0.20	-0.20
-10	-31.9	0.849	-0.528	3.84	-2.39	0.02	-0.28	-64	0.44	-0.90	0.01	-0.02	-0.12	0.25	-0.24	-0.14
-9	-36.2	0.807	-0.591	3.25	-2.36	0.02	-0.26	-72	0.31	-0.95	0.01	-0.02	-0.08	0.25	-0.24	-0.10
-8	-36.2	0.807	-0.591	2.78	-2.04	0.01	-0.22	-72	0.31	-0.95	0.00	-0.01	-0.07	0.21	-0.21	-0.08
-7	-28.7	0.877	-0.480	2.55	-1.40	0.01	-0.15	-57	0.54	-0.84	0.01	-0.01	-0.08	0.12	-0.11	-0.09
-6	-10.4	0.984	-0.181	2.67	-0.49	0.01	-0.05	-21	0.93	-0.36	0.01	0.00	-0.05	0.02	-0.01	-0.05
-5	8.1	0.990	0.141	3.08	0.44	0.01	-0.06	16	0.96	0.28	0.01	0.00	-0.06	-0.02	0.03	-0.06
-4	18.3	0.949	0.314	3.76	1.24	0.02	-0.15	37	0.80	0.60	0.02	0.01	-0.12	-0.09	0.11	-0.11
-3	21.2	0.932	0.362	4.58	1.78	0.02	-0.25	42	0.74	0.67	0.02	0.01	-0.18	-0.17	0.19	-0.17
-2	20.0	0.940	0.342	5.43	1.98	0.03	-0.32	40	0.77	0.64	0.02	0.02	-0.25	-0.21	0.23	-0.23
-1	16.5	0.959	0.284	6.16	1.82	0.04	-0.37	33	0.80	0.54	0.03	0.02	-0.30	-0.20	0.23	-0.28
→ 0	12.2	0.977	0.211	6.57	1.42	0.04	-0.40	24	0.91	0.41	0.04	0.02	-0.36	-0.16	0.20	-0.34
1	7.0	0.993	0.122	6.65	0.82	0.04	-0.39	14	0.97	0.24	0.04	0.01	-0.38	-0.09	0.13	-0.37
2	1.7	1.000	0.030	6.33	0.19	0.04	-0.35	3	1.00	0.05	0.04	0.00	-0.35	-0.02	0.06	-0.35
3	-4.4	0.997	-0.077	5.64	-0.44	0.03	-0.28	-9	0.99	-0.16	0.03	0.00	-0.28	0.05	-0.02	-0.28
4	-9.9	0.985	-0.172	4.73	-0.83	0.02	-0.22	-20	0.94	-0.34	0.02	-0.01	-0.21	0.07	-0.05	-0.22
5	-14.5	0.968	-0.250	3.64	-0.95	0.01	-0.14	-29	0.87	-0.48	0.01	0.00	-0.12	0.07	-0.06	-0.12
6	-15.9	0.962	-0.274	2.73	-0.78	0.01	-0.06	-32	0.85	-0.53	0.01	-0.01	-0.05	0.03	-0.02	-0.06
7	-10.1	0.985	-0.175	1.99	-0.35	0.00	0.03	-20	0.94	-0.34	0.00	0.00	0.03	-0.01	0.01	0.03
8	7.8	0.991	0.136	1.59	0.22	0.00	0.04	16	0.96	0.28	0.00	0.00	0.04	0.01	-0.01	0.04
9	31.0	0.857	0.515	1.58	0.91	0.00	-0.05	62	0.47	0.88	0.00	0.00	-0.02	-0.04	0.04	-0.02
10	37.7	0.791	0.612	1.81	1.40	0.01	-0.13	75	0.26	0.97	0.00	0.01	-0.03	-0.13	0.13	-0.02
11	34.0	0.829	0.559	2.44	1.64	0.01	-0.15	68	0.37	0.93	0.00	0.01	-0.05	-0.14	0.14	-0.04
12	26.4	0.896	0.445	3.22	1.60	0.01	-0.18	53	0.60	0.80	0.01	0.01	-0.11	-0.14	0.15	-0.10
13	16.8	0.957	0.289	4.01	1.21	0.02	-0.21	34	0.83	0.56	0.02	0.01	-0.17	-0.12	0.14	-0.16
14	7.0	0.993	0.122	4.69	0.58	0.02	-0.22	14	0.97	0.24	0.02	0.01	-0.21	-0.05	0.07	-0.20

CALCULATION OF a_0 AND λ, A_{10}, A_{12}, ...

A	a_0+a_4	A^2	Mult.	λA^2	a_0
4.95	8.05	24.5	1	0.02	8.03
2.71	7.97	7.3	-3	0.01	7.96
6.72	8.00	45.3	3	0.04	7.96
1.60	8.00	2.6	-1	0.00	8.00
	0.14	135.9		$\lambda = 0.0010$	$A_0 = 7.99$

d	A_1	B_1	A_3	B_3
0	-1.59	-1.40	-0.16	0.00
1	17.09	17.47	1.07	8.56
2	6.05	5.58	-0.23	-0.05
3	0.73	1.20	0.25	0.40
a	8.13	8.98	0.55	0.65
b	2.47	2.84	0.20	0.11
c	-1.13	-1.34	-0.28	0.59

	A_2	B_2	A_4	B_4
0	114.50	1.81	0.59	-4.47
1	23.24	11.25	1.72	-1.27
2	54.89	29.47	3.32	-4.07
3	...	...	...	...
4	-2.47	-2.07	0.22	0.17
a	10.66	-18.58	-1.89	-0.72
b	-21.08	41.50	3.72	1.87
c	...	...	...	...
d	1.11	-2.56	0.00	0.38

V

TIME ORIGIN:— 1923, DECEMBER 9, —00·03 HRS: $A_0 = 7.99$

A'_{10}	−0·34	A'_{20}	115·93
11	18·56	21	27·56
12	4·23	22	52·48
13	−0·58	24	0·10
B'_{1a}	9·44	B'_{2a}	−16·75
1b	1·02	2b	45·04
1c	−0·09	2d	−0·27
B'_{10}	−0·07	B'_{20}	8·49
11	14·75	21	13·58
12	3·97	22	28·33
13	−0·01	24	−0·72
A'_{1a}	4·53	A'_{2a}	9·74
1b	0·84	2b	−23·12
1c	0·03	2d	−0·08

	M_1	K_1	O_1	J_1	Q_1	OO_1	$2Q_1$	M_3	MK_3	MO_3
R cos r	−0·012	0·474	0·147	0·076	0·049	−0·011	−0·009	−0·001	0·028	0·005
R sin r	−0·002	0·163	0·470	0·041	0·080	−0·001	0·001	0·006	0·015	0·036
R	0·012	0·501	0·492	0·086	0·094	0·011	0·009	0·006	0·032	0·036
f		0·895	0·829	0·829	0·829	0·472	0·829		0·925	0·857
1+W		1·328	...	...	...	...	...		1·328	...
V_0		9·6	32·1							
V_0'		329·2	167·4							
V_0''		7·9	157·1							
V_0'''		−0·5	−0·4							
V		346·2	356·2	22·8	319·5	156·2	282·8		328·5	338·5
u		−4·1	5·5	−6·6	6·5	−19·8	6·5		−5·0	4·6
w		8·6	...	...	...	...	...		8·6	...
r	189·5	19·0	72·6	28·3	58·5	185·2	173·7	99·5	28·2	52·1
H		0·421	0·593	0·104	0·113	0·028	0·011		0·026	0·042
g		9·7	74·3	44·5	23·5	321·6	102·0		0·3	65·2

A'_{30}	−0·04	A'_{40}	0·70
31	0·98	41	1·98
32	−0·26	42	3·13
33	0·30	44	0·38
B'_{3a}	0·65	B'_{4a}	−0·65
3b	0·18	4b	2·02
3c	0·71	4d	0·43
B'_{30}	0·16	B'_{40}	−4·54
31	1·49	41	−1·59
32	−0·08	42	−3·94
33	0·41	44	−0·01
A'_{3a}	0·59	A'_{4a}	−1·73
3b	0·03	4b	4·08
3c	−0·21	4d	0·21

	M_2	L_2	N_2	S_2	μ_2	$2SM_2$		M_4	MN_4	MS_4
R cos r	3·997	0·161	0·760	1·587	*	−0·003		0·024	0·046	0·063
R sin r	0·086	0·060	0·410	0·838	*	−0·014		−0·157	−0·059	−0·131
R	3·998	0·172	0·861	1·795	0·056	0·014		0·159	0·075	0·155
f	1·034		1·034	...	1·034	1·034		1·069	1·069	1·034
1+W	...	...	1·114	0·888	...	...		...	1·114	0·888
V_0	41·7		153·2	0·0						
V_0'	136·6		92·9	0·0						
V_0''	164·9		60·4	0·0						
V_0'''	−0·9		−0·8	−0·9						
V	342·3		305·7	0·9	325·5	15·9		324·6	288·0	341·4
u	−0·9		−0·9	...	−0·9	0·9		−1·8	−1·8	−0·9
w	...		7·8	9·6	...	...		...	7·8	9·6
r	1·2	20·4	27·7	27·8	293·3	257·9		278·7	307·9	302·4
H	3·867		0·765	2·021	0·054	0·014		0·149	0·070	0·150
g	342·6		340·3	36·5	257·9	274·7		241·5	241·9	292·5

S_2 AND MS_4

K_1 : V = 346·2
K_1 : u = −4·1
V+u = 342·1
Tab. 7: S_2 : w/f = 12·5
" " W/f = −0·146
K_2 : f = 0·770
w = 9·6
W = −0·112
1+W = 0·888

K_1 AND MK_3

K_1 : 2V = 332·4
K_1 : u = −4·1
2V+u = 328·3
Tab. 7: K_1 : wf = 7·7
" " Wf = 0·293
K_1 : f = 0·895
w = 8·6
W = 0·328
1+W = 1·328

N_2 AND MN_4

M_2 : 3V = 1026·9
N_2 : 2V = 611·4
difference $(M_2 - N_2)$ = 415·4
Tab. 7: N_2 : w = 7·8
" " 1+W = 1·114

$2N_2$ R = 0·110 r = 54·2

μ_2	R cos r	R sin r
analysis	0·086	0·038
$2N_2$	0·064	0·089
μ_2	0·022	−0·051

μ_2 { R = 0·056, r = 293·3 }

* SEE SPECIAL CALCULATION

For W and (1+W), for 2 SM_2, see Sheet X, Part II.

THEORY AND EXPLANATION

1. — *Fundamental formulae.*

We shall use the following notation :

$$\left.\begin{array}{ll}
t & \text{the time, in mean solar hours.} \\
\sigma & \text{the speed of a constituent, in radians per mean solar hour.} \\
Z & \text{the height of high or low water.} \\
T & \text{the time of high or low water, in mean solar hours, relative to the time of high or low water of the semidiurnal tide.} \\
S & \text{the time of high water of the semidiurnal tide relative to the solar time scale.} \\
\tau & \text{a special lunar time scale.} \\
\sigma_0 & \text{the value of } \sigma \text{ for M, equal to 0.5059 radian per mean solar hour.} \\
n & \text{the value of } \sigma/\sigma_0 = 1.977\ \sigma \text{, a number.} \\
\theta & \text{the value of } \sigma_0 T \text{, in radians.} \\
T & \text{the value of } \theta/\sigma_0 = 1.977\ \theta \text{ in hours.}
\end{array}\right\} \quad \ldots(1)$$

The depth of water above datum will be denoted by

$$\Sigma\, a \cos \sigma t + \Sigma\, b \sin \sigma t \qquad (2)$$

and the times of high and low water occur when the gradient is zero, or when

$$\Sigma \sigma a \sin \sigma t - \Sigma \sigma b \cos \sigma t = 0 \qquad (3)$$

If the mean lunar semidiurnal constituent is supposed to be dominant then it is convenient to write

$$0 = \Sigma na \sin n\theta - \Sigma nb \cos n\theta \qquad (4)$$

$$Z = \Sigma a \cos n\theta + \Sigma b \sin n\theta \qquad (5)$$

It will be supposed that the origin of time is taken so closely to high water of the semidiurnal tide that θ may be taken as small, and on expanding $\cos n\theta$ and $\sin n\theta$ in terms of $n\theta$ we obtain with sufficient accuracy, by assuming that θ and therefore Σnb are small, and by retaining only the second order of small quantities,

$$\left.\begin{array}{l}
\theta \Sigma n^2 a = \Sigma nb - \ldots\ldots \\
Z = \Sigma a + \theta \Sigma nb - \dfrac{1}{2}\, \theta^2 \Sigma n^2 a - \ldots\ldots
\end{array}\right\} \qquad (6)$$

By elimination of $\theta \Sigma nb$ we obtain to the same degree of accuracy

$$Z = \Sigma a + \frac{1}{2}\, \theta^2 \Sigma n^2 a + \ldots\ldots \qquad (7)$$

The above formulae are quite general in that the summation applies to each constituent, but in what follows we shall consider as a whole the semidiurnal tide within a span of one day, and similarly with the other species. As the time-origin is at high water of this semidiurnal tide it follows that b_2 is zero, and as a_2 is a dominant quantity it is convenient to denote it by A. It is also convenient in future to exclude A from the summation sign so that we replace $\Sigma n^2 a$ by $A + \Sigma n^2 a$, and Σa by $A + \Sigma a$. We then obtain

$$T = 1.977\ \frac{\Sigma nb}{A} - \frac{2 \Sigma nb \Sigma n^2 a}{A} \qquad (8)$$

$$Z = A + \Sigma a + \frac{1}{2A}\, (\Sigma nb)^2 \qquad (9)$$

The last terms in each expression are regarded as "correction terms" in which it is permissible to introduce approximate values, so that we replaced 1.977 by 2. Actually, if we were to expand to higher terms we should find that it would be better to replace 1.977 by 2 throughout, and this will be done in future.

2. — *Application to a lunar tide*

Since the tide is approximately lunar in period we may treat it as truly so for the preliminary processes, and make corrections later. This means that we take the diurnal tide as an entity, the semidiurnal tide as an entity, and so on. Therefore, we can take n equal to half the species number.

Not all the terms in the expansions for the "correction terms" are of importance, and we neglect the contributions from the third-diurnals. Also we only have the larger values of a_4, b_4, when A is large and it is rare to have both the diurnal and quarter-diurnal tides relatively large at the same time. The method will be applicable to tides where the diurnal tide is about half the semidiurnal tide in range but then it is very probable that the quarter-diurnal tide is negligible. It is permissible therefore to neglect such products as a_1b_4, b_1a_4, etc.

With these simplifications we obtain

$$T = \frac{b_1+3b_3+4b_4}{A} - \frac{1}{A^2}\left(\frac{1}{4}a_1 b_1 + 16a_4 b_4\right) \qquad (10)$$

$$Z = A+a_0+a_1+a_3+a_4+ \frac{1}{A}\left(\frac{1}{8}b_1^2+2b_4^2\right) \qquad (11)$$

3. — *Notes on* α *and* β.

The first processes of analysis (Instructions I to VIII) consist of the separation of the species of tide. Since the diurnal contributions to high water are reversed in sign every twelve hours it follows that Z(2)+Z(0), using the notation Z(s) explained in I to III, will be approximately free from diurnal tide, but will contain contributions from the semidiurnal and quarter-diurnal tides. Similarly Z(0)+Z(—2) will be almost free from diurnal tide. The first combination, however, is symmetrical about s = +1, and the second combination is symmetrical about s = —1, and the residual diurnal tides are opposite in phase in the two contributions. Therefore the addition of the two combinations will give $\alpha(0)$ as defined in III, and it has a double advantage — it greatly diminishes the diurnal tide and is symmetrical about the high water with s = 0. It is our ultimate object to have such combinations symmetrical about s = 0, and therefore, though we have also to compute such functions as $\alpha(1)$, $\alpha(2)$, $\alpha(3)$ they are ultimately combined again symmetrically with s = 0, as in VI, VII, and VIII.

4. — *Formulae for* α, β, γ, δ

We have now to interpret the functions α and β according to the expansions (10) and (11). For this purpose we may treat the equations as though we were dealing purely with lunar constituents. It was for this reason that we have computed α and β, γ and δ so that they are related to a common time on s = 0. We may take Z(—1) = Z(3), and Z(—2) = Z(2), etc. The diurnal constants a_1 and b_1 are reversed in sign after 12 lunar hours, and so are a_3 and b_3. When the time origin is changed to low water of the semidiurnal tide then a_1, b_1 become b_1, — a_1, and a_3, b_3 become —b_3, a_3; A becomes —A; the quarter-diurnal constants are unchanged. We shall denote by S the time of high water of the semidiurnal tide relative to the solar time scale. Hence we obtain

$$
\left.\begin{aligned}
\alpha\,(0) &= A + a_0 + a_4 + \frac{1}{A}\left(\frac{1}{8}\,b_1^2 + 2\,b_4^2\right) = \alpha_0 + \alpha_2 + \alpha_4 \\
\alpha\,(1) &= -A + a_0 + a_4 - \frac{1}{A}\left(\frac{1}{8}\,a_1^2 + 2\,b_4^2\right) = \alpha_0 - \alpha_2 + \alpha_4 \\
\beta\,(0) &= S + \frac{4\,b_4}{A} - \frac{1}{A^2}\left(\frac{1}{4}\,a_1\,b_1 + 16 a_4\,b_4\right) = S + \beta_2 + \beta_4 \\
\beta\,(1) &= S - \frac{4\,b_4}{A} + \frac{1}{A^2}\left(\frac{1}{4}\,a_1\,b_1 - 16\,a_4\,b_4\right) = S - \beta_2 + \beta_4
\end{aligned}\right\} \quad (18)
$$

whence

$$
\left.\begin{aligned}
\alpha_0 + \alpha_4 &= a_0 + a_4 + \frac{1}{A}\left[\frac{1}{16}\,(b_1^2 - a_1^2)\right] \\
\alpha_2 &= A + \frac{1}{A}\left[\frac{1}{16}\,(a_1^2 + b_1^2) + 2\,b_4^2\right] \\
\beta_4 &= \frac{4\,b_4}{A} - \frac{1}{A^2}\left[\frac{1}{4}\,a_1\,b_1\right] \\
\beta_2 &= S - \frac{1}{A^2}\left[16\,a_4\,b_4\right]
\end{aligned}\right\} \quad (19)
$$

Similarly, we obtain

$$
\left.\begin{aligned}
\gamma(0) &= a_1 + a_3 \\
\gamma(1) &= b_1 - b_3 \\
\varepsilon(0) &= A\,\delta(0) = b_1 + 3b_3 \\
\varepsilon(1) &= A\,\delta(1) = a_1 - 3a_3
\end{aligned}\right\} \quad (20)
$$

The first equation in (19) shows the necessity for making an assumption regarding a_4. We shall therefore assume that with a sufficient degree of approximation

$$a_4 = \lambda A^2 = \lambda \alpha_2^2 \quad (21)$$

and the mode of computation of λ will be examined later.

We now replace the values of a and b in the correction terms by the appropriate values of α, β γ, δ and we write

$$
\left.\begin{aligned}
c_1 &= \frac{1}{16}\left[\delta^2(0) + \delta^2(1) + 2\beta_4^2\right] \\
c_2 &= \frac{1}{16}\left[\delta^2(1) - \delta^2(0)\right] \\
c_3 &= \frac{1}{4}\,\delta(0)\,\delta(1) \\
c_4 &= 16\,\lambda b_4
\end{aligned}\right\} \quad (22)
$$

and then obtain

$$
\left.\begin{aligned}
A &= \alpha_2 - \alpha_2 c_1 \\
a_0 + a_4 &= \alpha_0 + \alpha_4 + \alpha_2 c_2 \\
b_4 &= \frac{1}{4}\,A\,(\beta_4 + c_3) \\
S &= \beta_2 + c_4
\end{aligned}\right\} \quad (23)
$$

For the diurnal and third-diurnal tides we have

$$\left.\begin{array}{ll} a_3 = \frac{1}{4}\left[\gamma(0) - \varepsilon(1)\right] & a_1 = \gamma(0) - a_3 = \frac{1}{4}\left[3\gamma(0) + \varepsilon(1)\right] \\ b_3 = \frac{1}{4}\left[\varepsilon(0) - \gamma(1)\right] & b_1 = \gamma(1) + b_3 = \frac{1}{4}\left[3\gamma(1) + \varepsilon(0)\right] \end{array}\right\} \quad (24)$$

The above equations together with (15) - (17) give all the details of calculations required to obtain a, b for each species of tide.

5. — *Numerical tests of formulae.*

The accuracy of the formulae can be demonstrated by applying them to the values of Z and T for a lunar semidiurnal tide with unit amplitude and phase lag zero and a lunar diurnal tide with unit amplitude and phase lag g. Hence A = 1, $b_2 = 0$, $a_1 = \cos g$, $b_1 = \sin g$. The calculations of Z and T can be effected with great exactness, with results as follows :

g	a	b	Z (0)	Z (1)	Z (2)	Z (3)	T (0)	T (1)	T (2)	T (3)
0°	1.000	0.000	2.000	–1.125	0.000	–1.125	0.000	0.965	0.000	–0.965
20°	0.940	0.342	1.952	–0.779	0.079	–1.444	0.264	0.996	–0.429	–0.832
40°	0.766	0.648	1.809	–0.445	0.298	–1.706	0.516	0.884	–0.768	–0.632
60°	0.500	0.866	1.583	–0.174	0.607	–1.892	0.738	0.614	–0.959	–0.393
80°	0.174	0.985	1.290	–0.020	0.953	–1.988	0.909	0.220	–0.996	–0.133

From these we readily derive, with Z (–1) = Z (3), etc, the values of α, β, γ, and t as given below

g	$\alpha_0 + \alpha_4$	α_2	γ(0)	γ(1)	β_2	β_4	ε (0)	ε (1)
0°	–0.062	1.062	1.000	0.000	0.000	0.000	0.000	0.965
20°	–0.048	1.064	0.936	0.332	0.000	–0.082	0.346	0.914
40°	–0.011	1.065	0.755	0.630	0.000	–0.126	0.642	0.758
60°	0.031	1.064	0.488	0.859	0.000	–0.110	0.848	0.504
80°	0.059	1.063	0.168	0.984	0.000	–0.044	0.953	0.176

If we then apply the appropriate corrections we obtain results yielding maximum errors as follows.

0.005	0.002	0.012	0.012	0.000	0.001	0.035	0.035
in	in	in	in	in	in	in	in
a_0	A	a_1	b_1	b_2	b_4	b_1	a_1

(Actually the formulae for β and ε are not quite correct for here we are taking the speed of the constituent as 30° per m.s.h. whereas the formulae are based upon a speed of 29° per m.s.h. Hence the values of β and ε should be multiplied by 1.035, and the maximum errors in ε are then reduced to 0.023).

The errors in γ vary as the square of the amplitude of the diurnal tide relative to that of the semidiurnal tide, while those in ε vary as the cube of the relative amplitudes. This example has been chosen to demonstrate that even with the case of equality of diurnal and semidiurnal amplitudes the formulae are reasonably accurate. Normally it is not intended that this method will be used with such relatively large diurnal tides.

Similar numerical tests have been made with a quarter-diurnal tide whose amplitude is one-tenth of the amplitude of the semidiurnal tide, a value which is abnormally high. The maximum errors are then 0.001 in A, 0.003 in a_4, 0.005 in b_2, and 0.005 in b_4.

6. — *Calculation of* a_4 *and* a_0.

The assumption $a_4 = \lambda A^2 = \lambda\alpha_2^2$, approximately, has been well established, by theory and experience. An obvious method of determining α_0 and λ is to evaluate the terms of the equation

$$a_0 + \lambda A^2 = \alpha_0 + \alpha_4 + \alpha_2 c_2$$

for springs and neaps, but such a method can give erroneous results if there is a secular change in a_0. This secular change can be determined by the following method, which also gives a value of λ purified of secular perturbation.

The method is to solve for a_0 and λ from one equation for springs and the other from the mean of the equations for the adjacent neaps. Similarly, values for a_0 and λ can be derived from an equation for neaps and the mean equation from adjacent springs. The mean values of λ are then the best that can be obtained. It is readily shown that this procedure is equivalent to forming the four equations for springs and neaps (or neaps and springs) in due order and applying the factors —1, 3, —3, 1 to each of the terms, and to divide the sums of the products, as in the following artificial example :

α_2	$\alpha_0+\alpha_4$	c_2	a_0+a_4	α_2^2	Factors	$\lambda\ \alpha_2^2 = a_4$	a_0
5.00	8.18	-0.006	8.15	25	-1	0.25	7.90
2.72	8.04	-0.007	8.02	7	3	0.07	7.95
6.80	8.46	0.000	8.46	46	-3	0.46	8.00
1.60	8.10	-0.015	8.08	3	1	0.03	8.05
Sum of products with factors			-1.39	-139			
			Ratio $\lambda = 0.0100$				

The value of a_4 follows from $\lambda\alpha_2^2$ and on subtraction from (a_0+a_4) we get the values of a_0. If these show any marked secular variation then daily values can be interpolated.

It may be noted that the above procedure corresponds to taking third differences in (a_0+a_4) and in α_2^2, the reason being that if there is a secular variation it is eliminated by taking second differences, while the third difference corresponds to taking average values of λ ; thus differences may be formed as follows

8.15				25			
	—0.13				—18		
8.02		0.57		7		57	
	0.44		—1.39		39		—139
8.46		—0.82		46		—82	
	—0.38				—43		
8.08				3			

Evidently in this example the ratios of the second differences (0.57/57, 0.82/82) give identical values of λ.

7. — *Transference to lunar scale of time.*

The value of S is the time of high water of the semidiurnal tide, and this, for all values of a and b, has been taken as the daily origin of time. These values of a and b, however, are expressed relatively to a time scale which is not uniform. We could transfer each day the time origin to zero hour of the solar day, but that would involve a knowledge of the periods of the separate species of tides. Less error is incurred if we transform to a time scale which is lunar as we are only transferring through a small interval of time. The method of doing this is explained in (XIV) and (XV).

8. — *The transference of origin of time for each day.*

The diurnal tide may be considered as having a speed which is variable from day to day, and strictly speaking we ought to make use of this variable speed, but in practice it is sufficiently accurate to take the mean speed of $14^\circ.5$ per mean solar hour for the purpose of transferring a_1, b_1, to the new time origin, as the time-transfer is but small.

Relative to the special lunar time we have the diurnal tide defined by

$$a_1 \cos 14^\circ.5\ (t - \bar{\beta}) + b_1 \sin 14^\circ.5\ (t - \bar{\beta}) \qquad (25)$$

where $\bar{\beta}$ is the time, in mean solar hours, of the semidiurnal tide relative to the special lunar time.

The above expression is equal to

$$A_1 \cos 14^\circ.5 . t + B_1 \sin 14^\circ.5 . t \qquad (26)$$

where

$$\left.\begin{aligned} A_1 &= a_1 \cos \varphi_1 - b_1 \sin \varphi_1 \\ B_1 &= b_1 \cos \varphi_1 + a_1 \sin \varphi_1 \\ \varphi_1 &= 14^\circ.5\ \bar{\beta} \end{aligned}\right\} \qquad (27)$$

As the quantities A_1, B_1, are now referred to a uniform time scale they are expressible by harmonic terms whose phases increase uniformly in lunar time.

9. — *The interpretation of* A *and* B.

The quantities A and B are in reality the aggregate of the values of A and B for the constituents of that species of tide. If $V+u$ is the astronomical argument at the origin of time (S_0 solar hours on the central day) then for each constituent

$$\left.\begin{aligned} A &= R \cos (g - V - u - \mu L) \\ B &= R \sin (g - V - u - \mu L) \end{aligned}\right\} \qquad (28)$$

where L is the number of the lunar day, taken positively and negatively with respect to the central day $L = 0$, and μ is the increment of the phase per mean lunar day.

Hence

$$\left.\begin{aligned} A &= R \cos (r - \mu L) \\ B &= R \sin (r - \mu L) \\ r &= g - V - u \end{aligned}\right\} \qquad (29)$$

10. — *Values of phase-increments per lunar day.*

The values of μ for the more important constituents are given in Table 8. Where two constituents have equal and opposite values of μ, they are given together, and the second one in the pair has a negative value of μ. Such constituents are called "conjugates". The phase increments are in degrees per lunar day.

This present exposition only deals with observations covering 32 days of solar time (31 days of lunar time) so that theoretical considerations have to be used to separate P_1 from K_1, ν_2 from N_2, T_2 and K_2 from S_2, but as MS*f* is mainly due to shallow water tides it is not practicable to separate MS*f* and M*f* from one month's observations. (If twelve months' analyses were available then the analysis could be effected). To separate μ_2 and $2N_2$ we must firstly deduce $2N_2$ from N_2 and M_2 and then correct the analytical results to obtain μ_2.

No values of μ are given for sixth-diurnal constituents as they are inseparable from the semidiurnal constituents. The long-period constituents are not included in this analysis of a month's observations.

11. — *Analysis of A and B : monthly processes.*

The monthly processes of analysis of A and B are similar to those used in the author's standard methods. Daily multipliers ± 2, ± 1, 0, are used as in his standard method of analysis of hourly heights, (Doodson, 1928), in preference to the simpler set ± 1, 0 which he provided for the Admiralty Method. (The latter could be used but special tables of corrections would need to be calculated). The multipliers are given in Table 2 for $L = -14$ to $+14$.

The multipliers d_0, d_1, d_2, d_3, d_4, when applied to daily values of $\cos \mu L$ produce on summation quantities called D_0, D_1, D_2, D_3, D_4. They have zero result when applied to $\sin \mu L$. Similarly multipliers d_a to d_d have zero result when applied to $\cos \mu L$ but give quantities D_a, D_b, D_c, D_d with $\sin \mu L$.

The following table gives the values of D for the principal constituents.

	M_2	L_2	S_2	$2\,SM_2$	K_1	J_1	OO_1
	–	–	–	–	–	–	–
D_0	29.00	-2.36	0.48	0.48	-2.56	2.37	-2.29
D_1	—	29.15	-2.49	0.36	29.17	-6.99	6.60
D_2	—	3.07	31.56	0.75	3.32	30.45	-10.69
D_3	—	0.80	2.08	-1.04	0.73	6.47	27.73
D_4	—	-0.79	-1.12	30.96	-0.73	-2.68	10.98
D_a	—	-27.75	1.28	-1.00	-27.54	3.49	-1.62
D_b	—	-5.36	-29.14	0.54	-5.78	-25.61	6.07
D_c	—	4.59	-2.24	2.30	4.79	-9.17	-22.06
D_d	—	-0.46	1.31	-28.32	-0.58	3.84	-11.79

For the conjugate constituents the values of D_a to D_d are reversed in sign. The values for S_2 are applicable to MS_4 as they have the same value of μ, and similarly for other constituents.

If these operations are effected on A and B as expressed by (29) we get quantities such as A_{11}, A_{1a} where the first suffix indicates that A_1 has been used and the second suffixes indicate that d_1 and d_a have been used. From (29), therefore, we have

$$\left.\begin{aligned} A_{11} &= \Sigma D_1 R \cos r & B_{11} &= \Sigma D_1 R \sin r \\ A_{1a} &= \Sigma D_a R \sin r & B_{1a} &= -\Sigma D_a R \cos r \end{aligned}\right\} \quad (30)$$

with similar formulae for the remaining functions.

As each function derived from A_1 contains major contributions from one constituent and its conjugate, with minor contributions from all other diurnal constituents, it is essential to combine all the functions A_{10}, A_{11}, A_{12}, A_{13}, in order to eliminate the minor contributions. The formulae for this are given in Tables 3 and 4 ; for example, the combination

$$A'_{11} = 1.000\, A_{11} + 0.259\, A_{12} - 0.138\, A_{13} \quad (31)$$

is entirely free from diurnal contributions other than K_1 and its conjugate O_1. From (30) we see that we can write

$$A'_{11} = \Sigma D'_1\, R \cos r \quad (32)$$

with

$$D' = 1.000\, D_1 + 0.259\, D_2 - 0.138\, D_3$$

Similar interpretations and usages apply to all the formulae for D′ which are applicable to diurnal functions and also to third-diurnal functions. A second set of combinations, denoted by D″, applies to long-period, semidiurnal, and quarter-diurnal constituents, with similar interpretations.

12. — *Reconsideration of A and B.*

Two matters need to be considered :

(1) the effects of the mode of computation of $\alpha(0)$ and $\beta(0)$ and other quantities in (15) to (17) on the amplitude of any constituent ;

(2) the effects of n in the values of β.

Let ρ denote the speed of a constituent in degrees per mean lunar hour.Then a constituent can be defined by

$$A \cos \rho t + B \sin \rho t$$

and if we take t at intervals of 6 lunar hours, approximately, with $t = 0$ for Z(0), $t = 12$ for Z(2) and $t = -12$ for Z(—2) then it is evident that $\alpha(0)$ has no contribution from B and that the coefficient of A is

$$\frac{1}{4}\left(2+\cos 12\ \rho+\cos 12\ \rho\right) = \frac{1}{2}\left(1+\cos 12\ \rho\right) = \cos^2 6\rho$$

The same factor is obtained for $\beta(0)$, $\alpha(1)$, and $\beta(1)$. For $\alpha_1(0)$, $\beta_1(0)$, $\alpha_1(1)$, and $\beta_1(1)$ we have the factor

$$\frac{1}{4}\left(2 - \cos 12\ \rho - \cos 12\rho\right) = \frac{1}{2}\left(1 - \cos 12\ \rho\right) = \sin^2 6\rho$$

and in the same way we can find factors for all the processes in (15) to (17). As all these processes are centralized on $s = 0$ we have pure factors to deal with and have no complications from inter-actions as to phase. The factors are as follows :

	Coefficient	*Value of coefficient for*	
		S_2	K_1
$\alpha_0 + \alpha_4$, β_4	$\cos^2 6\rho . \cos^2 3\ \rho$	0.003	0.002
α_2, β_2	$\cos^2 6\rho . \sin^2 3\ \rho$	0.985	0.002
$\gamma(0)$, $\varepsilon(0)$	$\sin^4 6\ \rho$	0.000	0.993
$\gamma(1)$, $\varepsilon(1)$	$\sin^3 6\ \rho$	0.001	0.995

The coefficients for M_1, M_2, M_3, M_4 are, of course, either unity or zero, so that the processes are exact for these constituents, for which $\mu = 0$, and the departures from zero or unity vary as μ^2 in all cases except $\gamma(1)$, $\delta(1)$ where the departures vary as μ^3. The constituents S_2, K_1 and O_1 are the only ones whose perturbations could be of moment, and the above values show that their perturbations are negligible.

The factors near unity can be automatically included with those of the previous paragraph, for all the constituents.

If we re-examine the mode of derivation of a_1, b_1, a_3, b_3 in (24), we see that $\varepsilon(0)$ and $\varepsilon(1)$ were derived from (8) by taking the value of n equal to the mean value within the species. Let n_1 and n_3 denote the ratios of n to the mean value within the species, so that $n_1 = 2\sigma/\sigma_0$, $n_3 = 2\sigma/3\sigma_0$. Also let dashes to a and b denote the constituent values. Then we have

$$\varepsilon(0) = \Sigma n_1 b'_1 + 3\Sigma n_3 b'_3 \qquad \gamma(0) = \Sigma a'_1 + \Sigma a'_3$$

$$\varepsilon(1) = \Sigma n_1 a'_1 - 3\Sigma n_3 a'_3 \qquad \gamma(1) = \Sigma b'_1 - \Sigma b'_3$$

and in place of (24) we have

$$\left.\begin{aligned} a_1 &= \frac{1}{4}\Sigma(3+n_1)\,a'_1 - \frac{3}{4}\Sigma(n_3-1)\,a'_3 \\ b_1 &= \frac{1}{4}\Sigma(3+n_1)\,b'_1 + \frac{3}{4}\Sigma(n_3-1)\,b'_3 \\ a_3 &= \frac{1}{4}\Sigma(1+3n_3)\,a'_3 - \frac{1}{4}\Sigma(n_1-1)\,a'_1 \\ b_3 &= \frac{1}{4}\Sigma(1+3n_3)\,b'_3 + \frac{1}{4}\Sigma(n_1-1)\,b'_1 \end{aligned}\right\} \qquad (35)$$

As the third-diurnal constituents are small and (n_3-1) is very small, their corrections to a_1 and b_1 can be ignored. The diurnal constituents K_1 and O_1 will perturb a_3 and b_3 slightly ; the value of $\frac{1}{4}(n_1-1)$ is about 0.035 for each, so that the perturbation has an amplitude of 0.009 of those of K_1 and O_1. Owing, however, to the transference of time origin the perturbations will not have the same increments of phase per lunar day as MK_3 and MO_3 so that the effects of the perturbation of A_3 and B_3 may be ignored.

Hence we see that the diurnal constituents contributing to a_1 and b_1, and thence to A_1 and B_1, have a factor $\frac{1}{4}(3+n_1)$ and the third-diurnal constituents in A_3 and B_3 have a factor $\frac{1}{4}(1+3n_3)$. Actually, these third-diurnal constituents are so small that in practice this factor can be ignored.

In the same way we have

$$b_4 = \Sigma\, n_4\, b'_4 \qquad (36)$$

The transference of the time origin also needs consideration. For the diurnal constituents, each constituent should be computed by using $\cos n_1\,(14^\circ.5\bar{\beta})$ and $\sin n_1\,(14^\circ.5\bar{\beta})$, whereas we have taken n_1 to be unity. The error is about 0.009 K_1 or O_1, but the shift of the time origin introduces so many complications and spreads the error over so many constituents owing to the periodicity of $\bar{\beta}$ that the corrections may be ignored. The same conclusion applies to all other species, except for the semidiurnals. It should be noted that $\bar{\beta}$ is derived from semidiurnal constituents, and therefore B_2 will contain semidiurnal constituents with amplitudes proportional to n_2.

13. — *Calculation of* R cos r *and* R sin r.

The functions A'_{11}, etc., as computed in paragraph 11, and with the corrections indicated in paragraph 12 give results as indicated below for K_1 and O_1 ;

$$A'_{11} = 30.03\ (R \cos r)_K + 29.46\ (R \cos r)_O$$

$$B'_{1a} = 28.64\ (R \cos r)_K - 28.10\ (R \cos r)_O$$

No other constituents contribute to these functions. The solution of these two equations gives results as in (XXVII) and Table 5.

14. — *The calculation of* H *and* g.

The instructions in (XXVIII) are sufficient, but the computer is supposed to be familiar with normal tidal processes of this type. Standard Tables of V, f and u may be used, if available, or reference may be made to the author's paper on the Analysis of Tidal Observations, cited below.

15. — *Alternative methods of analysis of* A *and* B.

The multipliers d used in this analysis are those normally used by the Tidal Institute. The Admiralty Method uses multipliers limited to +1, —1, or 0. These could be used, but it would be necessary to evaluate corresponding tables to replace Tables 3, 4, 5. As the time-scale is lunar and not solar, the existing tables, as used in the Admiralty Method, are not applicable.

References

G.H. Darwin,1890 "On the harmonic analysis of tidal observations of high and low water ". (Proc. Roy. Soc., XLVIII, pp 278-340).

A.T. Doodson, 1928 "The analysis of tidal observations". (Phil. Trans. Roy. Soc., CCXXVII, pp. 223-279).

THE ANALYSIS OF HIGH AND LOW WATERS.

(Part II : Large Diurnal Inequalities).

by Dr A. T. Doodson, F.R.S.

INTRODUCTION

The Method of Analysis described in Part I is based upon certain series expansions, of which only the first few terms can be used computationally. The method has a very wide range of application as it can be used even when the diurnal tide on any day is as much as half the semidiurnal tide on that day. For greater diurnal inequalities it is necessary to devise a fresh method.

The method described in this part is more comprehensive than that given in Part I. It can be used in place of the previous method but it is a little more difficult for the computer, and normally the method of Part I will be used where possible. The new method, however, requires the existence of two high waters and two low waters per day, even though any adjacent tides tend to merge into one another. When the diurnal tide becomes predominant another method must be used.

A new principle of analysis has been developed for this method. It is shown that the complex tides can be replaced by equivalent solar tides each day, and the processes of analysis with these equivalent tides are of a very simple character. This new method is very powerful and can be applied to many kinds of tidal problems. The method depends upon interpolation in sequences of tides at inconstant intervals so as to obtain a corresponding sequence at constant solar intervals of time (whether lunar or solar does not matter, but in the present application the constant intervals are solar days). If this is done for all sequences then the reduced tides on a given day are all given in terms of astronomical data at zero hour and the reduced data define reduced tides which are solar in character.

The process of interpolation is simplified if all times are expressed in terms of a solar day, and this novel method is used in the present investigation.

Following the style of exposition adopted for Part I, detailed instructions for computation are given first, with the minimum of explanation, and the theory is given last.

Instructions for analysis (Part II)

(In order to avoid confusion with the computations in Part I the tables and sheets referred to in this part will be numbered consecutively to those of Part I).

I (Sheet VI). The solar times of high and low water in hours and minutes are converted by means of Table 9 to times with one mean solar day as unit, and the results, to three decimals, are entered in columns 1 to 4 in sequences $s = 0$ to 3 as in Part I, Instruction I, except that when the times pass through midnight there will be a gap left in the column, as usually occurs in tide-tables.

II. The corresponding heights are entered in columns 5 to 8, normally to 0.1 ft.

III. Considering the times for $s = 0$, let δ denote the increment of time on day $(x - 1)$ to the time on day x. We prepare a set of reduced (or interpolated) times, by dividing the time φ on day x by $(1 + \delta)$; the same process is applied down the column and for the other sequences, and the results are entered in columns 9 to 12. These values are really interpolates to zero hour of day x, and when φ passes through midnight there are two possible interpolates. An example will help to clarify the method.

Day	*Observed times*				$(1 + \delta)$				*Reduced times*			
	s=0	1	2	3								
13	0.427	0.713	0.937	0.132	—	—	—	—	—	—	—	—
14	0.445	0.751	—	0.159	1.018	1.038	—	1.027	0.437	0.724	0.941	0.155
15	0.464	0.789	0.002	0.186	1.019	1.038	1.065	1.027	0.455	0.760	0.002	0.181

Here, for $s = 1$, $\delta = 0.751 - 0.713 = 0.038$ on day 14, so that the reduced time is $0.751/1.038 = 0.724$. For $s = 2$ there is a gap and the time increment is really equivalent to $1.002 - 0.937$, if we allow for the elapsed day, so that the value of δ is 0.065, and the reduced value is 1.002/1.065 on day 14, but in the same sequence for day 15 the reduced time is 0.002/1.065. It is not necessary to make provision for writing down the value of $(1 + \delta)$ as it is obvious from inspection.

In certain cases the times change abruptly so that the values of δ become large. In such cases the method of interpolation explained above is not sufficiently accurate and a graphical method is preferable. If in sequence $s = 0$ we plotted the times 0.427, 0.445, 0.464 against the times 13.427, 14.445, 15.464 and then read off from the graph the values at times 14.000, 15.000,... we would obtain the required interpolates. It is only in rare cases that it is necessary to resort to the graphical method.

IV The corresponding reduced heights in columns 13 to 16 are obtained by interpolating in the observed heights according to the reduced times, as in the following example.

Day	*Observed heights*				*Increment per day*				*Reduced heights*			
13	17.45	9.25	12.95	7.85	—	—	-	-	—	-	—	—
14	17.05	8.5	—	9.55	-0.40	-0.75	-	1.70	17.2	9.0	12.9	9.3
15	16.7	7.75	12.95	11.25	-0.35	-0.75	0.0	1.70	16.9	8.3	12.9	10.9

The formula is

$$\left\{\begin{matrix}\text{Reduced}\\ \text{height}\end{matrix}\right\} = \left\{\begin{matrix}\text{Observed}\\ \text{height}\end{matrix}\right\} - \left\{\begin{matrix}\text{Increment from}\\ \text{previous day}\end{matrix}\right\} \times \left\{\begin{matrix}\text{Reduced}\\ \text{time}\end{matrix}\right\}$$

The process is exceedingly simple as the increments of height per day are usually quite small. In exceptional cases, where the height is changing very quickly, a graphical method is desirable, as with the times. Here the observed height is plotted against the reduced times, in days plus the decimals, and the values read off from the graphs at zero hour of each day.

V. It is necessary to determine the value of mean sea level for the month and also the formula for one phase of the quarter-diurnal tide. The method is like that used in Part I.

Let $z(s)$ denote values of reduced heights in sequence s, and in the space provided on Sheet IX compute the following quantities for springs and neaps (or neaps and springs) in due order.

$$e_0 = \frac{1}{4}[z(0) + z(1) + z(2) + z(3)]$$

$$e_2 = \frac{1}{4}[z(0) - z(1) + z(2) - z(3)]$$

$$e_1 = \frac{1}{2}[z(0) - z(2)]$$

$$f_1 = \frac{1}{2}[z(1) - z(3)]$$

Then we compute

$$(e_1^2 - f_1^2) / 16\, e_2$$

and add to e_0 to give

$$a_0 + a_4$$

As in Part I we assume

$$a_4 = \lambda\, e_2^2$$

and proceed to determine the value of λ and the mean value of a_0 as in Part I.

It is not always easy to determine the days on which springs and neaps occur but it is not essential to have actual spring or neap tides, for the calculation of λ only requires that the values of e_2 should be large in two cases and small in the other two cases.

VI. It will be sufficient to take a mean value of a_0 for the month, and then in columns 17 to 20 compute

$$Z = z - a_0$$

VII. In column 21 compute

τ = the approximate time of high water of the semidiurnal tide for the day. In normal cases this is obtained from the formula

$$4\tau = t(0) + [t(1) - 0.250] + [t(2) - 0.500] + [t(3) - 0.750]$$

or

$$\tau = \frac{1}{4}[t(0) + t(1) + t(2) + t(3) - 1.500]$$

Whenever t has passed through 1.000 the remaining values in the column must be increased by 1.000 for this calculation. Thus for the four values 0.445, 0.746, 0.017, 0.200, it is evident that they should be regarded as 0.445, 0.746, 1.017, 1.200.

VIII. In column 22 enter the values of $\tau + 0.250$.

(Columns 23 and 24 can be used later, if required, for the values of 2τ and 3τ.

IX. (Sheet VII). In columns 25 to 28 enter the values of the relative times given by

$$T = t - (\tau + 0.250\, s)$$

for the values s = 0 to 3. These give, very closely, the values of the times of high or low water relative to the times of high and low water of the semidiurnal tide for the day. For s = 2 it is easy to add 0.500 to the values of τ, and for s = 3 it is easy to add 0.500 to the values of $\tau + 0.250$, as given in columns 21 and 22, and then to perform the subtractions required to give T.

X. Using Table 10 for values of cos 2nT, compute

$$[\alpha] = Z \cos 2nT$$

and enter the results in columns 29 to 32. As usual, the values of Z are used with the corresponding values of T in the same sequence.

XI. In columns 33 to 35, using the values of $[\alpha]$ as the approximate values of α, compute to 2 decimals of a foot the approximate values of α_2, α_1, and α'_1 from the formulae

$$\alpha_2 = \frac{1}{4}[\alpha(0) - \alpha(1) + \alpha(2) - \alpha(3)]$$

$$\alpha_1 = \frac{1}{2}[\alpha(0) - \alpha(2)]$$

$$\alpha'_1 = \frac{1}{2}[\alpha(1) - \alpha(3)]$$

XII. — In columns 36 to 38 compute the approximate values of α_4, β_4, and β_2, from the formulae

$$\alpha_4 = \lambda\alpha_2^2 \ , \ \beta_4 = \mu\alpha_2^2 \ , \ \beta_2 = 8\alpha_4\beta_4/\alpha_2$$

$$\mu = [\overline{T}(0) - \overline{T}(1) + \overline{T}(2) - \overline{T}(3)] \times 1.56/\overline{\alpha}_2$$

where $\overline{T}$ and $\overline{\alpha}_2$ are the means in the month.

XIII. The approximate values of α_1, α'_1 α_4, β_4 are used to correct the values of $[\alpha]$, using Tables 12 and 13. Thus, from column 25, $s = 0$, use the first section of Table 10 or 11 according to whether T is positive or negative, and multiply the corresponding values of α_1, α'_1, α_4, β_4 by the tabular quantities on the calculating machine, and enter the sum of the products in column 39, to one decimal of a foot. Then for $s = 2$, with T in column 26, use the second sections of the tables for the multipliers, and enter the results in column 40 ; and so on.

XIV. Now apply the same formulae as are given in XI to the corrections to $[\alpha]$, and add the results, to two decimals of a foot, to the corresponding values of α_2, α_1, α'_1. The corrected values of α_2, α_1 and α'_1 are then entered in columns 43, 45 and 46. In column 44 we simply copy the values of α_4 from column 36.

XV. Compute the value of $(\alpha_0 + \alpha_4)$ from the mean of all the 8 quantities $[\alpha]$ and "corrections to $[\alpha]$", in columns 29 to 32 and 39 to 42, and enter the results to two decimals of a foot in column 47.

XVI. From columns 45 and 47, subtract the values of α_4 from the values of $(\alpha_0 + \alpha_4)$, and enter the results in column 48. These values should all be small, and they represent the variation of α_0 during the month. They are available for the analysis for long period tides if such are required, but unless there are many months of data available it is useless to analyse these values of α_0 as part of the normal routine.

Compute the mean value of α_0. This is a correction to the value used in VI.

XVII. (Sheet VIII). Now compute values of $\Sigma m^2\alpha$ in columns 49 to 52 from the formulae

$$s = 0 \qquad \Sigma m^2\alpha = \alpha_2 + 4\alpha_4 + \frac{1}{4}\alpha_1$$

$$s = 1 \qquad \Sigma m^2\alpha = -\alpha_2 + 4\alpha_4 + \frac{1}{4}\alpha'_1$$

$$s = 2 \qquad \Sigma m^2\alpha = \alpha_2 + 4\alpha_4 - \frac{1}{4}\alpha_1$$

$$s = 3 \qquad \Sigma m^2\alpha = -\alpha_2 + 4\alpha_4 - \frac{1}{4}\alpha'_1$$

XVIII. Using Table 11 for values of (2 sin nT) compute

$$[\beta] = (2 \sin nT) \,.\, \Sigma\, m^2\alpha$$

from values of T in columns 25 to 28 and corresponding values of $\Sigma m^2\alpha$ in columns 49 to 52. The values are given to two decimals of a foot.

XIX. Corrections to $[\beta]$ are computed in the same way as in XIII, using Tables 14 and 15, with approximate values of α_1, α'_1 α_4, β_4, β_2 as in columns 34 to 38. The results, to one decimal of a foot, are given in columns 57 to 60.

XX. Now compute β_2, β_4, β_1, β'_1 from the formulae

$$\beta_2 = \frac{1}{4}[\beta(0) + \beta(1) + \beta(2) + \beta(3)]$$

$$\beta_4 = \frac{1}{8}[\beta(0) - \beta(1) + \beta(2) - \beta(3)]$$

$$\beta_1 = [\beta(0) - \beta(2)]$$

$$\beta'_1 = [\beta(1) - \beta(3)]$$

and enter the results in columns 61 to 64. In the formulae the value of β is taken to be the sum of $[\beta]$ and its correction. As an example, let the values of $[\beta]$ and the corrections be

1.3 1.0 —1.4 —1.2 and 0.1 —0.1 —0.2 0.1

then the values required are as follows :

$$\beta_2 = \frac{1}{4}[(1.3 + 1.0 - 1.4 - 1.2) + (0.1 - 0.1 - 0.2 + 0.1)] = -0.10$$

$$\beta_4 = \frac{1}{8}[(1.3 - 1.0 - 1.4 + 1.2) + (0.1 + 0.1 - 0.2 - 0.1)] = 0.00$$

$$\beta_1 = [(1.3 + 1.4) + (0.1 + 0.2)] = 3.0$$

$$\beta'_1 = [(1.0 + 1.2) + (-0.1 - 0.1)] = 2.0$$

XXI. Compute a_3, b_3, a_1, b_1, in order from the formulae

$$a_3 = \frac{1}{4}[\alpha_1 + \beta'_1] \qquad b_3 = \frac{1}{4}[\beta_1 - \alpha'_1]$$

$$a_1 = \alpha_1 - a_3 \qquad b_1 = \alpha'_1 + b_3$$

using the data in columns 45 and 46, 63 and 64, and enter the results in columns 65 and 66, 69 and 70.

XXII. (Sheet IX). In columns 73 and 74, 77 and 78, enter the values of

$$a_2 = \alpha_2 \;,\; b_2 = \beta_2 \;,\; a_4 = \alpha_4 \;,\; b_4 = \beta_4$$

from columns 43 and 44, 61 and 62.

XXIII. It is now necessary to compute certain values of cosines and sines from angles derived from the mean times τ in column 21. A special table is given in Table 16. At the sides of this table are columns headed "cosine" and "sine", and these columns give the value of τ corresponding to direct readings from the table. Thus the sine corresponding to 0.127 is obtained by reading against 0.12 in the left-hand column headed "sine" and under 0.007 at the head of the columns. Similarly the sine corresponding to

0.327 comes from the tabular entry against 0.32 in the right-hand column headed "sine" and against 0.007 at the foot of the columns. Cosines come similarly from entries in the columns headed "cosine".

If τ is increased or decreased by 0.500 or 1.500 then the same table is used but negative signs are attached to the entries.

If τ is increased or decreased by 1.000 or 2.000 then the tabular entries take the positive signs.

Using the value of τ given in column 21 use Table 16 to obtain

$$c_1 = \cos\theta_1 \qquad s_1 = \sin\theta_1$$

and enter the values to three decimals in columns 71 and 72.

XXIV. — The values of 2τ can be computed from column 21 on a separate sheet and with these values we enter Table 16 to obtain

$$c_2 = \cos\theta_2 \quad , \quad s_2 = \sin\theta_2$$

and the values are entered to three decimals in columns 75 and 76.

XXV. Similarly we compute 3τ and 4τ, and thence the values of c_3, s_3, c_4, s_4 to two places of decimals only. The results are entered in columns 67 and 68, 79 and 80.

XXVI. Useful checks on the values of c and s are available, as follows :

$$c_2 = c_1^2 - s_1^2 \qquad s_2 = 2c_1 s_1$$

$$c_4 = c_2^2 - s_2^2 \qquad s_4 = 2c_2 s_2$$

$$c_3 = c_1 c_2 - s_1 s_2 \qquad s_3 = c_1 s_2 + s_1 c_2$$

These checks are particularly useful to check the signs.

XXVII. In columns 81 to 88 enter the values of A and B for the four species of tide, from the formulae

$$A = ac - bs \qquad B = bc + as$$

For a check

$$A + B = (a + b)c + (a - b)s$$

XXVIII. Examine the values of A and B for smoothness. Owing to the somewhat rapid changes which may occur in t and τ under certain conditions no real test of smoothness can be applied until this stage is reached. It may be found necessary to revise the interpolations where the changes of time of high or low water are rapid, but this can only affect at the most two or three days. It is permissible, however, to smooth the values without amending the earlier calculations, though it is always desirable to verify that errors have not occurred at any stage.

XXIX. The multipliers given in Table 17 are now to be applied to the values of A and B in the same manner as was explained in Part I. The table of multipliers has been extended from the corresponding table used in Part I because it is necessary to use the multipliers d_5 and d_e. The reason for this is that in this method we are working in intervals of a solar day whereas in Part I the intervals were lunar days.

XXX. The values of R cos r and R sin r for the required constituents are obtained from the values of A_{po}, A_{p1},...... B_{pa}, B_{pb}...... by the application of the multipliers of Table 18. The procedure differs somewhat from that used in Part I. The multipliers are placed alongside the values of A_{po}, etc., and the sums of the products, when divided by 10^5 give R cos r, or R sin r. The results are placed in the appropriate places on Sheet X, as in the example.

XXXI. The calculation of H and g is on exactly the same lines as in Part I, except that the time origin is always at zero hour on the central day, so that $V''' = 0$.

The value of mean sea level (A_0) is the value of a_0 used in columns 17 to 20 plus the mean value of a_0 in column 48.

Remarks on example of analysis (Part II)

The method of analysis explained in this part has been designed for tides in which the diurnal tide is so large, relatively to the semidiurnal tide, that the tides at certain states become very nearly diurnal. The example has been chosen to illustrate almost the extreme conditions under which the method can be used. The data are the predicted tides for Vancouver, in January 1950, but the predictions differ somewhat from the standard ones in that the sixth-diurnal tides have been omitted, and an arbitrary value of mean sea level has been chosen. The predictions were computed very carefully on the largest possible scale, and the times were expressed in units of a day. The heights were taken to the nearest 0.05 ft. The analysis has been completed in that H and g have been evaluated, but a more useful comparison is that between the values of R cos r, R sin r, as obtained by the analysis and as known from the settings on the predicting machine, for these comparisons indicate the errors in feet for the two quantities, and a comparison can be made with the errors obtained in Part I. The following table gives the exact values of R cos r and R sin r and the errors of the analytical results :

	R cos *r*	*R* sin *r*	*Errors.*			*R* cos *r*	*R* sin *r*	*Errors.*	
	–	–	–			–	–	–	
M_2	2.554	1.352	0.056	0.007	K_1	–3.414	1.535	0.001	0.011
S_2	–0.562	–0.409	0.001	0.012	O_1	1.303	1.202	0.008	0.043
N_2	–0.527	0.329	0.004	0.014	Q_1	–0.276	–0.090	0.012	0.006
L_2	0.036	0.147	0.033	0.009	J_1	0.138	0.103	0.127	0.023
μ_2	–0.006	0.017	0.021	0.033					
2 SM_2	–0.021	0.002	0.007	0.019					
M_4	0.000	–0.046	0.035	0.054	M_3	0.008	–0.004	0.031	0.037
MS_4	0.013	0.015	0.007	0.043	MK_3	–0.001	0.062	0.080	0.003
MN_4	0.013	0.014	0.035	0.008	MO_3	–0.070	–0.028	0.122	0.078

In addition to the above constituents the prediction included the effects of the long period constituents Sa, Ssa, Mm, MSf, and Mf with amplitudes equal to 0.23, 0.15, 0.05, 0.04 and 0.17 respectively.

It will be noted that the major constituents are quite well obtained, but there are some anomalies ; the large error in J_1 has not been explained, and the very small third-diurnal constituents are not well represented. The quarter-diurnal tides are about twice what they ought to be but the reason for this is that MSf and Mf have not been separable from the quarterdiurnal tides. The value of mean sea level ought to have been 12.00 when corrected for the contributions from Sa and Ssa but we obtained 12.10. The error is due to the over-large quarterdiurnal tide. It cannot be too often reiterated that it is impossible to obtain a perfect separation of the quarter diurnal and long-period tides.

There are some analytical results for constituents which were not represented in the predictions and these values of R cos r and R sin r are therefore errors. If we take the mean of all the errors, ignoring all signs, we obtain as the average errors from the separate species the following results. They represent the average error in any analytical quantity :

Semidiurnal constituents	0.018 ft.
Diurnal constituents..................	0.041 ft.
Third-diurnal constituents..........	0.059 ft.
Quarter-diurnal constituents	0.031 ft.
All constituents	0.034 ft.

The method has been applied to the same data as in Part I and the following table gives the errors in R cos r and R sin r by the two methods :

Errors in	*R* cos *r*		*R* sin *r*			*R* cos *r*		*R* sin *r*	
	I	II	I	II		I	II	I	II
M_2	0.003	0.016	0.023	0.027	K_1	0.005	0.016	0.002	0.019
L_2	0.002	0.014	0.004	0.018	O_1	0.009	0.007	0.011	0.008
N_2	0.014	0.025	0.001	0.004	J_1	0.004	0.008	0.000	0.005
S_2	0.001	0.000	0.015	0.016	Q_1	0.009	0.003	0.001	0.005
μ_2	0.006	0.020	0.003	0.019					
M_4	0.019	0.003	0.003	0.003	MK_3	0.012	0.001	0.002	0.002
MN_4	0.003	0.001	0.020	0.016	MO_3	0.010	0.006	0.004	0.011
MS_4	0.002	0.002	0.027	0.004					

The errors for other constituents not represented in the data are all very small. It is evident that the errors by the second method are nearly as small as those by the first method, and that no constituent is singled out as incurring a systematic error by either method. The error noted in J_1 for the first example of Part II cannot be explained on any systematic law of error, and evidently it must be due to casual error. If the calculations are examined it will be evident that the diurnal tides are more susceptible to error than any others as in deriving α_1 from α we only divide the combined results from α by 2, whereas we divide the combination for α_2 by 4 ; similarly when β_1 is derived from β it is taken direct from the combination whereas the combination for β_2 is divided by 4. Also β is derived from a formula which is largely dependent on the time T and as the times of high and low water are not capable of being expressed with the same exactness as the heights it must be expected that the diurnal tides will be susceptible to larger errors than other tides.

TABLE 9.

Reduction of times to decimals of a day. The table is given to 4 decimals. For times greater than 12 hours subtract 12 hours and add 0·5 to the entries.

mins.	hours. 0	1	2	3	4	5	6	7	8	9	10	11
0	0000	0417	0833	1250	1667	2083	2500	2917	3333	3750	4167	4583
2	0014	0431	0847	1264	1681	2097	2514	2931	3347	3764	4181	4597
4	0028	0444	0861	1278	1694	2111	2528	2944	3361	3778	4194	4611
6	0042	0458	0875	1292	1708	2125	2542	2958	3375	3792	4208	4625
8	0056	0472	0889	1306	1722	2139	2556	2972	3389	3806	4222	4639
10	0069	0486	0903	1319	1736	2153	2569	2986	3403	3819	4236	4653
12	0083	0500	0917	1333	1750	2167	2583	3000	3417	3833	4250	4667
14	0097	0514	0931	1347	1764	2181	2597	3014	3431	3847	4264	4681
16	0111	0528	0944	1361	1778	2194	2611	3028	3444	3861	4278	4694
18	0125	0542	0958	1375	1792	2208	2625	3042	3458	3875	4292	4708
20	0139	0556	0972	1389	1806	2222	2639	3056	3472	3889	4306	4722
22	0153	0569	0986	1403	1819	2236	2653	3069	3486	3903	4319	4736
24	0167	0583	1000	1417	1833	2250	2667	3083	3500	3917	4333	4750
26	0181	0597	1014	1431	1847	2264	2681	3097	3514	3931	4347	4764
28	0194	0611	1028	1444	1861	2278	2694	3111	3528	3944	4361	4778
30	0208	0625	1042	1458	1875	2292	2708	3125	3542	3958	4375	4792
32	0222	0639	1056	1472	1889	2306	2722	3139	3556	3972	4389	4806
34	0236	0653	1069	1486	1903	2319	2736	3153	3569	3986	4403	4819
36	0250	0667	1083	1500	1917	2333	2750	3167	3583	4000	4417	4833
38	0264	0681	1097	1514	1931	2347	2764	3181	3597	4014	4431	4847
40	0278	0694	1111	1528	1944	2361	2778	3194	3611	4028	4444	4861
42	0292	0708	1125	1542	1958	2375	2792	3208	3625	4042	4458	4875
44	0306	0722	1139	1556	1972	2389	2806	3222	3639	4056	4472	4889
46	0319	0736	1153	1569	1986	2403	2819	3236	3653	4069	4486	4903
48	0333	0750	1167	1583	2000	2417	2833	3250	3667	4083	4500	4917
50	0347	0764	1181	1597	2014	2431	2847	3264	3681	4097	4514	4931
52	0361	0778	1194	1611	2028	2444	2861	3278	3694	4111	4528	4944
54	0375	0792	1208	1625	2042	2458	2875	3292	3708	4125	4542	4958
56	0389	0806	1222	1639	2056	2472	2889	3306	3722	4139	4556	4972
58	0403	0819	1236	1653	2069	2486	2903	3319	3736	4153	4569	4986
60	0417	0833	1250	1667	2083	2500	2917	3333	3750	4167	4583	5000

TABLE 10.

Values of cos 2nT.

T	0·000	0·001	0·002	0·003	0·004	0·005	0·006	0·007	0·008	0·009
0·00	1·000	1·000	1·000	0·999	0·999	0·998	0·997	0·996	0·995	0·994
0·01	0·992	0·990	0·989	0·987	0·985	0·982	0·980	0·977	0·975	0·972
0·02	0·969	0·965	0·962	0·959	0·955	0·951	0·947	0·943	0·939	0·934
0·03	0·930	0·925	0·920	0·915	0·910	0·905	0·899	0·894	0·888	0·882
0·04	0·876	0·870	0·864	0·858	0·851	0·844	0·838	0·831	0·824	0·816
0·05	0·809	0·802	0·794	0·786	0·778	0·771	0·762	0·754	0·746	0·738
0·06	0·729	0·720	0·712	0·703	0·694	0·685	0·675	0·666	0·657	0·647
0·07	0·637	0·628	0·618	0·608	0·598	0·588	0·578	0·567	0·557	0·546
0·08	0·536	0·525	0·514	0·504	0·493	0·482	0·471	0·460	0·448	0·437
0·09	0·426	0·414	0·403	0·391	0·380	0·368	0·356	0·345	0·333	0·321

TABLE 11.

Values of 2 sin nT.

T	0·000	0·001	0·002	0·003	0·004	0·005	0·006	0·007	0·008	0·009
0·00	0·000	0·013	0·025	0·038	0·050	0·063	0·075	0·088	0·100	0·113
0·01	0·126	0·138	0·151	0·163	0·176	0·188	0·201	0·213	0·226	0·238
0·02	0·251	0·263	0·276	0·288	0·300	0·313	0·325	0·338	0·350	0·362
0·03	0·375	0·387	0·399	0·412	0·424	0·436	0·449	0·461	0·473	0·485
0·04	0·497	0·509	0·522	0·534	0·546	0·558	0·570	0·582	0·594	0·606
0·05	0·618	0·630	0·642	0·654	0·666	0·677	0·689	0·701	0·713	0·725
0·06	0·736	0·748	0·760	0·771	0·783	0·794	0·806	0·817	0·829	0·840
0·07	0·852	0·863	0·874	0·886	0·897	0·908	0·919	0·930	0·941	0·952
0·08	0·963	0·975	0·985	0·996	1·008	1·018	1·029	1·040	1·050	1·061
0·09	1·072	1·082	1·093	1·103	1·114	1·124	1·135	1·145	1·155	1·165

TABLE 12 :- Corrections to $[\alpha]$
Coefficients of α_1, α_1', α_4, β_4 to two decimals.

T	s = 0				s = 1				s = 2				s = 3			
	α_1	α_1'	α_4	β_4	α_1	α_1'	α_4	β_4	α_1	α_1'	α_4	β_4	α_1	α_1'	α_4	β_4
0·000	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00
0·003	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00
0·006	00	00	-01	00	00	00	-01	00	00	00	-01	00	00	00	-01	00
0·008	00	00	-02	00	00	00	-02	00	00	00	-02	00	00	00	-02	00
0·011	01	00	-03	-01	00	01	-03	-01	-01	00	-03	-01	00	-01	-03	-01
0·014	01	00	-04	-01	00	01	-04	-01	-01	00	-04	-01	00	-01	-04	-01
0·017	02	00	-06	-02	00	02	-06	-02	-02	00	-06	-02	00	-02	-06	-02
0·019	02	00	-08	-03	00	02	-08	-03	-02	00	-08	-03	00	-02	-08	-03
0·022	03	00	-11	-04	00	03	-11	-04	-03	00	-11	-04	00	-03	-11	-04
0·025	04	00	-13	-06	00	04	-13	-06	-04	00	-13	-06	00	-04	-13	-06
0·028	04	01	-16	-08	-01	04	-16	-08	-04	-01	-16	-08	01	-04	-16	-08
0·031	05	01	-19	-11	-01	05	-19	-11	-05	-01	-19	-11	01	-05	-19	-11
0·033	06	01	-22	-13	-01	06	-22	-13	-06	-01	-22	-13	01	-06	-22	-13
0·036	07	01	-24	-17	-01	07	-24	-17	-07	-01	-24	-17	01	-07	-24	-17
0·039	09	01	-27	-21	-01	09	-27	-21	-09	-01	-27	-21	01	-09	-27	-21
0·042	10	02	-30	-25	-02	10	-30	-25	-10	-02	-30	-25	02	-10	-30	-25
0·044	11	02	-32	-30	-02	11	-32	-30	-11	-02	-32	-30	02	-11	-32	-30
0·047	13	02	-35	-35	-02	13	-35	-35	-13	-02	-35	-35	02	-13	-35	-35
0·050	14	03	-37	-41	-03	14	-37	-41	-14	-03	-37	-41	03	-14	-37	-41
0·053	16	03	-39	-47	-03	16	-39	-47	-16	-03	-39	-47	03	-16	-39	-47
0·056	17	04	-40	-53	-04	17	-40	-53	-17	-04	-40	-53	04	-17	-40	-53
0·058	19	05	-41	-60	-05	19	-41	-60	-19	-05	-41	-60	05	-19	-41	-60
0·061	20	05	-41	-67	-05	20	-41	-67	-20	-05	-41	-67	05	-20	-41	-67
0·064	22	06	-41	-74	-06	22	-41	-74	-22	-06	-41	-74	06	-22	-41	-74
0·067	24	07	-41	-82	-07	24	-41	-82	-24	-07	-41	-82	07	-24	-41	-82
0·069	26	08	-40	-90	-08	26	-40	-90	-26	-08	-40	-90	08	-26	-40	-90
0·072	27	08	-38	-98	-08	27	-38	-98	-27	-08	-38	-98	08	-27	-38	-98
0·075	29	09	-36	-106	-09	29	-36	-106	-29	-09	-36	-106	09	-29	-36	-106
0·078	31	10	-33	-114	-10	31	-33	-114	-31	-10	-33	-114	10	-31	-33	-114
0·081	33	11	-29	-122	-11	33	-29	-122	-33	-11	-29	-122	11	-33	-29	-122
0·083	35	12	-25	-130	-12	35	-25	-130	-35	-12	-25	-130	12	-35	-25	-130
0·086	37	14	-20	-138	-14	37	-20	-138	-37	-14	-20	-138	14	-37	-20	-138
0·089	39	15	-15	-145	-15	39	-15	-145	-39	-15	-15	-145	15	-39	-15	-145
0·092	41	16	-09	-152	-16	41	-09	-152	-41	-16	-09	-152	16	-41	-09	-152
0·095	43	17	-02	-159	-17	43	-02	-159	-43	-17	-02	-159	17	-43	-02	-159
0·097	45	19	05	-166	-19	45	05	-166	-45	-19	05	-166	19	-45	05	-166
0·100	47	20	13	-172	-20	47	13	-172	-47	-20	13	-172	20	-47	13	-172

TABLE 13 :- Corrections to [α]

Coefficients of α_1 , α_1' , α_4 , β_4 to two decimals.

τ	s = 0				s = 1				s = 2				s = 3			
	α_1	α_1'	α_4	β_4	α_1	α_1'	α_4	β_4	α_1	α_1'	α_4	β_4	α_1	α_1'	α_4	β_4
0·000	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00
-0·003	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00
-0·006	00	00	-01	00	00	00	-01	00	00	00	-01	00	00	00	-01	00
-0·008	00	00	-02	00	00	00	-02	00	00	00	-02	00	00	00	-02	00
-0·011	01	00	-03	01	00	01	-03	01	-01	00	-03	01	00	-01	-03	01
-0·014	01	00	-04	01	00	01	-04	01	-01	00	-04	01	00	-01	-04	01
-0·017	02	00	-06	02	00	02	-06	02	-02	00	-06	02	00	-02	-06	02
-0·019	02	00	-08	03	00	02	-08	03	-02	00	-08	03	00	-02	-08	03
-0·022	03	00	-11	04	00	03	-11	04	-03	00	-11	04	00	-03	-11	04
-0·025	04	00	-13	06	00	04	-13	06	-04	00	-13	06	00	-04	-13	06
-0·028	04	-01	-16	08	01	04	-16	08	-04	01	-16	08	-01	-04	-16	08
-0·031	05	-01	-19	11	01	05	-19	11	-05	01	-19	11	-01	-05	-19	11
-0·033	06	-01	-22	13	01	06	-22	13	-06	01	-22	13	-01	-06	-22	13
-0·036	07	-01	-24	17	01	07	-24	17	-07	01	-24	17	-01	-07	-24	17
-0·039	09	-01	-27	21	01	09	-27	21	-09	01	-27	21	-01	-09	-27	21
-0·042	10	-02	-30	25	02	10	-30	25	-10	02	-30	25	-02	-10	-30	25
-0·044	11	-02	-32	30	02	11	-32	30	-11	02	-32	30	-02	-11	-32	30
-0·047	13	-02	-35	35	02	13	-35	35	-13	02	-35	35	-02	-13	-35	35
-0·050	14	-03	-37	41	03	14	-37	41	-14	03	-37	41	-03	-14	-37	41
-0·053	16	-03	-39	47	03	16	-39	47	-16	03	-39	47	-03	-16	-39	47
-0·056	17	-04	-40	53	04	17	-40	53	-17	04	-40	53	-04	-17	-40	53
-0·058	19	-05	-41	60	05	19	-41	60	-19	05	-41	60	-05	-19	-41	60
-0·061	20	-05	-41	67	05	20	-41	67	-20	05	-41	67	-05	-20	-41	67
-0·064	22	-06	-41	74	06	22	-41	74	-22	06	-41	74	-06	-22	-41	74
-0·067	24	-07	-41	82	07	24	-41	82	-24	07	-41	82	-07	-24	-41	82
-0·069	26	-08	-40	90	08	26	-40	90	-26	08	-40	90	-08	-26	-40	90
-0·072	27	-08	-38	98	08	27	-38	98	-27	08	-38	98	-08	-27	-38	98
-0·075	29	-09	-36	106	09	29	-36	106	-29	09	-36	106	-09	-29	-36	106
-0·078	31	-10	-33	114	10	31	-33	114	-31	10	-33	114	-10	-31	-33	114
-0·081	33	-11	-29	122	11	33	-29	122	-33	11	-29	122	-11	-33	-29	122
-0·083	35	-12	-25	130	12	35	-25	130	-35	12	-25	130	-12	-35	-25	130
-0·086	37	-14	-20	138	14	37	-20	138	-37	14	-20	138	-14	-37	-20	138
-0·089	39	-15	-15	145	15	39	-15	145	-39	15	-15	145	-15	-39	-15	145
-0·092	41	-16	-09	152	16	41	-09	152	-41	16	-09	152	-16	-41	-09	152
-0·095	43	-17	-02	159	17	43	-02	159	-43	17	-02	159	-17	-43	-02	159
-0·097	45	-19	05	166	19	45	05	166	-45	19	05	166	-19	-45	05	166
-0·100	47	-20	13	172	20	47	13	172	-47	20	13	172	-20	-47	13	172

TABLE 14 : Corrections to $[\beta]$

Coefficients of α_1, α_1', α_4, β_4, β_2 to two decimals.

T	s = 0					s = 1					s = 2					s = 3				
	α_1	α_1'	α_4	β_4	β_2	α_1	α_1'	α_4	β_4	β_2	α_1	α_1'	α_4	β_4	β_2	α_1	α_1'	α_4	β_4	β_2
0·000	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00
0·003	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00
0·006	00	00	00	02	00	00	00	00	02	00	00	00	00	02	00	00	00	00	02	00
0·008	00	00	00	04	00	00	00	00	04	00	00	00	00	04	00	00	00	00	04	00
0·011	00	00	-01	07	01	00	00	-01	07	-01	00	00	-01	07	01	00	00	-01	07	-01
0·014	00	00	-01	11	01	00	00	-01	11	-01	00	00	-01	11	01	00	00	-01	11	-01
0·017	00	00	-02	16	02	00	00	-02	16	-02	00	00	-02	16	02	00	00	-02	16	-02
0·019	00	00	-03	22	02	00	00	-03	22	-02	00	00	-03	22	02	00	00	-03	22	-02
0·022	00	00	-04	29	03	00	00	-04	29	-03	00	00	-04	29	03	00	00	-04	29	-03
0·025	00	00	-06	36	04	00	00	-06	36	-04	00	00	-06	36	04	00	00	-06	36	-04
0·028	00	00	-08	44	05	00	00	-08	44	-05	00	00	-08	44	05	00	00	-08	44	-05
0·031	00	00	-11	53	06	00	00	-11	53	-06	00	00	-11	53	06	00	00	-11	53	-06
0·033	00	00	-14	63	07	00	00	-14	63	-07	00	00	-14	63	07	00	00	-14	63	-07
0·036	00	00	-18	74	08	00	00	-18	74	-08	00	00	-18	74	08	00	00	-18	74	-08
0·039	00	00	-23	85	09	00	00	-23	85	-09	00	00	-23	85	09	00	00	-23	85	-09
0·042	00	00	-28	96	10	00	00	-28	96	-10	00	00	-28	96	10	00	00	-28	96	-10
0·044	01	00	-33	109	12	00	01	-33	109	-12	-01	00	-33	109	12	00	-01	-33	109	-12
0·047	01	00	-40	122	13	00	01	-40	122	-13	-01	00	-40	122	13	00	-01	-40	122	-13
0·050	01	00	-47	135	15	00	01	-47	135	-15	-01	00	-47	135	15	00	-01	-47	135	-15
0·053	01	00	-55	149	17	00	01	-55	149	-17	-01	00	-55	149	17	00	-01	-55	149	-17
0·056	01	00	-64	163	18	00	01	-64	163	-18	-01	00	-64	163	18	00	-01	-64	163	-18
0·058	01	00	-74	178	20	00	01	-74	178	-20	-01	00	-74	178	20	00	-01	-74	178	-20
0·061	01	00	-84	192	22	00	01	-84	192	-22	-01	00	-84	192	22	00	-01	-84	192	-22
0·064	02	00	-95	208	25	00	02	-95	208	-25	-02	00	-95	208	25	00	-02	-95	208	-25
0·067	02	00	-108	223	27	00	02	-108	223	-27	-02	00	-108	223	27	00	-02	-108	223	-27
0·069	02	00	-121	238	29	00	02	-121	238	-29	-02	00	-121	238	29	00	-02	-121	238	-29
0·072	02	00	-135	254	31	00	02	-135	254	-31	-02	00	-135	254	31	00	-02	-135	254	-31
0·075	03	00	-150	269	34	00	03	-150	269	-34	-03	00	-150	269	34	00	-03	-150	269	-34
0·078	03	00	-166	285	37	00	03	-166	285	-37	-03	00	-166	285	37	00	-03	-166	285	-37
0·081	03	00	-182	300	39	00	03	-182	300	-39	-03	00	-182	300	39	00	-03	-182	300	-39
0·083	04	00	-200	315	42	00	04	-200	315	-42	-04	00	-200	315	42	00	-04	-200	315	-42
0·086	04	00	-219	330	45	00	04	-219	330	-45	-04	00	-219	330	45	00	-04	-219	330	-45
0·089	05	00	-238	345	48	00	05	-238	345	-48	-05	00	-238	345	48	00	-05	-238	345	-48
0·092	05	00	-258	360	52	00	05	-258	360	-52	-05	00	-258	360	52	00	-05	-258	360	-52
0·095	06	00	-280	374	55	00	06	-280	374	-55	-06	00	-280	374	55	00	-06	-280	374	-55
0·097	06	00	-302	387	58	00	06	-302	387	-58	-06	00	-302	387	58	00	-06	-302	387	-58
0·100	07	00	-325	400	62	00	07	-325	400	-62	-07	00	-325	400	62	00	-07	-325	400	-62

TABLE 15 : Corrections to $[\beta]$

Coefficients of α_1 , α_1' , α_4 , β_4 , β_2 to two decimals.

τ	s = 0					s = 1					s = 2					s = 3				
	α_1	α_1'	α_4	β_4	β_2	α_1	α_1'	α_4	β_4	β_2	α_1	α_1'	α_4	β_4	β_2	α_1	α_1'	α_4	β_4	β_2
0·000	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00
-0·003	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00	00
-0·006	00	00	00	02	00	00	00	00	02	00	00	00	00	02	00	00	00	00	02	00
-0·008	00	00	00	04	00	00	00	00	04	00	00	00	00	04	00	00	00	00	04	00
-0·011	00	00	01	07	01	00	00	01	07	-01	00	00	01	07	01	00	00	01	07	-01
-0·014	00	00	01	11	01	00	00	01	11	-01	00	00	01	11	01	00	00	01	11	-01
-0·017	00	00	02	16	02	00	00	02	16	-02	00	00	02	16	02	00	00	02	16	-02
-0·019	00	00	03	22	02	00	00	03	22	-02	00	00	03	22	02	00	00	03	22	-02
-0·022	00	00	04	29	03	00	00	04	29	-03	00	00	04	29	03	00	00	04	29	-03
-0·025	00	00	06	36	04	00	00	06	36	-04	00	00	06	36	04	00	00	06	36	-04
-0·028	00	00	08	44	05	00	00	08	44	-05	00	00	08	44	05	00	00	08	44	-05
-0·031	00	00	11	53	06	00	00	11	53	-06	00	00	11	53	06	00	00	11	53	-06
-0·033	00	00	14	63	07	00	00	14	63	-07	00	00	14	63	07	00	00	14	63	-07
-0·036	00	00	18	74	08	00	00	18	74	-08	00	00	18	74	08	00	00	18	74	-08
-0·039	00	00	23	85	09	00	00	23	85	-09	00	00	23	85	09	00	00	23	85	-09
-0·042	00	00	28	96	10	00	00	28	96	-10	00	00	28	96	10	00	00	28	96	-10
-0·044	-01	00	33	109	12	00	-01	33	109	-12	01	00	33	109	12	00	01	33	109	-12
-0·047	-01	00	40	122	13	00	-01	40	122	-13	01	00	40	122	13	00	01	40	109	-13
-C·050	-01	00	47	135	15	00	-01	47	135	-15	01	00	47	135	15	00	01	47	135	-15
-0·053	-01	00	55	149	17	00	-01	55	149	-17	01	00	55	149	17	00	01	55	149	-17
-0·056	-01	00	64	163	18	00	-01	64	163	-18	01	00	64	163	18	00	01	64	163	-18
-0·058	-01	00	74	178	20	00	-01	74	178	-20	01	00	74	178	20	00	01	74	178	-20
-0·061	-01	00	84	192	22	00	-01	84	192	-22	01	00	84	192	22	00	01	84	192	-22
-0·064	-02	00	95	208	25	00	-02	95	208	-25	02	00	95	208	25	00	02	95	208	-25
-0·067	-02	00	108	223	27	00	-02	108	223	-27	02	00	108	223	27	00	02	108	223	-27
-0·069	-02	00	121	238	29	00	-02	121	238	-29	02	00	121	238	29	00	02	121	238	-29
-0·072	-02	00	135	254	31	00	-02	135	254	-31	02	00	135	254	31	00	02	135	254	-31
-0·075	-03	00	150	269	34	00	-03	150	269	-34	03	00	150	269	34	00	03	150	269	-34
-0·078	-03	00	166	285	37	00	-03	166	285	-37	03	00	166	285	37	00	03	166	285	-37
-0·081	-03	00	182	300	39	00	-03	182	300	-39	03	00	182	300	39	00	03	182	300	-39
-0·083	-04	00	200	315	42	00	-04	200	315	-42	04	00	200	315	42	00	04	200	315	-42
-0·086	-04	00	219	330	45	00	-04	219	330	-45	04	00	219	330	45	00	04	219	330	-45
-0·089	-05	00	238	345	48	00	-05	238	345	-48	05	00	238	345	48	00	05	238	345	-48
-0·092	-05	00	258	360	52	00	-05	258	360	-52	05	00	258	360	52	00	05	258	360	-52
-0·095	-06	00	280	374	55	00	-06	280	374	-55	06	00	280	374	55	00	06	280	374	-55
-0·097	-06	00	302	387	58	00	-06	302	387	-58	06	00	302	387	58	00	06	302	387	-58
-0·100	-07	00	325	400	62	00	-07	325	400	-62	07	00	325	400	62	00	07	325	400	-62

TABLE 16.

Cosines and Sines of 360°r (3 decimals)

For Cosines τ	For Sines τ	0·000	0·001	0·002	0·003	0·004	0·005	0·006	0·007	0·008	0·009		For Sines τ	For Cosines τ
0·75	0·00	000	006	013	019	025	031	038	044	050	057	063	0·49	0·24
0·76	0·01	063	069	075	082	088	094	100	107	113	119	125	0·48	0·23
0·77	0·02	125	132	138	144	150	156	163	169	175	181	187	0·47	0·22
0·78	0·03	187	194	200	206	212	218	224	230	236	243	249	0·46	0·21
0·79	0·04	249	255	261	267	273	279	285	291	297	303	309	0·45	0·20
0·80	0·05	309	315	321	327	333	339	345	351	356	362	368	0·44	0·19
0·81	0·06	368	374	380	386	391	397	403	409	414	420	426	0·43	0·18
0·82	0·07	426	431	437	443	448	454	460	465	471	476	482	0·42	0·17
0·83	0·08	482	487	493	498	504	509	514	520	525	531	536	0·41	0·16
0·84	0·09	536	541	546	552	557	562	567	572	578	583	588	0·40	0·15
0·85	0·10	588	593	598	603	608	613	618	623	628	633	637	0·39	0·14
0·86	0·11	637	642	647	652	657	661	666	671	675	679	685	0·38	0·13
0·87	0·12	685	689	694	698	703	707	712	716	720	725	729	0·37	0·12
0·88	0·13	729	733	738	742	746	750	754	758	762	766	771	0·36	0·11
0·89	0·14	771	775	778	782	786	790	794	798	802	805	809	0·35	0·10
0·90	0·15	809	813	816	820	824	827	831	834	838	841	844	0·34	0·09
0·91	0·16	844	848	851	854	858	861	864	867	870	873	876	0·33	0·08
0·92	0·17	876	879	882	885	888	891	894	897	899	902	905	0·32	0·07
0·93	0·18	905	907	910	913	915	918	920	923	925	927	930	0·31	0·06
0·94	0·19	930	932	934	937	939	941	943	945	947	949	951	0·30	0·05
0·95	0·20	951	953	955	957	959	960	962	964	965	967	969	0·29	0·04
0·96	0·21	969	970	972	973	975	976	977	979	980	981	982	0·28	0·03
0·97	0·22	982	983	985	986	987	988	989	990	990	991	992	0·27	0·02
0·98	0·23	992	993	994	994	995	996	996	997	997	998	998	0·26	0·01
0·99	0·24	998	998	999	999	999	1000	1000	1000	1000	1000	1000	0·25	0·00
			0·009	0·008	0·007	0·006	0·005	0·004	0·003	0·002	0·001	0·000		

A change of 0·5, 1·5, 2·5, ...τ adds a negative sign
A change of 1·0, 2·0, 3·0, ...τ leaves sign unchanged.

TABLE 17.

L	d_o	d_1	d_2	d_3	d_4	d_5	d_a	d_b	d_c	d_d	d_e
-14	1	-2	2	-2	2	-2	0	-1	1	-1	1
-13	1	-2	2	-1	1	0	1	-1	2	-1	2
-12	1	-2	1	0	-2	2	1	-2	2	-2	1
-11	1	-1	0	1	-2	1	1	-2	1	0	-1
-10	1	-1	-1	2	-1	0	2	-2	0	1	-2
- 9	1	-1	-2	2	0	-2	2	-1	-1	2	-1
- 8	1	0	-2	1	2	-1	2	-1	-2	1	1
- 7	1	0	-2	-1	2	1	2	1	-2	0	2
- 6	1	1	-2	-1	1	2	2	1	-1	-2	0
- 5	1	1	-1	-2	-1	1	2	2	-1	-2	-2
- 4	1	1	0	-2	-2	-1	1	2	1	-1	-2
- 3	1	1	1	-1	-2	-2	1	2	2	1	0
- 2	1	2	1	1	0	-1	1	1	2	2	2
- 1	1	2	2	2	1	1	1	1	1	2	2
0	1	2	2	2	2	2	0	0	0	0	0
1	1	2	2	2	1	1	-1	-1	-1	-2	-2
2	1	2	1	1	0	-1	-1	-1	-2	-2	-2
3	1	1	1	-1	-2	-2	-1	-2	-2	-1	0
4	1	1	0	-2	-2	-1	-1	-2	-1	1	2
5	1	1	-1	-2	-1	1	-2	-2	1	2	2
6	1	1	-2	-1	1	2	-2	-1	1	2	0
7	1	0	-2	-1	2	1	-2	-1	2	0	-2
8	1	0	-2	1	2	-1	-2	1	2	-1	-1
9	1	-1	-2	2	0	-2	-2	1	1	-2	1
10	1	-1	-1	2	-1	0	-2	2	0	-1	2
11	1	-1	0	1	-2	1	-1	2	-1	0	1
12	1	-2	1	0	-2	2	-1	2	-2	2	-1
13	1	-2	2	-1	1	0	-1	1	-2	1	-2
14	1	-2	2	-2	2	-2	0	1	-1	1	-1

TABLE 18.

Combinations for $10^5R \cos r$ and $10^5R \sin r$.

		p = 1							p = 2						p = 3			p = 4		
		$2Q_1$	Q_1	O_1	M_1	K_1	J_1	OO_1	μ_2	N_2	M_2	L_2	S_2	$2SM_2$	MO_3	M_3	MK_3	MN_4	M_4	MS_4
For $10^5R \cos r$	Apo	-88	122	-141	261	3438	-267	226	...	...	...	...	3448	...	-57	66	-130	...	...	...
	1	139	-168	45	1662	192	1758	-370	132	-189	126	1735	-75	-214	...	...	...	-87	132	126
	2	68	-83	1519	198	-142	382	1641	23	-37	1600	-15	60	1602	-30	121	1649	114	23	1600
	3	-219	1685	291	-78	40	-76	76	-68	1699	167	-27	12	22	40	1683	54	-124	-68	167
	4	1712	186	-31	-25	-45	30	-78	1688	-108	104	-114	42	29	1675	18	47	7	1688	104
	5	...	...	...	...	...	...	...	-80	-79	-19	-117	1	-24	...	...	...	1704	-80	-19
	Bpa	17	-210	99	-1782	246	1662	-356	-64	-81	-52	-1747	245	-116	...	...	...	-89	-64	-52
	b	-138	193	-1745	-46	-76	306	1833	-7	-29	-1668	11	-2	1666	30	-121	-1649	-50	-7	-1668
	c	219	-1685	-291	78	-40	76	-76	68	-1699	-167	27	-12	-22	-40	-1683	-54	124	68	-167
	d	-1712	-186	31	25	45	-30	78	-1688	108	-104	114	-42	-29	-1675	-18	-47	-7	-1688	-104
	e	...	...	...	...	...	...	...	80	79	19	117	-1	24	...	...	...	-1704	80	19
For $10^5R \sin r$	Bpo	-88	122	-141	261	3438	-267	226	...	...	...	...	3448	...	-57	66	-130	...	...	...
	1	139	-168	45	1662	192	1758	-370	132	-189	126	1735	-75	-214	...	...	...	-87	132	126
	2	68	-83	1519	198	-142	382	1641	23	-37	1600	-15	60	1602	-30	121	1649	114	23	1600
	3	-219	1685	291	-78	40	-76	76	-68	1699	167	-27	12	22	40	1683	54	-124	-68	167
	4	1712	186	-31	-25	-45	30	-78	1688	-108	104	-114	42	29	1675	18	47	7	1688	104
	5	...	...	...	...	...	...	...	-80	-79	-19	-117	1	-24	...	...	...	1704	-80	-19
	Apa	-17	210	-99	1782	-246	-1662	356	64	81	52	1747	-245	116	...	...	...	89	64	52
	b	138	-193	1745	46	76	-306	-1833	7	29	1668	-11	2	-1666	-30	121	1649	50	7	1668
	c	-219	1685	291	-78	40	-76	76	-68	1699	167	-27	12	22	40	1683	54	-124	-68	167
	d	1712	186	-31	-25	-45	30	-78	1688	-108	104	-114	42	29	1675	18	47	7	1688	104
	e	...	...	...	...	...	...	...	-80	-79	-19	-117	1	-24	...	...	...	1704	-80	-19

EXAMPLE OF ANALYSIS. (PART II) VI

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
6	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3				
L	ϑ OBSERVED TIMES				S OBSERVED HEIGHTS				t REDUCED TIMES				z REDUCED HEIGHTS				Z = z − 12.5				τ + 0.250			
	0.038	0.224	0.490	0.818	13.9	11.4	17.05	7.05																
−14	0.110	0.273	0.517	0.855	14.15	12.9	17.2	5.6	0.102	0.260	0.503	0.824	14.1	12.5	17.1	6.8	1.6	0.0	4.6	−5.7	0.047	0.297		
−13	0.169	0.331	0.548	0.891	15.4	14.05	17.25	4.25	0.159	0.313	0.532	0.860	15.2	13.7	17.2	5.4	2.7	1.2	4.7	−7.1	0.091	0.341		
−12	0.215	0.389	0.581	0.925	16.6	14.7	17.15	3.25	0.206	0.368	0.562	0.895	16.4	14.5	17.2	4.2	3.9	2.0	4.7	−8.3	0.133	0.383		
−11	0.251	0.441	0.617	0.958	17.5	14.8	16.9	2.6	0.242	0.419	0.596	0.927	17.3	14.8	17.1	3.2	4.8	2.3	4.6	−9.3	0.171	0.421		
−10	0.283	0.488	0.656	0.990	18.1	14.45	16.5	2.45	0.274	0.466	0.631	0.959	17.9	14.5	16.8	2.6	5.4	2.0	4.3	−9.9	0.208	0.458		
−9	0.312	0.529	0.697	...	18.5	13.8	15.95	...	0.303	0.508	0.670	0.990	18.4	14.1	16.3	2.4	5.9	1.6	3.8	−10.1	0.243	0.493		
−8	0.339	0.567	0.740	0.020	18.55	13.0	15.3	2.75	0.330	0.546	0.709	0.019	18.5	13.4	15.8	2.7	6.0	0.9	3.3	−9.8	0.276	0.526		
−7	0.363	0.603	0.785	0.050	18.45	12.05	14.6	3.50	0.354	0.582	0.751	0.049	18.5	12.6	15.1	3.5	6.0	0.1	2.6	−9.0	0.309	0.559		
−6	0.386	0.639	0.832	0.078	18.2	11.05	13.95	4.70	0.377	0.617	0.795	0.076	18.3	11.7	14.5	4.6	5.8	−0.8	2.0	−7.9	0.341	0.591		
−5	0.407	0.675	0.881	0.105	17.85	10.1	13.35	6.2	0.399	0.652	0.840	0.102	18.0	10.7	13.8	6.1	5.5	−1.8	1.3	−6.4	0.373	0.623		
−4	0.427	0.713	0.937	0.132	17.45	9.25	12.95	7.85	0.419	0.687	0.887	0.129	17.6	9.8	13.3	7.6	5.1	−2.7	0.8	−4.9	0.406	0.656		
−3	0.445	0.751	...	0.159	17.05	8.5	...	9.55	0.437	0.724	0.941	0.155	17.2	9.0	12.9	9.3	4.7	−3.5	0.4	−3.2	0.439	0.689		
−2	0.464	0.789	0.002	0.186	16.7	7.75	12.95	11.25	0.455	0.760	0.002	0.181	16.9	8.3	12.9	10.9	4.4	−4.2	0.4	−1.6	0.474	0.724		
−1	0.483	0.825	0.078	0.221	16.4	7.05	13.45	12.75	0.474	0.796	0.072	0.214	16.5	7.6	13.4	12.4	4.0	−4.9	0.9	−0.1	0.514	0.764		
0	0.503	0.858	0.145	0.273	16.1	6.4	14.4	13.95	0.493	0.831	0.136	0.260	16.3	6.9	14.3	13.6	3.8	−5.6	1.8	1.1	0.555	0.805		
1	0.526	0.889	0.191	0.341	15.85	5.8	15.4	14.65	0.514	0.862	0.183	0.319	16.0	6.3	15.2	14.4	3.5	−6.2	2.7	1.9	0.594	0.844		
2	0.554	0.916	0.223	0.403	15.6	5.3	16.25	14.85	0.539	0.892	0.216	0.379	15.7	5.8	16.1	14.8	3.2	−6.7	3.6	2.3	0.632	0.882		
3	0.586	0.941	0.249	0.449	15.3	4.85	16.9	14.75	0.568	0.918	0.243	0.429	15.4	5.3	16.7	14.8	2.9	−7.2	4.2	2.3	0.664	0.914		
4	0.621	0.966	0.272	0.483	15.1	4.55	17.3	14.45	0.600	0.942	0.266	0.467	15.2	4.8	17.2	14.6	2.7	−7.7	4.7	2.1	0.694	0.944		
5	0.657	0.990	0.298	0.510	15.0	4.4	17.5	14.0	0.634	0.967	0.287	0.497	15.1	4.5	17.4	14.2	2.6	−8.0	4.9	1.7	0.721	0.971		
6	0.693	...	0.318	0.535	14.9	...	17.5	13.45	0.669	0.990	0.307	0.522	15.0	4.4	17.5	13.7	2.5	−8.1	5.0	1.2	0.747	0.997		
7	0.731	0.013	0.330	0.558	14.8	4.5	17.4	12.8	0.704	0.013	0.324	0.545	14.9	4.5	17.4	13.1	2.4	−8.0	4.9	0.6	0.772	0.022		
8	0.769	0.036	0.345	0.581	14.7	4.9	17.2	12.05	0.741	0.035	0.340	0.568	14.8	4.9	17.3	12.5	2.3	−7.6	4.8	0.0	0.796	0.046		
9	0.808	0.059	0.360	0.606	14.5	5.55	17.0	11.15	0.778	0.058	0.355	0.591	14.6	5.5	17.1	11.7	2.1	−7.0	4.6	−0.8	0.820	0.070		
10	0.851	0.083	0.378	0.634	14.2	6.55	16.9	10.2	0.816	0.081	0.368	0.617	14.4	6.5	16.9	10.8	1.9	−6.0	4.4	−1.7	0.846	0.096		
11	0.901	0.108	0.387	0.667	13.9	7.85	16.9	9.2	0.858	0.105	0.382	0.646	14.2	7.7	16.9	9.8	1.7	−4.8	4.4	−2.7	0.873	0.123		
12	0.963	0.134	0.404	0.705	13.6	9.4	16.95	8.2	0.907	0.131	0.397	0.679	13.9	9.2	16.9	8.9	1.4	−3.3	4.4	−3.6	0.904	0.154		
13	...	0.162	0.424	0.746	...	11.1	17.0	7.15	0.966	0.158	0.416	0.717	13.6	10.8	17.0	7.9	1.1	−1.7	4.5	−4.6	0.939	0.189		
14	0.041	0.199	0.448	0.793	13.8	12.65	17.0	6.05	0.038	0.192	0.438	0.759	13.8	12.4	17.0	6.9	1.3	−0.1	4.5	−5.6	−0.018	0.232		

VII

	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
S	0	1	2	3	0	1	2	3	$\lambda = -0.013$		$\mu = -0.0011$				0	1	2	3	0	1	2	3		
L	T RELATIVE TIMES 3 decs.				[α] = Z cos 2nT.				APPROXIMATE VALUES						CORRECTIONS TO [α]				CORRECTED VALUES					
									α_2	α_1	α_1'	α_4	β_{44}	β_2					α_2	α_4	α_1	α_1'	$\alpha_0+\alpha_4$	a_0
-14	0.055	-037	-044	027	1.2	0.0	3.9	-5.4	2.62	-1.40	2.70	-0.09	-0.01	..	-0.1	0.2	0.2	-0.1	2.62	-0.09	-1.55	2.85	-0.02	0.07
-13	0.068	-028	-059	019	1.8	1.1	2.5	-6.9	2.78	-0.85	4.00	-0.10	-0.01	..	0.1	0.2	0.4	-0.1	2.88	-0.10	-1.00	4.15	0.02	0.12
-12	0.073	-015	-071	012	2.4	2.0	3.0	-6.2	2.90	-0.30	5.10	-0.11	-0.01	..	0.4	0.1	0.5	0.0	3.10	-0.11	-0.35	5.15	0.05	0.16
-11	0.071	-002	-075	006	3.0	2.3	2.7	-9.3	3.18	0.15	5.30	-0.13	-0.01	..	0.6	0.0	0.5	0.0	3.46	-0.13	0.20	5.80	-0.05	0.08
-10	0.066	008	-077	001	3.6	2.0	2.4	-9.9	3.48	0.60	5.95	-0.16	-0.01	..	0.6	0.0	0.4	0.0	3.73	-0.16	0.70	5.95	-0.22	-0.06
-9	0.060	015	-073	-003	4.3	1.6	2.3	-10.1	3.78	1.00	5.85	-0.18	-0.02	0.01	0.6	0.1	0.2	0.0	3.96	-0.18	1.20	5.90	-0.25	-0.07
-8	0.054	020	-067	-007	4.7	0.9	2.2	-9.8	3.95	1.25	5.35	-0.20	-0.02	0.01	0.4	0.1	0.1	0.0	4.05	-0.20	1.40	5.40	-0.40	-0.20
-7	0.045	023	-058	-010	5.1	0.1	1.9	-8.9	3.95	1.60	4.50	-0.20	-0.02	0.01	0.3	0.2	0.0	0.0	3.97	-0.20	1.75	4.60	-0.32	-0.12
-6	0.036	026	-046	-015	5.2	-0.8	1.7	-7.8	3.88	1.75	3.50	-0.19	-0.02	0.01	0.2	0.2	-0.1	0.0	3.86	-0.19	1.90	3.60	-0.40	-0.21
-5	0.026	029	-033	-021	5.2	-1.7	1.2	-6.2	3.58	2.00	2.25	-0.17	-0.01	..	0.1	0.1	-0.1	0.0	3.56	-0.17	2.10	2.30	-0.40	-0.23
-4	0.013	031	-019	-027	5.0	-2.5	0.8	-4.6	3.22	2.10	1.05	-0.13	-0.01	..	0.0	0.1	0.0	0.0	3.20	-0.13	2.10	1.10	-0.30	-0.17
-3	-0.002	035	002	-034	4.7	-3.2	0.4	-2.9	2.80	2.15	-0.45	-0.10	-0.01	..	0.0	0.0	0.0	0.0	2.80	-0.10	2.15	-0.45	-0.25	-0.15
-2	-0.019	036	026	-043	4.3	-3.8	0.4	-1.4	2.48	1.95	-1.20	-0.08	-0.01	..	0.0	-0.1	-0.1	0.1	2.46	-0.08	2.00	-1.30	-0.15	-0.07
-1	-0.040	032	058	-050	3.5	-4.5	0.7	-0.1	2.20	1.45	-2.20	-0.06	-0.01	..	0.2	-0.1	-0.1	0.3	2.18	-0.06	1.55	-2.40	-0.02	0.04
0	-0.062	026	081	-045	2.7	-5.3	0.9	0.9	2.00	0.90	-3.10	-0.05	0.00	..	0.4	-0.1	0.1	0.3	2.08	-0.05	1.05	-3.30	-0.02	0.03
1	-0.080	018	089	-025	1.9	-6.0	1.2	1.8	1.82	0.35	-3.90	-0.04	0.00	..	0.6	-0.1	0.4	0.2	2.04	-0.04	0.45	-4.05	0.00	0.04
2	-0.093	010	084	-003	1.3	-6.6	1.8	2.3	1.85	-0.25	-4.45	-0.03	0.00	..	0.6	0.0	0.5	0.0	2.13	-0.03	-0.20	-4.45	-0.02	0.01
3	-0.096	004	079	015	1.0	-7.2	2.3	2.8	2.05	-0.65	-4.75	-0.05	0.00	..	0.6	0.0	0.7	0.0	2.37	-0.05	-0.70	-4.75	-0.08	-0.03
4	-0.094	-002	072	023	1.0	-7.7	2.9	2.0	2.40	-0.95	-4.85	-0.07	-0.01	..	0.5	0.0	0.7	0.1	2.68	-0.07	-1.05	-4.90	-0.12	-0.05
5	-0.087	-004	066	026	1.2	-8.0	3.3	1.6	2.72	-1.05	-4.80	-0.10	-0.01	..	0.3	0.0	0.6	0.2	2.90	-0.10	-1.20	-4.90	-0.20	-0.10
6	-0.078	-007	060	025	1.4	-8.1	3.6	1.1	3.00	-1.10	-4.60	-0.12	-0.01	..	0.1	0.0	0.5	0.2	3.10	-0.12	-1.30	-4.70	-0.30	-0.18
7	-0.068	-009	052	023	1.6	-8.0	3.9	0.6	3.22	-1.15	-4.30	-0.14	-0.01	..	0.1	0.0	0.4	0.1	3.32	-0.14	-1.30	-4.35	-0.32	-0.18
8	-0.055	-011	044	022	1.8	-7.5	4.1	0.0	3.35	-1.15	-3.75	-0.15	-0.01	..	0.0	0.0	0.3	0.1	3.40	-0.15	-1.30	-3.80	-0.30	-0.15
9	-0.042	-012	035	021	1.8	-6.9	4.2	-0.8	3.42	-1.20	-3.05	-0.15	-0.01	..	0.0	0.0	0.2	0.1	3.44	-0.15	-1.30	-3.10	-0.35	-0.20
10	-0.030	-015	022	021	1.8	-5.9	4.2	-1.6	3.38	-1.20	-2.15	-0.15	-0.01	..	0.0	0.0	0.1	0.1	3.38	-0.15	-1.25	-2.20	-0.32	-0.17
11	-0.015	-018	009	023	1.7	-4.7	4.4	-2.6	3.35	-1.35	-1.05	-0.15	-0.01	..	0.0	0.0	0.0	0.0	3.35	-0.15	-1.35	-1.05	-0.30	-0.15
12	0.003	-023	-007	025	1.4	-3.2	4.4	-3.4	3.10	-1.50	0.10	-0.12	-0.01	..	0.0	0.0	0.0	0.0	3.10	-0.12	-1.50	0.10	-0.20	-0.08
13	0.027	-031	-023	028	1.0	-4.6	4.3	-4.3	2.80	-1.65	1.35	-0.11	-0.01	..	0.0	0.1	0.1	-0.1	2.82	-0.11	-1.70	1.45	-0.12	-0.01
14	0.056	-040	-044	027	1.0	-0.1	3.8	-5.8	2.55	-1.90	2.60	-0.08	-0.01	..	-0.2	0.2	0.3	-0.1	2.55	-0.08	-2.15	2.75	-0.15	-0.07

MEAN: -0.072 020 026 021

MEAN: 2.96

MEAN: -0.07

VIII

	49	50	51	52	53	54	55	56	57	58	59	60
S	0	1	2	3	0	1	2	3	0	1	2	3
L	$\Sigma m^2 \alpha$				$[\beta] = (2 \sin nT) . \Sigma m^2 \alpha$				CORRECTIONS TO $[\beta]$			
-14	1.57	-2.27	2.65	-3.69	1.3	1.0	-1.4	-1.2	0.0	0.0	-0.1	0.0
-13	2.23	-2.24	2.73	-4.12	1.8	0.8	-2.0	-1.0	0.1	0.0	-0.1	0.0
-12	2.57	-2.25	2.75	-4.83	2.3	0.4	-2.0	-0.7	0.1	0.0	-0.2	0.0
-11	2.99	-2.53	2.89	-5.43	2.6	0.1	-2.6	-0.4	0.2	0.0	-0.2	0.0
-10	3.27	-2.88	2.91	-5.86	2.6	-0.3	-2.7	-0.1	0.2	0.0	-0.2	0.0
-9	3.54	-3.20	2.94	-6.16	2.6	-0.6	-2.6	0.2	0.1	0.0	-0.2	0.0
-8	3.60	-3.50	2.90	-6.20	2.4	-0.9	-2.4	0.5	0.1	0.0	-0.2	0.0
-7	3.61	-3.62	2.73	-5.92	2.0	-1.0	-1.9	0.7	0.1	0.0	-0.2	0.0
-6	3.58	-3.72	2.62	-5.52	1.6	-1.2	-1.5	1.0	0.0	0.0	-0.1	0.0
-5	3.40	-3.66	2.36	-4.82	1.1	-1.3	-1.0	1.3	0.0	0.0	0.0	0.0
-4	3.20	-3.44	2.16	-4.00	0.5	-1.3	-0.5	1.4	0.0	0.0	0.0	0.0
-3	2.94	-3.24	1.86	-3.16	-0.1	-1.4	0.0	1.3	0.0	0.0	0.0	0.0
-2	2.64	-3.10	1.64	-2.46	-0.6	-1.4	0.5	1.3	0.0	0.0	0.0	0.0
-1	2.33	-3.02	1.55	-1.82	-1.2	-1.2	1.1	1.1	0.0	0.0	0.0	0.0
0	2.14	-3.10	1.62	-1.46	-1.6	-1.0	1.6	0.8	-0.1	0.0	0.1	0.0
1	1.99	-3.21	1.77	-1.19	-1.9	-0.7	1.9	0.4	-0.1	0.0	0.1	0.0
2	1.96	-3.36	2.06	-1.14	-2.2	-0.4	2.1	0.0	-0.1	0.0	0.1	0.0
3	1.99	-3.76	2.35	-1.38	-2.3	-0.2	2.2	-0.3	-0.1	0.0	0.1	0.0
4	2.14	-4.18	2.66	-1.74	-2.4	0.1	2.3	-0.5	-0.2	0.0	0.1	0.0
5	2.20	-4.52	2.80	-2.08	-2.3	0.4	2.3	-0.7	-0.2	0.0	0.1	0.0
6	2.30	-4.76	2.94	-2.40	-2.2	0.4	2.2	-0.8	-0.2	0.0	0.1	0.0
7	2.44	-4.97	3.08	-2.79	-2.0	0.6	2.0	-0.8	-0.2	0.0	0.1	0.0
8	2.48	-4.95	3.12	-3.05	-1.7	0.7	1.7	-0.8	-0.1	0.0	0.1	0.0
9	2.52	-4.82	3.16	-3.26	-1.3	0.7	1.4	-0.9	-0.1	0.0	0.0	0.0
10	2.47	-4.53	3.09	-3.43	-0.9	0.9	0.9	-0.9	0.0	0.0	0.0	0.0
11	2.41	-4.21	3.09	-3.69	-0.5	0.9	0.3	-1.1	0.0	0.0	0.0	0.0
12	2.24	-3.56	3.00	-3.60	0.1	1.0	-0.3	-1.1	0.0	0.0	0.0	0.0
13	1.96	-2.90	2.80	-3.62	0.7	1.1	-0.8	-1.3	0.0	0.0	0.0	0.0
14	1.69	-2.18	2.77	-3.56	1.2	1.1	-1.5	-1.2	0.0	0.0	0.0	0.0

	61	62	63	64	65	66	67	68	69	70	71	72
L	β_2	β_4	β_1	β_1'	a_3	b_3	c_3	s_3	a_1	b_1	c_1	s_1
-14	0.00	-0.05	2.8	2.2	0.16	-0.01	0.63	0.26	-1.71	2.84	0.957	0.291
-13	0.00	-0.05	4.0	1.8	0.20	-0.04	-0.14	0.49	-1.20	4.11	0.841	0.541
-12	0.02	-0.06	5.0	1.1	0.19	-0.04	-0.80	0.59	-0.56	5.11	0.671	0.742
-11	0.08	-0.04	5.6	0.5	0.18	-0.05	-1.00	-0.08	0.02	5.75	0.476	0.879
-10	0.08	-0.06	5.7	-0.2	0.12	-0.06	-0.71	-0.70	0.58	5.89	0.261	0.965
-9	0.08	-0.06	5.5	-0.8	0.10	-0.10	-0.13	-0.99	1.10	5.80	0.044	0.999
-8	0.08	-0.06	5.1	-1.4	0.00	-0.08	0.47	-0.88	1.40	5.32	-0.163	0.987
-7	0.08	-0.04	4.2	-1.7	0.01	-0.10	0.90	-0.44	1.74	4.50	-0.362	0.932
-6	0.05	-0.02	3.2	-2.2	-0.08	-0.10	0.99	0.14	1.98	3.50	-0.541	0.841
-5	0.02	0.01	2.1	-2.6	-0.12	-0.05	0.73	0.68	2.22	2.25	-0.698	0.716
-4	-0.02	0.01	1.0	-2.7	-0.15	-0.02	0.20	0.98	2.25	1.08	-0.831	0.557
-3	0.00	-0.02	-0.1	-2.7	-0.14	0.01	-0.41	0.91	2.29	-0.14	-0.927	0.374
-2	0.00	-0.02	-1.1	-2.7	-0.18	0.05	-0.88	0.47	2.18	-1.25	-0.987	0.163
-1	0.00	-0.02	-2.3	-2.3	-0.19	0.02	-0.97	-0.26	1.74	-2.38	-0.996	-0.088
0	0.05	-0.02	-3.4	-1.8	-0.19	-0.02	-0.51	-0.86	1.24	-3.32	-0.941	-0.339
1	0.08	-0.04	-4.0	-1.1	-0.16	0.01	0.20	-0.98	0.61	-4.04	-0.831	-0.557
2	0.08	-0.06	-4.5	-0.4	-0.15	-0.01	0.79	-0.61	-0.05	-4.46	-0.675	-0.738
3	0.10	-0.08	-4.7	0.1	-0.15	0.01	1.00	-0.05	-0.55	-4.74	-0.514	-0.858
4	0.05	-0.07	-5.0	0.6	-0.11	-0.02	0.87	0.49	-0.94	-4.92	-0.345	-0.939
5	0.10	-0.07	-4.9	0.9	-0.08	0.00	0.52	0.85	-1.12	-4.90	-0.181	-0.983
6	0.08	-0.06	-4.7	1.2	-0.02	0.00	0.06	1.00	-1.28	-4.70	-0.019	-1.000
7	0.02	-0.04	-4.3	1.4	0.02	0.01	-0.40	0.92	-1.32	-4.34	0.138	-0.990
8	0.02	-0.01	-3.6	1.5	0.05	0.05	-0.76	0.65	-1.35	-3.75	0.285	-0.959
9	0.05	-0.02	-2.8	1.6	0.08	0.08	-0.97	0.25	-1.38	-3.02	0.426	-0.905
10	0.00	0.00	-1.8	1.8	0.14	0.10	-0.97	-0.24	-1.39	-2.10	0.567	-0.823
11	0.00	-0.05	-0.8	2.0	0.16	0.06	-0.73	-0.68	-1.51	-0.99	0.698	-0.716
12	-0.02	-0.04	0.4	2.1	0.15	0.08	-0.24	-0.97	-1.65	0.18	0.823	-0.567
13	0.02	-0.04	1.5	2.4	0.18	0.01	0.48	-0.91	-1.88	1.66	0.927	-0.374
14	-0.05	-0.05	2.7	2.3	0.04	-0.01	0.94	-0.88	-2.19	2.74	0.993	-0.113

IX

·	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88
L	a_2	b_2	c_2	s_2	a_4	b_4	c_4	s_4	A_1	B_1	A_2	B_2	A_3	B_3	A_4	B_4
-14	2.62	0.00	0.831	0.557	-0.09	-0.05	0.38	0.92	-2.46	2.22	2.18	1.06	0.10	0.04	0.01	-0.11
-13	2.88	0.00	0.414	0.910	-0.10	-0.05	-0.66	0.75	-2.23	4.81	1.19	2.62	0.01	0.20	0.10	-0.04
-12	3.10	0.02	-0.100	0.995	-0.11	-0.06	-0.98	-0.20	-4.75	3.03	-0.33	3.08	-0.12	0.14	0.10	0.08
-11	3.46	0.08	-0.546	0.838	-0.13	-0.04	-0.40	-0.92	-5.04	2.75	-1.96	4.86	-0.19	0.04	0.02	0.14
-10	3.73	0.08	-0.864	0.504	-0.16	-0.06	0.49	-0.87	-5.53	2.10	-3.26	1.81	-0.13	-0.04	-0.10	0.11
-9	3.96	0.08	-0.996	0.088	-0.18	-0.06	0.98	-0.18	-5.75	1.35	-3.45	0.27	-0.10	-0.10	-0.19	-0.03
-8	4.05	0.08	-0.947	-0.321	-0.20	-0.06	0.79	0.61	-5.48	0.51	-3.81	-1.38	-0.07	-0.04	-0.12	-0.17
-7	3.97	0.08	-0.738	-0.675	-0.20	-0.04	0.09	1.00	-4.82	0.00	-2.88	-2.74	-0.04	-0.10	0.02	-0.21
-6	3.86	0.05	-0.414	-0.910	-0.19	-0.02	-0.66	0.75	-4.01	-0.23	-1.55	-3.53	-0.07	-0.12	0.14	-0.13
-5	3.56	0.02	-0.025	-1.000	-0.17	0.01	-1.00	0.05	-3.16	0.02	-0.07	-3.56	-0.06	-0.12	0.17	-0.02
-4	3.20	-0.02	0.380	-0.925	-0.13	0.01	-0.71	-0.70	-2.47	0.36	1.20	-2.97	-0.01	-0.15	0.10	0.08
-3	2.80	0.00	0.720	-0.694	-0.10	-0.02	0.04	-1.00	-2.07	0.99	2.02	-1.94	0.08	-0.12	-0.03	0.10
-2	2.46	0.00	0.947	-0.321	-0.08	-0.02	0.79	-0.61	-1.95	1.59	2.33	-0.79	0.13	-0.13	-0.08	0.03
-1	2.18	0.00	0.985	0.175	-0.06	-0.02	0.94	0.34	-1.94	2.22	2.15	0.38	0.19	0.03	-0.04	-0.04
0	2.08	0.05	0.771	0.637	-0.05	-0.02	0.19	0.98	-2.29	2.70	1.57	1.36	0.08	0.17	0.01	-0.06
1	2.04	0.08	0.380	0.925	-0.04	-0.04	-0.71	0.70	-2.76	3.02	0.70	1.92	-0.02	0.16	0.06	0.00
2	2.13	0.08	-0.088	0.996	-0.03	-0.06	-0.98	-0.18	-3.26	3.05	-0.27	2.11	-0.12	0.08	0.02	0.06
3	2.37	0.10	-0.471	0.882	-0.05	-0.07	-0.56	-0.83	-3.78	2.91	-1.20	2.04	-0.15	0.02	-0.03	0.08
4	2.68	0.05	-0.762	0.647	-0.07	-0.07	0.16	-0.99	-4.30	2.58	-2.07	1.70	-0.09	-0.07	-0.08	0.06
5	2.90	0.10	-0.934	0.356	-0.10	-0.07	0.85	-0.67	-4.61	1.99	-2.74	0.94	-0.04	-0.07	-0.13	0.01
6	3.10	0.08	-0.999	0.038	-0.12	-0.06	1.00	-0.08	-4.68	1.37	-3.10	0.04	0.00	-0.02	-0.13	-0.05
7	3.32	0.02	-0.962	-0.273	-0.14	-0.08	0.85	0.52	-4.48	0.71	-3.19	-0.93	-0.02	0.01	-0.10	-0.11
8	3.40	0.02	-0.838	-0.546	-0.15	-0.01	0.40	0.92	-3.98	0.83	-2.84	-1.67	-0.07	-0.01	-0.05	-0.14
9	3.44	0.05	-0.637	-0.771	-0.15	-0.02	-0.19	0.98	-3.32	-0.04	-2.15	-2.68	-0.10	-0.06	0.05	-0.14
10	3.38	0.00	-0.356	-0.934	-0.15	0.00	-0.85	0.67	-2.52	-0.05	-1.20	-3.16	-0.12	-0.13	0.11	-0.11
11	3.35	0.00	-0.025	-1.000	-0.15	-0.05	-1.00	0.05	-1.76	0.39	-0.08	-3.35	-0.08	-0.15	0.15	0.04
12	3.10	-0.02	0.356	-0.934	-0.12	-0.04	-0.75	-0.67	-1.26	1.08	1.08	-2.90	0.04	-0.17	0.06	0.11
13	2.82	0.02	0.720	-0.694	-0.11	-0.04	0.04	-1.00	-1.20	2.06	2.04	-1.44	0.08	-0.16	-0.04	0.11
14	2.55	-0.05	0.975	-0.224	-0.08	-0.05	0.90	-0.44	-1.57	1.97	2.48	-0.62	0.04	-0.02	-0.09	-0.01

L	e_0	e_2	e_3	f_1	$\frac{e_1^2-f_1^2}{16e_2}$	a_0+a_4	e_2^2
-8	12.60	4.55	1.35	5.35	-0.37	12.23	20.7
-1	12.48	2.48	1.55	-2.40	-0.08	12.40	6.2
6	12.65	3.60	-1.25	-4.65	-0.35	12.30	13.0
13	12.32	2.98	-1.70	1.45	0.02	12.34	8.9

$\lambda = -0.41/32.2 = -0.013$

$a_0 = 12.32 - \lambda\,12.20 = 12.48$

	p=1	2	3	4
A_{po}	-98.13	-17.71	-0.89	-0.12
1	-1.22	0.77	0.56	-0.34
2	40.89	82.46	2.07	0.80
3	-3.68	-14.20	-0.53	-0.20
4	-3.10	-1.61	1.70	-1.15
5	-1.41	-6.03	0.47	1.28
B_{pa}	-2.26	0.72	-0.18	0.07
b	-31.38	-80.52	-2.62	-0.64
c	0.27	17.16	1.15	-0.14
d	4.23	0.71	-1.50	0.80
e	-2.63	4.41	0.09	-1.23
B_{po}	44.69	-11.77	-0.89	-0.36
1	0.31	8.53	0.35	-0.18
2	86.79	48.57	1.90	8.43
3	0.22	9.86	0.92	-0.35
4	-2.31	-3.02	1.47	-3.14
5	-2.42	-2.24	0.60	0.01
A_{pa}	-12.85	5.50	0.15	0.31
b	37.35	38.85	1.44	1.88
c	-1.03	8.56	0.79	-0.22
d	-1.07	-3.05	1.53	-2.83
e	-0.29	0.26	0.65	0.08

VANCOUVER CENTRAL DAY : JANUARY 17, 1951 **X**

	$2Q_1$	Q_1	O_1	M_1	K_1	J_1	OO_1	MO_3	M_3	MK_3
$R\cos r$	0.039	−0.288	1.295	−0.136	−3.414	0.265	−0.111	0.052	−0.023	0.079
$R\sin r$	−0.016	−0.096	1.159	−0.016	1.546	0.126	−0.025	0.050	0.033	0.059
R	0.042	0.304	1.773	0.137	3.748	0.293	0.113	0.072	0.040	0.099
f	1.182	1.162	1.162		1.112	1.164	1.707	1.139	0.946	1.071
$1+W$					1.207					1.207
V_0		218.0	162.3		9.8	314.1			78.2	
V_0'		0.0	0.0		0.0	0.0			0.0	
V_0''		105.1	314.1		15.8	224.8			134.8	
V_0'''										
V	169.8	323.1	116.4		25.6	178.9	114.8	96.1	213.0	167.6
u	−1.2	−1.2	−1.2		1.1	1.5	4.5	−0.9	[illegible]	1.4
w					−11.1					−11.1
r	337.7	198.4	41.8	186.7	155.6	25.4	192.7	43.9	124.9	36.8
H	0.04	0.26	1.50		2.80	0.25	0.07	0.06	0.04	0.08
g	146°	160°	157°		171°	207°	312°	139°	338°	195°

	μ_2	N_2	M_2	L_2	S_2	$2SM_2$		MN_4	M_4	MS_4
$R\cos r$	...	−0.531	2.610	0.003	−0.563	−0.028		0.048	−0.035	0.020
$R\sin r$	...	0.315	1.345	0.156	−0.397	0.021		0.006	−0.100	0.058
R	0.022	0.617	2.936	0.156	0.689	0.035		0.048	0.106	0.061
f	0.964	0.964	0.964	1.282	1.000	0.964		0.928	0.928	0.964
$1+W$		1.022			0.903	0.815		1.022		0.903
V_0	344.3	227.9	172.1	296.4	0.0					
V_0'	0.0	0.0	0.0	0.0	0.0					
V_0''	299.8	120.9	329.9	178.9	0.0					
V_0'''										
V	284.1	348.8	142.0	116.3	0.0	218.0		130.8	284.1	142.0
u	0.8	0.8	0.3	−9.2		−0.3		0.6	0.6	0.3
w		10.2			−18.9	−37.8		10.2		−18.9
r	313.2	149.2	[illegible]	88.9	215.2	143.1		9.1	250.7	70.9
H	0.02	0.63	3.05	0.12	0.76	0.04		0.05	0.11	0.08
g	235°	145°	170°	195°	196°	323°		149°	175°	194°

$\underline{S_2}$ AND $\underline{MS_4}$:

K_1 : $V+u$ = 25.6

TAB. 7 ; S_2 : w/f = −14.4

W/f = −0.074

K_2 f = 1.314

w = −18.9°

W = −0.097

$1+W$ = 0.908

$\underline{2SM_2}$:

w = $2w$ for S_2

$1+W$ = $(1+W)^2$ for S_2

$\underline{K_1}$ AND $\underline{MK_3}$:

K_1 : $2V+u$ = 52.3

TAB. 7 : K_1 : wf = −12.3

Wf = 0.230

K_1 f = 1.112

w = −11.1°

W = 0.207

$1+W$ = 1.207

$\underline{N_2}$ AND $\underline{MN_4}$:

M_2 : $3V$ = 426.0

N_2 : $2V$ = 697.6

DIFF. $(M_2 - N_2)$ = 88.4

TAB. 7 . N_2 w = 10.2°

$1+W$ = 1.022

$\underline{2N_2}$: R = 0.077 r = 271.1°

$\underline{\mu_2}$

	$R\cos r$	$R\sin r$
ANALYSIS	0.016	−0.093
$2N_2$	0.001	−0.077
μ_2	0.015	−0.016

: μ_2 { R = 0.022, r = 313.2° }

Theory and Explanation (Part II)

1. — *Reduced times and heights of high and low water.*

A sequence of times of high water has been defined in Part I. If this sequence is plotted so that the times of high water are given in terms of days and the time of high water in each day then the smooth curve expresses the time of high water as a simple function of the time. Now any high water is a simple function of the astronomical conditions prevailing at that time, and it is legitimate to conclude that if from the curve the times of high water are read off at zero hour of each day then those times are functions of the astronomical conditions at zero hour each day. The new sequence of times will therefore be expressed at intervals of 24 mean solar hours in the longitudes, declinations, and parallaxes of sun and moon, and are thus suited for direct analysis.

The graphical interpretation mentioned above is somewhat tedious and it is simpler to interpolate numerically except for those occasions when the times are changing very rapidly. It is normally sufficient to interpolate linearly.

Let φ denote the time of high water on day x in a sequence. It will be supposed that φ is measured in units of a day. The values of φ in the sequence are, of course measured each day from zero hour of the day, and the increment in φ from one day to the next is usually small; the increment from day (x—1) to day x will be denoted by δ. The true increment, of course, is $(1 + \delta)$. Then linear interpolation to zero hour of the day gives

$$t = \varphi - \frac{\varphi}{1+\delta} . \delta \tag{1}$$

and this reduces to

$$t = \frac{\varphi}{1+\delta} \tag{2}$$

When the time passes through midnight there are two occasions of zero hour, and one interpolation is effected by formula (2); and the other from

$$t = \frac{1+\varphi}{1+\delta} \tag{3}$$

The former applies to day x and the latter to the previous day.

If d is the corresponding increment in height, then the interpolated value of height is obtained from the sequence of high waters denoted by ζ, with the formula

$$z = \zeta - \frac{\varphi}{1+\delta} . d$$

and this is obviously equal, by (2) to

$$z = \zeta - td \tag{4}$$

where t is the reduced time.

The formulae are very simple to apply. This simplicity is largely due to the choice of the day as unit of time, for if the times φ had been expressed in hours and decimals thereof it would have been necessary to replace (2) and (4) by

$$t = \frac{24}{24 + \delta} . \varphi \text{ and } z = \zeta - \frac{t}{24} . d \tag{5}$$

where φ and t are in hours. Obviously this would have rendered the computations much more troublesome.

The times are very readily expressed in units of a day by means of a table and this process is nearly as simple as expressing the times in hours and decimals.

In most cases, when δ is not large, the interpolations can be effected mentally by the use of the approximation

$$t = \varphi - \varphi . \delta$$

in place of (2).

2. — *Interpretation of the reduced times and heights.*

The height of tide can be expressed by a number of constituents, each of which can be given in the form

$$A \cos \sigma t + B \sin \sigma t$$

where A and B are constant for a constituent, σ is the speed of the constituent, and t is the time. This is the usual method of expressing the constituent but an alternative form is

$$C \cos 15^{o} pt + D \sin 15^{o} pt \qquad (6)$$

where p is the species number, and now C and D are not constant for the constituent but vary slowly with time. If we take a sequence of values of the constituent for the same hour of the day, we can interpolate in C and D for zero hour. If the interpolation is effected for each hourly sequence then we obtain the same values C_0 and D_0 from each, so that we get an interpolated expression

$$C_0 \cos 15^{o} pt + D_0 \sin 15^{o} pt \qquad (7)$$

But as C_0 and D_0 are constant for the day then the above expression is an apparent solar tide of species given by p. That is, we have replaced the constituent, whatever its speed, by an equivalent solar constituent.

The same results follow for a large number of constituents taken together so that the interpolations of the sequences of tide at a given hour of the day give for each day 24 reduced values of height which can be analyzed as though they are derived from pure solar constituents.

The extension of these ideas to the interpolated values of times and heights of high and low water is obvious, and it appears that the interpolated values may be attributed to the equivalent solar constituents on the day in question, with all astronomical conditions as at zero hour of the day. The processes of analysis are thus enormously simplified.

It is recognized that these ideas are somewhat novel in tidal theory and analysis, but they can be readily tested as has been done by the author, who has found that if the interpolations are accurately effected, then they are in exact conformity with calculations, whether on hourly heights or on high and low waters. Small errors can, and do, occur when linear interpolation is used ; for instance if times are passing through a maximum point then the interpolated value will be less than the correct value. As, however, the correction for curvature would be much less than the average errors of observation or even of prediction it is judged to be unnecessary to complicate the method by providing for more exact interpolations.

3. — *The analysis of mixed solar tides.*

For this section the following notation will be used :

Z	:	the height of high or low water above or below mean sea level for the day.	(8)
Z(s)	:	a sequence value as defined in Part I.	
T	:	the time of high or low water, relatively to the corresponding time of the semidiurnal tide for the day (or approximately so).	
n	:	a speed of 360° per day.	
m	:	half the species number.	
a, b	:	constituent numbers.	

With the above notation we have the four solar constituents expressed by

$$Z = \Sigma\, a \cos 2mnT + \Sigma\, b \sin 2mnT \qquad (9)$$

and the condition for high or low water is

$$0 = \Sigma ma \sin 2mnT - \Sigma mb \cos 2mnT \qquad (10)$$

Multiply (9) by cos 2nT and (10) by sin 2nT and add together, to give

$$Z \cos 2nT = \Sigma a\,(\cos 2mnT \cos 2nT + m \sin 2mnT \sin 2nT) +$$
$$\Sigma b\,(\sin 2mnT \cos 2nT - m \cos 2mnT \sin 2nT) \qquad (11)$$

It is convenient to write

$$Z \cos 2nT = [\alpha]$$

The multipliers were chosen so as to eliminate any contributions from the semidiurnal tide to

$$\Sigma a - [\alpha] \qquad (12)$$

and if we ignore the effects of the very small third-diurnal tide we can consider (12) to be a linear function of a_1, b_1, a_4 and b_4, with small coefficients. It is sufficient to use approximate values of a and b in the expansion of (12) so as to give corrections to $[\alpha]$ to yield Σa. The following table shows specimen values of the corrections for the sequence s = 0.

CORRECTIONS TO $[\alpha]$

T	Coefficients of a_1	b_1	a_4	b_4	
0.02	0.02	0.00	—0.09	—0.03	
0.04	0.09	0.02	—0.28	—0.22	(13)
0.06	0.20	0.05	—0.41	—0.64	
0.08	0.33	0.11	—0.30	—1.20	

It is rare for T to exceed 0.08, as then the diurnal tide will be dominant, and the methods of Part III will have to be applied. This present part is dealing with tides in which there are two high waters and two low waters per day. When T is large it is due, not so much to the diurnal tide becoming very large, but to the smallness of the semidiurnal tide, in which case the quarter-diurnal tides become very small indeed. Hence there is no need to be troubled by the large coefficients of a_4 and b_4 when T is large.

Returning to equation (9), this can be re-written as

$$\Sigma mb \cos nT - \Sigma m^2 a \sin 2nT =$$
$$\Sigma mb\,(\cos nT - \cos 2mnT) - \Sigma ma\,(m \sin 2nT - \sin 2mnT) \qquad (14)$$

and on dividing by cos nT we obtain

$$\Sigma mb - 2 \sin nT.\,\Sigma m^2 a =$$
$$\Sigma mb\,(1 - \cos 2mnT \sec nT) - \Sigma ma\,(2m \sin nT - \sin 2mnT \sec nT) \quad (15)$$

It is convenient to write

$$2 \sin nT.\Sigma m^2 a = [\beta]$$

The coefficients on the right of (15) are very small as a rule, and the following table shows some specimen values, for sequence s = 0.

CORRECTIONS TO [β]

Coefficients of

T	a_1	b_1	a_4	b_4	b_2	
0.02	0.00	0.00	—0.03	0.23	0.02	
0.04	0.00	0.00	—0.25	0.89	0.10	
0.06	0.01	0.00	—0.80	1.86	0.22	(16)
0.08	0.03	0.00	—1.79	2.97	0.39	

The formula (15) has practically eliminated the diurnal tide, and if there is no quarter-diurnal tide then b_2 is zero because the values of T have been taken as relative to the approximate times of high or low water of the semi-diurnal tide. In practice we can only take the mean of the four times each day and it needs to be corrected slightly on account of the quarterdiurnal tide. It is simpler to utilize the small value of b_2 in the corrections rather than to revise the calculations of T. The only important corrections are those derived from the quarterdiurnal tide, but it may be remarked that T can only be large for one tide in the day and its effects are spread over all the four constituents by the processes of analysis so that no serious errors can occur. If, however, there is any uncertainty as to the validity of the operation on any one day it can be omitted and the values for the day filled in by interpolation.

It may be noted that the above formulae have points of comparison with the formulae used in Part I, but they are more accurate since the new formulae are finite series and the older ones were parts of infinite series. The one can be used through a much wider range of T than the other, for an infinite series can only be conveniently used when the variable is small.

4. — *The correction terms.*

Instruction XI gives the formula for obtaining approximate values of a_2, a_1, b_1, as α_2, α_1, α'_1 from [α] on the assumption that the correction teims may be ignored for the approximations. The value of α_4 could also, at first sight, be obtained by similar methods, but there are very good reasons for preferring to obtain this quantity from the formula

$$\alpha_4 = \lambda\alpha_2^2 \qquad (17)$$

The method used for the determination of λ is similar to that used in Part I, and there are full instructions for the calculation in Instruction V. The value of λ so obtained may be used in (17) though α_2 differs somewhat from e_2.

The approximate values of β_4 and β_2 are obtained as in Instruction XII. The daily values of β_4 could be derived from the daily values of T(s) in much the same way as in Instruction XII, but they would require very complicated correction for the effects of the diurnal tides and it is better to work from the mean values of T(s) for the month and to assume that β_4 varies with the semi-diurnal tide according to the formula

$$\beta_4 = \mu\alpha_2^2 \qquad (18)$$

This approximation is theoretically justified on the same basis as (17). The value of β_2 is always small, and would be zero by the choice of daily time-origin if the quarterdiurnal tide were zero.

It must be recognized that these values of a and b are only approximations, and it may be discovered later that they are not very good ones, but it is always possible to revise the calculations by using better approximations.

5. — *Tables of corrections to* [α] and [β].

The corrections to [α] and [β] are obtained by the use of Tables 12 to 15. It will be noted that these are given for the four sequences. The formulae for the corrections are given on the assumption that the values of a and b refer to sequence $s = 0$, but when we wish to compute corrections to sequence $s = 1$ we have to replace a_1 and b_1 by b_1 and $- a_1$ with similar changes for the other sequences.

It is convenient for computation to give these tables fully so that the computer has not to be troubled by the changes due to the transference of time origin from one sequence to another.

Separate tables are given for positive and negative values of T, and T is given at intervals of 0.003 or 0.002 (the reason for this being that the original calculations on which these tables were based were made for values of nT at intervals of one degree. It is not necessary to interpolate in the tables, but for any given value of T the nearest tabular entry should be taken.

6. — *The completion of the analysis.*

There are no special comments to make on the later processes of analysis, as these are similar to those used in Part I. The first procedure is to transfer the temporary daily time-origin, which has been taken as approximately the time of high water of the semidiurnal tide, to zero hour of the day. The new constituent-numbers A and B are functions of harmonic terms which are taken at intervals of a mean solar day, whereas the corresponding quantities in Part I were taken at intervals of a lunar day. The combinations of A_{p_0}, A_{p_1} ...... to give R cos r and R sin r require an appropriate set of tables, and there is no advantage in calculating A'_{p_0},......... as in Part I.

As usual, the value of mean sea level for the month can be adjusted to allow for any known annual and semiannual variations, as experienced regionally.

THE ANALYSIS OF HIGH AND LOW WATERS.

(Part III : Diurnal Tides Predominant).

by D[r] A.T. Doodson, F.R.S.

INTRODUCTION.

When the diurnal tide is very large relatively to the semidiurnal tide there are occasions when the diurnal tide is so predominant that there are only one high water and one low water per day. When this occurs it is not possible to use the methods of Part II, for the time of semi-diurnal high water cannot be determined on such occasions. For this reason a special method is here described. It can be adapted for all the problems hitherto considered, but in this Part it is applied only to the analysis of Higher High Waters and Lower Low Waters (HHW and LLW).

If there are only two heights and two times available in the day it is impossible to obtain from them four quantities necessary to specify the diurnal and semidiurnal tides and also to obtain a value of mean sea level, and if the tide is characteristically diurnal every day then a precise analysis is not possible. If there is any evidence from other sources then such may be used to specify mean sea level ; for instance if hourly heights for a day or two are available then an approximate value of mean sea level may be obtained, or if on any occasion the semidiurnal tides become evident then the four tides of the day be used. If there is no such evidence then a value of mean sea level may be assumed, and harmonic constants may be deduced, but a different assumption will produce different results. The associated constants may be used to give predictions as they will reproduce the data, since all the data have been fully used, and possibly it might be able to say that one set of constants is more reasonable than another, but it is not easy to give instructions for doing this. We shall therefore assume that there are either a few days in the month when there are two high and two low waters each day, or else hourly heights for a day or two are available.

Obviously it is entirely impossible to determine the third and quarter diurnal tides, so that the method of analysis is simplified on that account.

Instructions for Analysis (Part III).

(The tables and sheets for this part are numbered consecutively with those of Parts I and II).

I. (Sheet XI). The instructions for entering results in columns 1 to 16 are the same as in Part II, Instructions I to IV. There may be some difficulty in determining the sequences as with very large diurnal tides all the sequences are discontinuous, but no real troubles arise so long as the obvious sequences are in the columns pertaining to high or low water as the case may be. It is supposed that for a number of days in the month there are two high waters and two low waters in the day. If the tides are such as to give only one tide per day for the whole of the month the analysis cannot be made.

II. After obtaining the reduced times and heights choose on each day one high water and one low water, normally the higher high water and the lower low water, and for any other data in the day enclose the values in brackets to show that they will not be used except for the determination of mean sea level.

III. On the days for which there are two high and two low waters follow the instructions of Part II, Instruction V, for about five or six of these days and compute e_0, e_2, e_1, f_1, and the corrections to be added to e_0 to give a_0. We do not compute the quarterdiurnal tide, and we only require the mean value of a_0. There is provision on Sheet XII for these calculations.

IV. Subtract the value of mean sea level from the values of the reduced heights and enter the results Z in columns 17 and 18, for HHW and LLW. When there is a change of sequence in HHW or LLW indicate this by a separating line across the column. Similar separating lines should be used in columns 19 - 32.

V. In columns 19 and 20 enter the reduced times corresponding to the values of Z for HHW and LLW. It is essential to take care that the proper sequences are used. Also, we desire to make the HHW time greater than the LLW time and to do this we add 1.000 to the HHW time wherever necessary.

VI. In column 21 compute γ, equal to half the interval of time from HHW to LLW.

VII. In column 22 compute τ by adding γ to the time of LLW. Check these last operations by subtracting γ from the HHW time, which should give τ.

VIII. In column 23 enter the value of X, the sum of the values of Z for HHW and LLW.

IX. In column 24 enter the value of Y, by subtracting the value of Z for LLW from the value of Z for HHW.

X. (Sheet XII). In column 25 enter values of X_1, by multiplying the values of X in column 23 by factors given in Table 19 according to the values of γ in column 21.

XI. Similarly compute Y_1, X_2, and Y_2, using Tables 20 to 22, and enter the results in columns 26, 29 and 30.

XII. Following the instructions given in Part II, XXIII and XXIV, compute c_1, s_1, c_2, and s_2, from the value of τ and enter the results to three decimals in columns 27, 28, 31, and 32. The values may be checked as indicated in Part II, XXVI.

XIII. In columns 33 to 36 enter the values of A and B for the two species, from the formulae

$$A = Xc - Ys \quad , \quad B = Yc + Xs$$

and check the results by

$$A + B = (X + Y)c + (X - Y)s.$$

XIV. Examine the values of A and B for smoothness. Owing to the discontinuous sequences no adequate check by smoothness can be applied until this stage is reached. Any values of doubtful accuracy may be checked by revising the calculations in the earlier stages.

XV. The remaining processes are identical with those of Part II, except that only two species of tides are involved. Sheet XIII is similar to Sheet X with the columns for the third-diurnal and quarter-diurnal tides omitted.

Remarks on the example of Analysis (Part III).

The data for this example are the same as in Part II but in order to simulate the tidal conditions when the tide becomes diurnal in character a number of observations given in Part II have been deleted. It was judged that this artificial example was more useful than any other because it is possible to compare the results of analysis by the two methods. The example has not been completed, as Sheet XIII is not given because of its similarity to Sheet X. The following table gives the results of analysis as far as R cos r and R sin r and it also gives the errors of these quantities. (In comparing with the results for Part II note that the values of R cos r and R sin r given there under the remarks on the example are the exact values ; the values from the analysis are found on Sheet X).

	R cos *r*	*R* sin *r*	*Errors*			*R* cos *r*	*R* sin *r*	*Errors*	
	–	–	–			–	–	–	
M_2	2.49	1.34	0.06	0.01	K_1	–3.52	1.38	0.11	0.15
S_2	–0.74	–0.43	0.18	0.02	O_1	1.34	1.35	0.04	0.15
N_2	–0.49	0.33	0.04	0.00	Q_1	–0.32	–0.15	0.04	0.06
L_2	0.04	0.18	0.00	0.03	J_1	0.28	0.16	0.13	0.10
μ_2	0.02	0.03	0.03	0.01					
2 SM_2	0.02	0.02	0.00	0.02					

The mean error for the semidiurnal constituents, in any analytical quantity, is 0.03, and for the diurnal constituents it is 0.09 ; that is, the errors are about double what they are by the method of Part II, but we must take into consideration the halving of the data and also the fact that the constituents obtained must be perturbed by the constituents of the species which have had to be ignored. The results may be taken as being very satisfactory.

The value of mean sea level was accurately obtained, and it may be noted that the constituent J_1 is not outstanding as regards errors of determination. It may be taken that the errors in Parts II and III are almost wholly due to casual error in the observed quantities, and improvement in the values of the harmonic constants derived by the analyses can only be obtained by using more observations. In the introduction to Part I it was emphasized that it must not be expected that the results of analysis of high and low waters will be as accurate as from the analysis of hourly heights, partly from the scantier data in a month of observation and partly because of the impossibility of separation of all species.

TABLE 19 : X_1/X
(3 decimals)

γ	000	001	002	003	004	005	006	007	008	009		γ
0·12						707	707	707	707	707	707	0·37
0·13	707	707	706	706	706	706	705	705	705	704	703	0·36
0·14	703	703	702	701	701	700	699	698	697	696	695	0·35
0·15	695	694	693	692	690	689	687	686	684	682	681	0·34
0·16	681	679	677	675	673	671	668	666	663	661	658	0·33
0·17	658	655	652	649	646	643	640	636	633	629	625	0·32
0·18	625	621	617	613	608	604	599	594	589	584	579	0·31
0·19	579	574	569	563	557	551	545	538	532	526	519	0·30
0·20	519	512	505	498	491	483	475	467	459	450	442	0·29
0·21	442	434	425	416	408	399	389	380	370	360	350	0·28
0·22	350	340	330	319	309	298	288	277	266	255	243	0·27
0·23	243	232	220	208	196	185	173	161	149	137	125	0·26
0·24	125	112	100	087	075	062	050	037	025	012	000	0·25
		009	008	007	006	005	004	003	002	001	000	

(Apply negative sign to values for $\gamma > 0{\cdot}250$)

TABLE 20 : Y_1/Y
(3 decimals)

γ	000	001	002	003	004	005	006	007	008	009		γ
0·12						000	017	034	050	065	080	0·37
0·13	080	095	110	123	136	149	161	172	183	194	205	0·36
0·14	205	215	224	234	243	252	261	269	277	284	292	0·35
0·15	292	299	305	312	319	325	331	337	343	348	354	0·34
0·16	354	359	364	369	373	378	382	386	390	394	398	0·33
0·17	398	402	405	409	412	415	419	422	425	427	430	0·32
0·18	430	433	436	438	440	443	445	447	449	451	453	0·31
0·19	453	455	457	459	461	463	464	466	467	469	470	0·30
0·20	470	472	473	474	475	477	478	479	480	481	482	0·29
0·21	482	483	484	485	486	487	487	488	489	490	491	0·28
0·22	491	491	492	492	493	494	494	495	495	495	496	0·27
0·23	496	496	497	497	497	498	498	498	499	499	499	0·26
0·24	499	499	499	499	500	500	500	500	500	500	500	0·25
		009	008	007	006	005	004	003	002	001	000	

TABLE 21. X_2/X

(3 decimals)

r	000	001	002	003	004	005	006	007	008	009		r
0·12						-250	-252	-253	-255	-256	-258	0·37
0·13	-258	-260	-262	-263	-265	-267	-268	-270	-272	-274	-276	0·36
0·14	-276	-278	-280	-282	-284	-286	-288	-290	-292	-294	-296	0·35
0·15	-296	-298	-300	-302	-304	-307	-309	-311	-313	-315	-317	0·34
0·16	-317	-320	-322	-324	-327	-329	-332	-334	-337	-339	-341	0·33
0·17	-341	-344	-346	-349	-352	-354	-357	-359	-362	-364	-367	0·32
0·18	-367	-369	-372	-375	-377	-380	-383	-385	-388	-391	-393	0·31
0·19	-393	-396	-399	-401	-404	-407	-409	-412	-415	-417	-420	0·30
0·20	-420	-422	-425	-427	-430	-432	-435	-438	-440	-442	-445	0·29
0·21	-445	-447	-450	-452	-454	-457	-459	-461	-463	-465	-467	0·28
0·22	-467	-469	-471	-473	-475	-477	-478	-480	-482	-483	-485	0·27
0·23	-485	-486	-487	-489	-490	-491	-492	-493	-494	-495	-496	0·26
0·24	-496	-497	-497	-498	-498	-499	-499	-500	-500	-500	-500	0·25
		009	008	007	006	005	004	003	002	001	000	

TABLE 22. Y_2/Y

(3 decimals)

r	000	001	002	003	004	005	006	007	008	009		r
0·12						500	488	476	464	453	442	0·37
0·13	442	431	421	411	401	392	383	374	365	357	349	0·36
0·14	349	341	333	325	318	311	303	297	290	284	278	0·35
0·15	278	271	265	260	254	248	243	238	233	227	223	0·34
0·16	223	218	213	208	204	200	195	191	187	183	179	0·33
0·17	179	175	171	168	164	160	157	154	150	147	144	0·32
0·18	144	141	137	134	131	128	125	123	120	117	115	0·31
0·19	115	112	109	107	104	102	099	097	094	092	090	0·30
0·20	090	088	085	083	081	079	077	074	072	070	068	0·29
0·21	068	066	064	062	061	059	057	055	053	051	049	0·28
0·22	049	048	046	044	042	041	039	037	036	034	032	0·27
0·23	032	031	029	027	026	024	022	021	019	017	016	0·26
0·24	016	014	013	011	010	008	006	005	003	002	000	0·25
		009	008	007	006	005	004	003	002	001	000	

(Apply negative sign to values for r > 0·250)

EXAMPLE OF ANALYSIS (PART III) XI

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
s	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3								
L	φ OBSERVED TIMES				ζ OBSERVED HEIGHTS				t REDUCED TIMES				z REDUCED HEIGHTS				Z = z − 12·4		t		ϑ	τ	X	Y
	0·038	0·224	0·490	0·818	13·3	11·4	17·05	7·05									HHW	LLW	HHW	LLW				
−14	0·110	0·273	0·517	0·855	14·2	12·9	17·2	5·6	(0·102)	(0·260)	0·503	0·824	(14·1)	(12·5)	17·1	6·8	4·7	−5·6	1·503	0·824	0·340	1·164	−0·9	10·3
−13	0·169	0·331	0·546	0·891	15·4	14·05	17·25	4·25	(0·159)	(0·313)	0·532	0·860	(15·2)	(13·7)	17·2	5·4	4·8	−7·0	1·532	0·860	0·336	1·196	−2·2	11·8
−12	0·215	0·359	0·581	0·925	16·6	14·7	17·15	3·25	(0·206)	(0·368)	0·562	0·895	(16·4)	(14·5)	17·2	4·2	4·8	−8·2	1·562	0·895	0·334	1·229	−3·4	13·0
−11	0·251			0·958	17·5			2·6	0·242			0·927	17·3			3·2	4·9	−9·2	1·242	0·927	0·158	1·085	−4·3	14·8
−10	0·283			0·990	16·1			2·45	0·274			0·959	17·9			2·6	5·5	−9·8	1·274	0·959	0·158	1·117	−4·3	15·3
−9	0·312			...	18·5			–	0·303			0·990	18·4			2·4	6·0	−10·0	1·303	0·990	0·156	1·146	−4·0	16·0
−8	0·339			0·020	18·55			2·75	0·330			0·019	18·5			2·7	6·1	−9·7	·330	0·019	0·156	0·175	−3·6	15·8
−7	0·363			0·050	18·45			3·50	0·354			0·049	18·5			3·5	6·1	−8·9	·354	0·049	0·152	0·201	−2·8	15·0
−6	0·386			0·078	18·2			4·70	0·377			0·076	18·3			4·6	5·9	−7·8	·377	0·076	0·150	0·226	−1·9	13·7
−5	0·407			0·105	17·85			6·20	0·399			0·102	18·0			6·1	5·6	−6·3	·399	0·102	0·148	0·250	−0·7	11·9
−4	0·427			0·132	17·45			7·45	0·419			0·129	17·6			7·6	5·2	−4·8	·419	0·129	0·145	0·274	0·4	10·0
−3	0·445	0·751	..	0·459	17·05	8·5	..	9·55	0·437			0·455	17·2			9·3	4·8	−3·1	·437	0·455	0·141	0·296	1·7	7·9
−2	0·464	0·789	0·002	0·136	16·7	8·75	12·95	11·25	0·455	0·760		(0·181)	16·9	8·9		(10·9)	4·5	−4·1	1·455	0·760	0·348	1·108	0·4	8·6
−1	0·483	0·825	0·078	0·221	16·4	7·05	13·45	12·75	0·474	0·796	(0·072)	(0·244)	16·5	7·6	(13·4)	(12·4)	4·1	−4·8	1·474	0·796	0·339	1·135	−0·7	8·9
0	0·503	0·858	0·145	0·273	16·1	6·4	14·4	13·95	0·493	0·831	(0·136)	(0·260)	16·3	6·9	(14·3)	(13·6)	3·9	−5·5	1·493	0·831	0·331	1·162	−1·6	9·4
1	0·526	0·889	0·191	0·341	15·85	5·8	15·4	14·85	0·514	0·862	(0·183)	(0·319)	16·0	6·3	(15·2)	(14·4)	3·6	−6·1	1·514	0·862	0·326	1·188	−2·5	9·7
2		0·916	0·223			5·3	16·25			0·892	0·216			5·8	16·1		3·7	−6·6	1·216	0·892	0·162	1·054	−2·9	10·3
3		0·941	0·249			4·85	16·9			0·918	0·243			5·3	16·7		4·3	−7·1	1·243	0·918	0·162	1·080	−2·8	11·4
4		0·966	0·272			4·55	17·3			0·942	0·266			4·8	17·2		4·8	−7·6	1·266	0·942	0·162	1·104	−2·8	12·4
5		0·990	0·293			4·4	17·8			0·967	0·287			4·5	17·4		5·0	−7·9	1·287	0·967	0·160	1·127	−2·9	12·9
6		...	0·313			...	17·5			0·990	0·307			4·4	17·5		5·1	−8·0	1·307	0·990	0·158	1·148	−2·9	13·1
7		0·013	0·330			4·5	17·4			0·013	0·324			4·5	17·4		5·0	−7·9	·324	0·013	0·156	0·169	−2·9	12·9
8		0·036	0·345			4·9	17·2			0·035	0·340			4·9	17·3		4·9	−7·5	·340	0·035	0·152	0·187	−2·6	12·4
9		0·059	0·360			5·55	17·0			0·058	0·355			5·5	17·1		4·7	−6·9	·355	0·058	0·148	0·206	−2·2	11·6
10		0·083	0·378			6·55	16·9			0·081	0·368			6·5	16·9		4·5	−5·9	·368	0·081	0·144	0·225	−1·4	10·4
11	0·901	0·408	0·387	0·667	13·9	7·65	16·9	9·2		0·105	0·382			7·7	16·9		4·5	−4·7	·382	0·105	0·138	0·243	−0·2	9·2
12	0·963	0·434	0·404	0·705	13·6	8·4	16·95	8·2	(0·907)	(0·131)	0·397	0·679	(13·9)	(9·2)	16·9	8·9	4·5	−3·5	1·397	0·679	0·359	1·036	1·0	8·0
13	–	0·462	0·424	0·748	...	11·1	17·0	7·65	(0·966)	(0·158)	0·416	0·717	(13·6)	(10·8)	17·0	7·9	4·6	−4·5	1·416	0·717	0·350	1·067	0·1	9·1
14	0·041	0·499	0·448	0·793	13·8	12·45	17·0	6·05	(0·038)	(0·192)	0·438	0·759	(13·8)	(12·4)	17·0	6·9	4·6	−5·5	1·438	0·759	0·340	1·099	−0·9	10·1

XII

	25	26	27	28	29	30	31	32	33	34	35	36
L	X_1	Y_1	c_1	s_1	X_2	Y_2	c_2	s_2	A_1	B_1	A_2	B_2
-14	0.6	3.6	0.514	0.858	0.3	-2.3	-0.471	0.882	-2.8	3.4	1.9	4.3
-13	1.5	4.4	0.333	0.943	0.7	-3.4	-0.778	0.628	-3.6	2.9	1.0	2.1
-12	2.3	4.9	0.132	0.991	1.1	-3.8	-0.965	0.261	-2.6	2.9	-0.4	2.7
-11	-2.9	4.9	0.841	0.539	1.3	3.2	0.482	0.876	-5.0	2.7	-2.8	2.7
-10	-2.9	5.3	0.742	0.671	1.3	3.5	0.100	0.995	-5.7	1.0	-2.4	1.5
-9	-2.7	5.3	0.608	0.794	1.2	3.9	-0.261	0.965	-5.8	1.1	-4.1	0.1
-8	-2.5	5.2	0.454	0.891	1.1	3.9	-0.588	0.809	-5.8	0.1	-3.6	-1.4
-7	-1.9	4.6	0.303	0.953	0.6	4.0	-0.816	0.578	-5.0	-0.4	3.0	-2.8
-6	-1.3	4.0	0.150	0.989	0.6	3.8	-0.955	0.297	-4.2	-0.7	-1.7	-3.5
-5	-0.5	3.3	0.000	1.000	0.2	3.4	-1.000	0.000	-3.3	-0.5	-0.2	-3.4
-4	0.3	2.6	-0.150	0.989	-0.1	3.1	-0.955	-0.297	-2.5	-0.4	1.0	-3.0
-3	1.2	1.7	-0.285	0.959	-0.5	2.7	-0.839	-0.546	-1.0	0.7	1.9	-2.0
-2	-0.3	2.6	0.778	0.628	-0.1	-2.4	0.212	0.977	-1.9	1.8	2.3	-0.6
-1	0.5	3.2	0.661	0.750	0.2	-2.0	-0.125	0.992	-2.1	2.5	2.0	0.4
0	1.1	3.7	0.525	0.851	0.5	-1.7	-0.448	0.894	-2.6	2.9	1.3	1.2
1	1.6	4.0	0.380	0.925	0.9	-1.6	-0.694	0.720	-3.1	3.0	0.5	1.6
2	-2.0	3.4	0.943	0.333	0.9	2.2	0.778	0.628	-3.2	2.9	-0.7	2.2
3	-1.9	4.2	0.876	0.482	0.9	2.4	0.536	0.844	-3.7	2.8	-1.5	2.0
4	-1.9	4.5	0.794	0.608	0.9	2.6	0.261	0.965	-4.2	2.4	-2.3	1.5
5	-2.0	4.6	0.698	0.716	0.9	2.9	-0.025	1.000	-4.7	1.8	-2.9	0.8
6	-2.0	4.5	0.598	0.802	0.9	3.0	-0.285	0.959	-4.6	1.4	-3.1	0.0
7	-2.0	4.3	0.487	0.873	0.9	3.1	-0.525	0.851	-4.7	0.8	-3.1	-0.9
8	-1.8	3.8	0.386	0.923	0.8	3.3	-0.703	0.720	-4.2	-0.2	-2.9	-1.6
9	-1.5	3.2	0.233	0.962	0.6	3.4	-0.851	0.525	-3.5	-0.6	-2.3	-2.6
10	-1.0	2.5	0.076	0.985	0.4	3.4	-0.951	0.309	-2.6	-0.6	-1.6	-3.1
11	-0.1	1.7	0.049	0.999	0.1	3.4	-0.996	0.068	-1.7	0.0	-0.4	-3.4
12	-0.7	1.7	0.972	0.236	-0.3	-2.9	0.688	0.460	-1.1	1.8	1.0	-2.5
13	-0.4	2.9	0.913	0.409	0.0	-2.5	0.666	0.746	-1.2	2.4	1.9	-1.7
14	0.4	3.6	0.813	0.588	0.8	-2.2	0.321	0.947	-1.6	3.3	2.2	-0.4

L	c_0	e_2	g_1	f_1	$\frac{e_1^2-f_1^2}{16e_2}$	a_0
-14	12.62	2.98	-1.5	2.88	-0.12	12.50
-13	12.88	3.32	-1.0	4.15	-0.32	12.56
-8	12.48	2.48	1.55	-2.8	-0.14	12.34
0	12.78	2.52	1.0	-3.35	-0.25	12.53
12	12.22	3.18	-1.5	0.15	0.04	12.26
13	12.32	2.98	-1.7	1.45	0.02	12.34
					MEAN	12.42

	p = 1	p = 2
A_{p0}	-101.2	-22.6
1	-1.1	0.8
2	41.8	80.3
3	-8.2	-14.7
4	-6.5	-0.3
5	-8.6	-4.9
B_{pa}	-2.0	-1.4
b	-32.9	-75.4
c	-0.6	14.3
d	9.3	2.0
e	-2.8	2.1
B_{p0}	40.4	-12.6
1	-1.9	4.2
2	47.3	41.2
3	0.2	9.5
4	-0.5	-2.3
5	-1.5	-2.7
A_{pa}	-14.1	6.5
b	39.5	36.7
c	-8.8	10.2
d	1.9	-14
e	-1.5	0.0

Theory and explanation (Part III).

1. *Analysis of Mixed Solar Tides.*

The following notation will be used :

Z : the height of high or low water above or below mean sea level.
t : the time of high or low water, in units of a day.
τ : a mean time.
n : a speed of 360° per day.
p : the species number.
γ : the time interval from the mean time to the next stationary value.
A, B : constituent numbers at epoch zero hour.
C, D : constituent numbers at epoch τ.

The tide will be expressed as :

$$Z = \Sigma A_p \cos pnt + \Sigma B_p \sin pnt \qquad (1)$$

$$= \Sigma C_p \cos pn(t-\tau) + \Sigma D_p \cos pn(t-\tau) \qquad (2)$$

and the condition for these representing high or low water is :

$$0 = \Sigma p C_p \sin pn(t-\tau) - \Sigma p D_p \cos pn(t-\tau) \qquad (3)$$

Let there be two stationary values at times $(\tau+\gamma)$ and $(\tau-\gamma)$ and let the corresponding heights be denoted by $Z(\gamma)$ and $Z(-\gamma)$ respectively. Then we readily find

$$X = Z(\gamma) + Z(-\gamma) = 2\,\Sigma C_p \cos pn\gamma \qquad (4)$$

$$Y = Z(\gamma) - Z(-\gamma) = 2\,\Sigma D_p \sin pn\gamma \qquad (5)$$

$$0 = \Sigma pC_p \sin pn\gamma \qquad (6)$$

$$0 = \Sigma pD_p \cos pn\gamma \qquad (7)$$

From these pairs of equations a constituent with $p = q$, say, can readily be eliminated, yielding

$$qX \sin qn\gamma = -\Sigma\,[(p+q)\sin(p-q)n\gamma + (p-q)\sin(p+q)n\gamma]C_p \qquad (8)$$

$$qY \cos qn\gamma = \Sigma\,[(p+q)\sin(p-q)n\gamma - (p-q)\sin(p+q)n\gamma]D_p \qquad (9)$$

It is convenient to write

$$\frac{X_1}{X} = \frac{2\sin 2n\gamma}{3\sin n\gamma + \sin 3n\gamma} \qquad \frac{Y_1}{Y} = -\frac{2\cos 2n\gamma}{3\sin n\gamma - \sin 3n\gamma} \qquad (10)$$

$$\frac{X_2}{X} = -\frac{\sin n\gamma}{3\sin n\gamma + \sin 3n\gamma} \qquad \frac{Y_2}{Y} = \frac{\cos n\gamma}{3\sin n\gamma - \sin 3n\gamma} \qquad (11)$$

For the present we shall ignore the terms in (8) and (9) which are neither diurnal nor semidiurnal, and it is then evident that (10) and (11) give

$$X_1 = C_1\,,\ Y_1 = D_1\,,\ X_2 = C_2\,,\ Y_2 = D_2 \qquad (12)$$

Hence, by taking a special time origin half way between two stationary values it is possible to prepare tables of multipliers which give the harmonic numbers for the diurnal and semidiurnal tides without any further correction if the third-diurnal and quarterdiurnal tides can be ignored. The transference from the special origin of time to zero hour of the solar day is a standard process (see Instruction XIII).

2. *Tables of* X_1/X, Y_1/Y, X_2/X and Y_2/Y.

These tables are very easily prepared in terms of γ, with $n = 360^o$ and γ in units of a solar day. The value of γ varies according to the stationary values. The theory just given has not specified whether the stationary values are two high waters or two low waters, adjacent low and high waters, or HHW and LLW. It is a general theory, but in the application we shall take the stationary values as pertaining to HHW and LLW. The range of values of γ depends upon the choice of stationary values and in those chosen here the value of γ varies between 0.125 day (3 hours) when the semidiurnal tide is dominant to 0.25 day when the diurnal tide is dominant, but the interval may be 0.375 with semidiurnal tides and vary to 0.25, according to whether the tides are adjacent or are separated by LHW and HLW. Hence the tables are prepared for the interval $\gamma = 0.125$ to 0.375, but there is either symmetry or asymmetrd about $\gamma = 0.250$. As they are given at intervals of 0.001 in γ there is no neey for interpolations.

3. *Remarks on the calculation of* γ *and* τ.

It is essential to note that these two quantities are intimately related and they should not be computed independently of each other. It is desirable to have a convention that the HHW should follow the LLW, and this can always be assured by adding 1.000 to the value of t for HHW. This is permissible, of course, because we are dealing with reduced tides, which are solar in character, and therefore repeat themselves after the interval of a day. The practice recommended in the instructions is firstly to compute γ and then to add this to the time of LLW. As a check, the value of τ can also be computed by subtracting γ from the time of HHW.

4. *Remarks on the sequences of observations.*

Discontinuities in the sequences of tides may cause some perplexity, but it is only necessary to ensure that a high water sequence is placed under $s = 0$ or 2. It is not necessary to relate the sequence to lunar transits, and in fact it is sometimes almost impossible to do so as the times of high and low water may change very quickly. The original version of this method was in lunar time but so many difficulties were encountered that it was found to be simplest to work in solar time as in Part II and Part III. The choice of HHW and LLW may also be perplexing, but this need not be so. It is not even necessary to choose HHW or LLW. It may happen that a sequence of HHW varies rapidly at beginning or ending. If there is any other alternative it can be chosen, so long as there is a reasonable separation between the heights of the two stationary values. It would be foolish to apply this method to LHW and HLW, for the difference in height is too small to give accurate results. There is thus a measure of flexibility in the method, the only essential things being that the sequences may be easily interpolated, and the quantities analysed separated reasonably in time and height, and that the times and heights be taken together (that is, they pertain to one another).

THE ANALYSIS OF HIGH AND LOW WATERS.

Appendix : An Outline of a General Method.

1. *Introduction.*

It was stated in the introduction to Part II that the method there given is more general than the method of Part I, and that it can deal with the problems dealt with in Part I, but that it could not cope with the problems considered in Part III. In the same way, the method of Part III can be adapted as a general method capable of dealing with all the types of problems considered in the three Parts. It was with some reluctance that the author gave three different methods, and before doing so he spent much time searching for and developing a more general method, but had to decide that it was not practicable to use it without making available some extensive tables. The actual operations required by the method are not appreciably more complicated than those of Part II but it was thought well to have simple methods with restricted fields of application rather than one apparently complex method covering the whole field. In view of the importance of the subject, however, it seems desirable to indicate the way in which the general problem may be solved.

2. *The General Equations.*

The general solution is contained in the expressions given in (8) to (11) in Part III.

If we take $q = 2$ in (8) and multiply the equation by the denominator of X_1/X in (10), we have an equation which is independent of the semidiurnal tide but involves the third and quarterdiurnal tides. Let the coefficients of C_3 and C_4 in the equation be denoted by λ_{13} and λ_{14} ; then we have

$$X_1 = C_1 + \lambda_{13}C_3 + \lambda_{14}C_4 \tag{1}$$

and similarly

$$Y_1 = D_1 + \mu_{13}D_3 + \mu_{14}D_4 \tag{2}$$

$$X_2 = C_2 + \lambda_{23}C_3 + \lambda_{24}C_4 \tag{3}$$

$$Y_2 = D_2 + \mu_{23}D_3 + \mu_{24}D_4 \tag{4}$$

The values of the coefficients are given by the equations

$$\left.\begin{aligned}
(3 \sin n\gamma + \sin 3n\gamma)\lambda_{13} &= -(5 \sin n\gamma + \sin 5n\gamma)\\
(3 \sin n\gamma + \sin 3n\gamma)\lambda_{14} &= -(6 \sin 2n\gamma + 2 \sin 6n\gamma)\\
(3 \sin n\gamma + \sin 3n\gamma)\lambda_{23} &= (4 \sin 2n\gamma + 2 \sin 4n\gamma)\\
(3 \sin n\gamma + \sin 3n\gamma)\lambda_{24} &= (5 \sin 3n\gamma + 3 \sin 5n\gamma)
\end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned}
(3 \sin n\gamma - \sin 3n\gamma)\mu_{13} &= -(5 \sin n\gamma - \sin 5n\gamma)\\
(3 \sin n\gamma - \sin 3n\gamma)\mu_{14} &= -(6 \sin 2n\gamma - 2 \sin 6n\gamma)\\
(3 \sin n\gamma - \sin 3n\gamma)\mu_{23} &= (4 \sin 2n\gamma - 2 \sin 4n\gamma)\\
(3 \sin n\gamma - \sin 3n\gamma)\mu_{24} &= (5 \sin 3n\gamma - 3 \sin 5n\gamma)
\end{aligned}\right\} \tag{6}$$

The above equations are derived for a pair of stationary values but as there are normally two such pairs in the day we shall denote the second pair by variables with single dashes. In order to combine the two sets of results it is necessary to have a common and central time origin. This will be the mean value of the two times τ and τ', and we may denote it by

$$\tau'' = \frac{1}{2}(\tau + \tau') \tag{7}$$

We may also write γ'' for half the time interval between τ and τ'', and so that

$$\tau'' = \tau + \gamma'' = \tau' - \gamma'' \tag{8}$$

Let E, F be the harmonic numbers at epoch τ''; so that

$$E_p = C_p \cos p\gamma'' + D_p \sin p\gamma'' , \quad F_p = -C_p \sin p\gamma'' + D_p \cos p\gamma'' \tag{9}$$

or

$$C_p = E_p \cos p\gamma'' - F_p \sin p\gamma'' , \quad D_p = E_p \sin p\gamma'' + F_p \cos p\gamma'' \tag{10}$$

We now substitute by (10) into equations (1) to (4) and then eliminate E_1 or F_1, E_2 or F_2 by the following combinations :

$$\left.\begin{aligned} G_{1a} &= X_1 \cos n\gamma'' + Y_1 \sin n\gamma'' \\ G_{1b} &= Y_1 \cos n\gamma'' - X_1 \sin n\gamma'' \\ G_{2a} &= X_2 \cos 2\,n\gamma'' + Y_2 \sin 2\,n\gamma'' \\ G_{2b} &= Y_2 \cos 2\,n\gamma'' - X_2 \sin 2\,n\gamma'' \end{aligned}\right\} \tag{11}$$

and obtain

$$\begin{aligned} G_{1a} &= E_1 + L_{13a}E_3 + L_{13b}F_3 + L_{14a}E_4 + L_{14b}F_4 \\ G_{1b} &= F_1 + M_{13a}E_3 + M_{13b}F_3 + M_{14a}E_4 + M_{14b}F_4 \\ G_{2a} &= E_2 + L_{23a}E_3 + L_{23b}F_3 + L_{24a}E_4 + L_{24b}F_4 \\ G_{2b} &= F_2 + M_{23a}E_3 + M_{23b}F_3 + M_{24a}E_4 + M_{24b}F_4 \end{aligned} \tag{12}$$

The definitions of L and M are given below in terms of l, m, and $n\gamma''$, where

$$l = \frac{1}{2}(\lambda + \mu) , \quad m = \frac{1}{2}(\lambda - \mu) \tag{13}$$

$$\left.\begin{aligned} L_{13a} &= l_{13} \cos 2n\gamma'' + m_{13} \cos 4n\gamma'' & M_{13a} &= l_{13} \sin 2n\gamma'' - m_{13} \sin 4n\gamma'' \\ L_{13b} &= -(l_{13} \sin 2n\gamma'' + m_{13} \sin 4n\gamma'') & M_{13b} &= l_{13} \cos 2n\gamma'' - m_{13} \cos 4n\gamma'' \\ L_{14a} &= (l_{14} \cos 3n\gamma'' + m_{14} \cos 5n\gamma'') & M_{14a} &= l_{14} \sin 3n\gamma'' - m_{14} \sin 5n\gamma'' \\ L_{14b} &= -(l_{14} \sin 3n\gamma'' + m_{14} \sin 5n\gamma'') & M_{14b} &= l_{14} \cos 3n\gamma'' - m_{14} \cos 5n\gamma'' \\ \\ L_{23a} &= l_{23} \cos n\gamma'' + m_{23} \cos 5n\gamma'' & M_{23a} &= l_{23} \sin n\gamma'' - m_{23} \sin 5n\gamma'' \\ L_{23b} &= -(l_{23} \sin n\gamma'' + m_{23} \sin 5n\gamma'') & M_{23b} &= l_{23} \cos n\gamma'' - m_{23} \cos 5n\gamma'' \\ L_{24a} &= l_{24} \cos 2n\gamma'' + m_{24} \cos 6n\gamma'' & M_{24a} &= l_{24} \sin 2n\gamma'' - m_{24} \sin 6n\gamma'' \\ L_{24b} &= -(l_{24} \sin 2n\gamma'' + m_{24} \sin 6n\gamma'') & M_{24b} &= l_{24} \cos 2n\gamma'' - m_{24} \cos 6n\gamma'' \end{aligned}\right\} \tag{14}$$

The above expressions are given for one pair of stationary values, and for the other pair, indicated by dashed variables, we get similar expressions to those given above, in terms of L′ and M′, l′ and m′, and $-\gamma''$ in place of γ''.

It will be noted that G_{1a} and G'_{1a} each have E_1 as the principal term and we therefore write with appropriate suffixes

$$H = G' - G \tag{15}$$

and thus obtain four equations for E_3, F_3, E_4, and F_4. The coefficients of these equations are too complex for trigonometrical analysis, and the simplest procedure is to obtain numerical solutions in terms of γ, γ', and γ''. These solutions are in terms of H_{1a}, H_{1b}, H_{2a}, and H_{2b} ; and the solutions are effected once for all, and tabulated for use by the computers.

Similarly we may write, with appropriate suffixes,

$$J = G' + G \tag{16}$$

and thus obtain four equations for E_1, F_1, E_2, and F_2 in terms of J and the four values of H. The solutions are tabulated once for all in terms of γ, γ', and γ''. The tables are therefore triple-entry tables. Finally, the time origin is transferred back again to zero hour of the day to give A and B as in Part II.

3. *Remarks on the numerical analysis.*

The processes of analysis are by no means so complicated as the theory suggests. The early processes are the same as in Part II as far as X_1, Y_1, X_2, and Y_2, except that in the general method we may have two high waters and two low waters each day. In the normal method we take the two high waters for one pair of stationary values and the two low waters for the second pair. This avoids the possibility of combining a high water and a low water which are almost merged. The tables for the computation of X_1/X, etc., need to be extended as the range of γ is increased. The required tables have all been computed by the author, as well as the tables for E and F for the four species of tides, and the method has been proved to be simple enough in operation. The only difficulty is the re-production of the tables. They are necessarily somewhat bulky as interpolations in triple-entry tables are not to be recommended. From the computer's point of view the method is as simple as that of Part II, and it has the merit of being general and avoids the use of approximate values in the corrections.