THE USE OF ALGEBRAIC METHODS IN THE HARMONIC ANALYSIS OF TIDAL OBSERVATIONS

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1. *The aim of this study*

The phenomenon of the tides is caused by the disturbing influence of the sun and the moon. The product of the generating forces produced by these bodies springs from a potential which can be expressed, in a given instance, in terms of time T as follows:

$$\Sigma\ A \cos (\sigma t - \alpha)$$

the well-known constants A, σ and α depending on astronomical elements.

According to Laplace's first principle, each of these strictly periodic terms gives rise to fluctuations of water molecules of the same period; according to his second principle, as the forces springing from each of the terms of the potential are small, the total movement is formed by the sum of all the partial movements they create. It is possible, therefore, to express the height of water, in a given instance, in the same terms as the potential, as follows:

$$h_t = \Sigma\ R \cos (\sigma t - \varepsilon).$$

But amplitude R and phase ε can only be arrived at by experiment; they cannot be calculated theoretically. In practice, of course, it will not be necessary to study more than a few waves, the sum of which will, to avoid confusion with other sums used later on, be expressed in terms of Σ_o.

$$h_t = \Sigma_o\ R \cos (\sigma t - \varepsilon).$$

The σ speeds are expressed in degrees per hour.

Observations are based on hourly measurements of the height of water. This, accordingly, leads to the working out of such an equation system as the following:

$$h_t = \Sigma_o\ R \cos (\sigma t - \varepsilon) \qquad [t]$$

in which t assumes positive or negative integral values and in which R and ε are unknown quantities.

If $\cos (\sigma t - \varepsilon)$ is developed, then the equations become linear in $R \cos \varepsilon$ and $R \sin \varepsilon$ and form a system which inevitably must always consist of a superfluity of terms. Indeed, although the $\sigma°$ hourly speeds have been established with all required accuracy, this is not the case with h_t subject to meteorological forces that are difficult to reckon, and the separation of the waves, under these conditions, takes a very long time to accomplish.

The most feasible solution is to be found in the least-square method, but this has the disadvantage of not entirely isolating each unknown quantity in the corresponding standard equation. The working out of a system of this kind — already laborious enough if the period of observation is long — should hardly be contemplated.

A compromise is, therefore, called for. It is not necessary here to build up a complete method, but only to explain and outline certain ideas suggested by the work of the « Tidal Institute » (particularly the work of A. T. Doodson) (1), which are not too clearly explained.

More precisely, the aim proposed here is to find, for hourly equations, linear combinations of small, integral coefficients, eliminating the terms relative to certain waves in such a way as to produce a new and simpler system than the first. In other words, the aim is to outline simplification processes without becoming involved in incorporating them in a general method and without, consequently, being disturbed by the loss of accuracy which, in comparison with the least-square method, can result from their use. This loss of accuracy, incidentally, is very slight, as the systems contemplated are generally comparatively coherent.

2. *Principles*

The symbol [t] stands for the equation $\Sigma_o \cos(\sigma t - \varepsilon) = h_t$. Let t_o be a given time and assume that among the equations $[t_o]$, $[t_o+1]$, ... $[t_o+p]$, ..., $[t_o+n]$, the combination

$$\sum_{p=0}^{p=n} d_p \times [t_o+p]$$

eliminates a wave of angular speed σ ; this then means that, whatever the value of ε,

$$\Sigma_p d_p \cos [\sigma (t_o+p) - \varepsilon] = 0.$$

Or again,

$$\cos(\sigma t_o - \varepsilon) \Sigma_p d_p \cos p\sigma + \sin(\sigma t_o - \varepsilon) \Sigma_p d_p \sin p\sigma = 0$$

which means that $\Sigma_p d_p \cos p\sigma = 0$ and that $\Sigma_p d_p \sin p\sigma = 0$, or, using the Euler formulae in which the notation e is preserved, that

$$\Sigma_p d_p (e^{\pm i\sigma})^p = 0,$$

even if x is expressed in degrees.

In others words, if the combination $\Sigma_p d_p [t_o+p]$ eliminates the wave of angular speed σ, this is because the equation $\Sigma_p d_p z^p = 0$ has $e^{\pm i\sigma}$ as its root,

(1) Dr. A.T. Doodson. The analysis of tidal observations. Phil. Trans. of the Royal Soc. of London. Series A, Vol. 227, p. 223-279, 1928.

3. *The search for combinations*

The practical problem is not to eliminate any one particular wave, but, on the contrary, to isolate one wave in so far as possible. On the basis of the previous paragraph, the search for combinations of integral coefficients leading to such an isolation is reduced to an algebraic problem which may be posed as follows: to find a polynomial based on integral coefficients which cancel each other out when the variable takes on the $e^{\pm i\sigma}$ values of the different waves to be eliminated. The actual coefficients of this polynomial supply, unmodified, those for the combination sought.

In practice, of course, it will not be necessary to insist on a strict elimination, which is generally impossible anyhow, and it will be sufficient to obtain the polynomials in which the $e^{\pm i\sigma}$ values are close to zero. Neither will it be necessary to try to isolate a single wave, but rather a more or less narrow band, according to the case in hand. Consequently, depending on the end in view (i. e., degree of selectivity, accuracy and simplicity required), numerous combinations can be considered. Here, however, only the most important example — that of isolating as a body the waves belonging to the same diurnal, semi-diurnal, ... or long-period group — will be dealt with, and this example can be used as a model for other applications (1) of the process.

Let us take the nearest a15 number as the closest value of the hourly speed of a wave in degrees (« a » being a positive integer), which will merely mean that the wave belongs to the 1/a diurnal group; we will restrict ourselves to studying the waves up to the sixth-diurnal; to isolate a group of waves, it is necessary to form a polynomial which cancels itself out when $z = e^{\pm ia15}$, whatever the values of a from 1 to 6, with the exception of one.

All these numbers are roots of $1 - z^{24} = 0$. We are, therefore, obliged to reduce $1 - z^{24}$ to a product of integral coefficient polynomials of the lowest possible degree. This reduction may be expressed as follows:

$$1 - z^{24} = (1 - z)(1 + z + z^2)(1 + z)(1 - z + z^2)(1 + z^2)(1 - z^2 + z^4)(1 + z^4)(1 - z^4 + z^8).$$

The roots may be divided amongst these polynomials as follows:

$1 - z = 0$	a = 0
$1 + z + z^2 = 0$	a = 8, 16
$1 + z = 0$	a = 12
$1 - z + z^2 = 0$	a = 4, 20
$1 + z^2 = 0$	a = 6, 18
$1 - z^2 + z^4 = 0$	a = 2, 10, 14, 22
$1 + z^4 = 0$	a = 3, 9, 15, 21
$1 - z^4 + z^8 = 0$	a = 1, 5, 7, 11, 13, 17, 19, 23.

(1) A more detailed article on this subject will appear in the « Annales Hydrographiques ».

All integral coefficients factors of $1 - z^{24}$ are products of certain of these six polynomials. The following is a list of the most conveniently-handled factors of which the roots given are of the low order for $a \leq 8$:

No	Polynomial	Zeros	No	Polynomial	Zeros
(1)	$1+z^2$	$a = 6$	(11)	$1 - z^8$	$a = 0,3,6$
(2)	$1+z^3$	$a = 4$	(12)	$1 - z^{12}$	$a = 0,2,4,6,8$
(3)	$1+z^4$	$a = 3$	(13)	$1 - z^{24}$	$a = 0,1,2,3, \dots$
(4)	$1+z^6$	$a = 2,6$	(14)	$1+z+z^2$	$a = 8$
(5)	$1+z^{12}$	$a = 1,3,5,7$	(15)	$1+z^2+z^4$	$a = 4,8$
(6)	$1 - z$	$a = 0$	(16)	$1+z^4+z^8$	$a = 2,4,8$
(7)	$1 - z^2$	$a = 0$	(17)	$1+z^8+z^{16}$	$a = 1,2,4,5,7,8$
(8)	$1 - z^3$	$a = 0,8$	(18)	$1 - z+z^2$	$a = 4$
(9)	$1 - z^4$	$a = 0,6$	(19)	$1 - z^2+z^4$	$a = 2$
(10)	$1 - z^6$	$a = 0,4,8$	(20)	$1 - z^4+z^8$	$a = 1,5,7$

Let $a_0 15$ be the approximate speed of the waves to be isolated; then select from the above table various polynomials of which one at least cancels itself out when $z = e^{\pm ia15}$ if $a \neq a_0$, and of which not one cancels itself out if $a = a_0$. *As their product cancels itself out for all values of* $e^{\pm ia15}$ *in which* $a \neq a_0$, *it is sufficient that it be developed for suitable combinations of coefficients to be arrived at.*

For example, to isolate the quarter-diurnal waves, it is possible, by choosing the factors (4), (5) and (8), to arrive at the product

$$(1+z^6)(1+z^{12})(1 - z^3) = 1 - z^3+z^6 - z^9+z^{12} - z^{15}+z^{18} - z^{21}.$$

But this choice is not the only one and it is determined by the accuracy of the elimination and the simplicity desired. It is only through numerical calculation that we can arrive at the best choice. It should be noted, however, that it is possible to improve the elimination of a particularly large group (generally the semi-diurnals) by raising the corresponding polynomial to the square. In fact, as the values of the modulus of this factor for these waves are nearly zero, their squares are even smaller. In the preceding case, for example, the combination $(1+z^6)^2(1+z^{12})(1 - z^3)$ would be better than the first. In this matter, the only obstacles that might be encountered are the complexity of the coefficients and the length of the combinations.

Then again, it is useful to obtain polynomials which eliminate more effectively certain waves with speeds rather different from those of solar waves. For example, the following two polynomials correspond to certain combinations used by A. T. Doodson:

(21) $$1+z^9+z^{18} = \frac{1 - z^{27}}{1 - z^9}$$ eliminating 0_1 and

(22) $$1+z^5+z^{10}+z^{15}+z^{20} = \frac{1 - z^{25}}{1 - z^5}$$ eliminating approximately

the lunar waves.

4. *Return to the trigonometric form*

The first member of the equation

$$\Sigma_p\, d_p\, [t_o + p]$$

should be calculated when the coefficients d_p are arrived at by the preceding method. This is the total of terms, having the form of $\Sigma\, d_p\, R \cos\, [\sigma\, (t_o + p) - \varepsilon]$, which are expressed as

$$\frac{R}{2}\, e^{i(\sigma t_0 - \varepsilon)}\, \Sigma_p\, d_p\, (e^{i\sigma})^p + \frac{R}{2}\, e^{-i(\sigma\, t_0 - \varepsilon)}\, \Sigma_p\, d_p\, (e^{-i\sigma})^p \qquad (1)$$

Now, by its very formation, the polynomial $\Sigma_p d_p z^p$ reduces itself to factors drawn from the preceding table. When $z = e^{i\sigma}$, each of these factors assumes the value $re^{i\omega}$, of which r is the modulus without taking the sign into account (as r is not assumed to be positive) and ω is the argument $\pm$ 180°. We shall return later to the calculation (which is an easy one) of these numbers for each factor. In any case, $\Sigma\, d_p\, (e^{i\sigma})^p$ can also be expressed as $Je^{i\eta}$, J being the product of the r's of the different factors and η the total of the ω's. The first term of (1) can, therefore, be expressed as

$$\frac{1}{2} R\, J\, e^{i\, (\sigma\, t_0 - \varepsilon + \eta)}$$

As the second is precisely the conjugate imaginary expression, then

$$\Sigma_p\, d_p\, R \cos\, [\sigma\, (t_o + p) - \varepsilon] = J\, R \cos\, (\sigma\, t_o - \varepsilon + \eta).$$

The calculation of r and ω is very easy. In the case of the factors 1 to 12, such as $1 + z^3$, for example, it is sufficient to write

$$1 + z^3 = z^{3/2}\, (z^{-3/2} + z^{3/2})$$

which, when $z = e^{i\sigma}$, is expressed as $2 \cos \frac{3}{2}\, \sigma\, e^{3i\sigma/2}$ that is $r = 2 \cos \frac{3}{2}\, \sigma$ and $\omega = 3\, \sigma/2$.

Or yet again $1 - z^4 = z^2\, (z^{-2} - z^2)$, which becomes

$$-2\, i \sin 2\, \sigma\, e^{2i\sigma} = -2 \sin 2\, \sigma\, e^{2i\sigma + 90°i}$$

in which case $r = 2 \cos\, (2\, \sigma + 90°)$ and $\omega = 2\, \sigma + 90°$.

As to the last factors, it could, for example, be said that

$$1 + z^2 + z^4 = \frac{1 - z^6}{1 - z^2} = z^2\, \frac{z^{-3} - z^3}{z^{-1} - z} \text{ and that, where } z = e^{i\sigma},$$

$$re^{i\omega} = e^{2i\sigma}\, \frac{\sin 3\sigma}{\sin \sigma}.$$

The results arrived at for the different polynomials are collected in the table given below.

To sum up, the first member of a combination is expressed as

$$\Sigma_0 \, R \, J \cos (\sigma \, t_0 - \varepsilon + \eta)$$

in which J and η are the total and the product, respectively, of the r and ω terms given in the following table and corresponding to the factors which occur in the formation of the combination.

No.	r	ω	No.	r	ω
1	$2 \cos \sigma$	σ	12	$2 \cos (6\sigma + 90)$	$6\sigma + 90$
2	$2 \cos \frac{3}{2} \sigma$	$\frac{3}{2} \sigma$	13	$2 \cos (12\sigma + 90)$	$12\sigma + 90$
3	$2 \cos 2\sigma$	2σ	14	$\sin \frac{3\sigma}{2} / \sin \frac{\sigma}{2}$	σ
4	$2 \cos 3\sigma$	3σ	15	$\sin 3\sigma / \sin \sigma$	2σ
5	$2 \cos 6\sigma$	6σ	16	$\sin 6\sigma / \sin 2\sigma$	4σ
6	$2 \cos (\frac{\sigma}{2} + 90)$	$\frac{\sigma}{2} + 90$	17	$\sin 12\sigma / \sin 4\sigma$	8σ
7	$2 \cos (\sigma + 90)$	$\sigma + 90$	18	$\cos \frac{3\sigma}{2} / \cos \frac{\sigma}{2}$	σ
8	$2 \cos (\frac{3\sigma}{2} + 90)$	$\frac{3\sigma}{2} + 90$	19	$\cos 3\sigma / \cos \sigma$	2σ
9	$2 \cos (2\sigma + 90)$	$2\sigma + 90$	20	$\cos 6\sigma / \cos 2\sigma$	4σ
10	$2 \cos (3\sigma + 90)$	$3\sigma + 90$	21	$\sin \frac{27}{2} \sigma / \sin \frac{9}{2} \sigma$	9σ
11	$2 \cos (4\sigma + 90)$	$4\sigma + 90$	22	$\sin \frac{25\sigma}{2} / \sin \frac{5\sigma}{2}$	10σ

5. *British methods*

Combinations based on the preceding statement, but arrived at, it would seem, by less systematic methods, are used by Great Britain for the harmonic analysis of tides. In particular, the groupings adopted by Dr. Doodson for the « daily » level of his analysis lead to the following polynomial combinations:

Long-period waves	(1)	(17)	(22)		
Diurnal waves	(1)	(3)	(4)	(12)	(12)
Semi-diurnal waves	(1)	(5)	(9)	(10)	(15)
Third-diurnal waves	(9)	(12)	(17)	(21)	
Quarter-diurnal waves	(4)	(4)	(5)	(8)	
Sixth-diurnal waves	(5)	(7)	(16)	(16)	

It should be noted that in these polynomials, with the exception, of course, of the one to be used for the calculation of the semi-diurnal waves, at least two factors cancel themselves out for M_2. Dr. Doodson's choice was governed by numerical tests carried out by means of the trigonometric formulae set out in the preceding paragraph. These combinations give very good results, as is shown by the table which is to be found on page 234 of Doodson's treatise. The presence of certain factors, such as (1) in x_1, would appear to serve no other purpose than to increase the number of measurements in the formulae.

(1) Admiralty Tide Tables Part III — 2nd Edition 1941 — p. 52-136.

The daily combinations used by the British Admiralty (1) for the analysis of an observation period of from 15 to 29 days appear to have been arrived at either for reasons of symmetry or as an approximation to the least-square method. They are, however, incorporated in the preceding outline and correspond to the following polynomials :

Diurnal waves	(14)	(2)	(10)	(12)	and	(12)	(12)	(6)	
Semi-diurnal waves	(14)	(8)	(10)	(5)	and	(14)	(2)	(10)	(5)
Quartier-diurnal waves	(7)	(8)	(4)	(5)	and	(14)	(8)	(4)	(5)

Here again, polynomial (14) is used for taking systematic advantage of all the measurements worked out.

6. *Other methods of application*

By means of combinations covering a long period but introducing very few measurements, it is possible practically to isolate a wave. If the heights used are satisfactory, the results thus arrived at can be absolutely correct.

Utilizing factors of the lowest possible degrees, combinations can be set up suitable for cases in which not more than twelve hours' observations daily are possible. For example, for diurnal waves, it is quite acceptable to use the polynomial

$$(1 - z^2 + z^4)^2 \ (1 + z^3) \ (1 - z).$$

Finally, by this process it is possible to build up identically null combinations allowing for the calculation of a height (the other heights, of course, being known) and for the conclusion of incomplete observations or the checking of certain values.

For example, the polynomial $(1 - z^{24})\ (1 - z^{25})^2$ supplies a combination which strictly eliminates the solar waves and in which the coefficient J assumes the values 0,002 and 0,016 for M_2 and O_1 respectively. In other words, the total

$$h_0 - h_{24} - 2\,h_{25} + 2\,h_{49} + h_{50} - h_{74}$$

is very clearly null, thus allowing for the calculation of h_{74} in terms of the preceding days' observations.

It is only in special cases such as these that the type of methods outlined above will continue to be of interest, as the harmonic analysis of the tides under conditions applicable to the principal general systems will belong, from now on, to the field of electronic computers.