A METHOD OF INFERRING THE AMPLITUDES OF THE TIDAL CONSTITUENTS M_2 AND $(K_1 + O_1)$

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In the classification of tides and in many other phases of tidal work, it is important to know the amplitude of the principal diurnal wave, represented by the sum of the amplitudes of the tidal constituents K_1 and O_1, and the amplitude of the principal semi-diurnal wave, as represented by M_2.

The constituent amplitudes are usually determined by a harmonic analysis of observed hourly heights. Since there are many series of observations that have not been analyzed, it is desirable to have a method of inferring the amplitudes $K_1 + O_1$ and M_2 from non-harmonic data.

R.A. Harris describes a method of obtaining $K_1 + O_1$, S_2 and M_2 from non-harmonic quantities on page 168 of *Manual of Tides*, Part III (U.S. Coast and Geodetic Survey publication, now out of print). His formula for $K_1 + O_1$ requires the use of tropic inequalities (the largest inequalities which occur at the extreme declination of the moon). His formula for S_2, which in turn is needed for M_2, requires observed spring and neap ranges. Since tropic inequalities and spring and neap ranges are ordinarily not obtained in the reduction of observations, the required data for his formulas are usually not available.

The method presented here for inferring the amplitudes of $(K_1 + O_1)$ and M_2 requires the mean diurnal inequalities, the mean range and two tables which are furnished. DHQ, the mean diurnal high water inequality, is the difference between mean higher high water and mean high water. DLQ, the mean diurnal low water inequality, is the difference between mean low water and mean lower low water. Mn is the mean range. In solving for $K_1 + O_1$, which must be obtained before M_2, it is necessary to choose between two solutions depending on whether DHQ is greater or less than DLQ.

To obtain $K_1 + O_1$:

(a) If $DHQ \geq DLQ$,

find $\dfrac{DHQ}{DLQ}$ to one decimal place and obtain factor F_1 from Table 1.

$$\text{Then } K_1 + O_1 = \frac{DHQ}{F_1} \qquad (1)$$

(b) If $DHQ < DLQ$,

find $\dfrac{DLQ}{DHQ}$ to one decimal place and obtain factor F_1 from Table 1.

$$\text{Then } K_1 + O_1 = \frac{DLQ}{F_1} \qquad (2)$$

To obtain M_2 :

Find $\dfrac{2.2\,(K_1 + O_1)}{Mn}$ to one decimal place and obtain factor F_2 from Table 2.

$$M_2 = \frac{Mn}{2.19 + F_2} \qquad (3)$$

DERIVATION OF FORMULAS AND TABLES

The procedures used in the method just described are primarily a reversal of the processes used by the Coast and Geodetic Survey in deriving non-harmonic tidal data from the harmonic constants. The procedures and tables used in that process are given in *Special Publication No. 260, Manual of Harmonic Constant Reductions,* 1952.

To obtain $K_1 + O_1$:

Table 13 of the Special Publication No. 260 supplies factors F_H and F_L which, multiplied by the sum of the amplitudes of K_1 and O_1, give the mean diurnal high and low water inequalities, DHQ and DLQ. The arguments used in the table are (1) the ratio of the amplitude O_1/K_1, and (2) the phase $P = | n\,\pi - (MKO - \frac{1}{2}\,v) |$ for DHQ and $P = | n\,\pi - (MKO - \frac{1}{2}\,w) | \pm 90°$ for DLQ. n is either 0, 1 or 2 ; MKO is one half the phase difference, $M_2° - K_1° - O_1°$; and v and w are the angular changes in the mean high and low water lunitidal intervals introduced by M_4 and M_6.

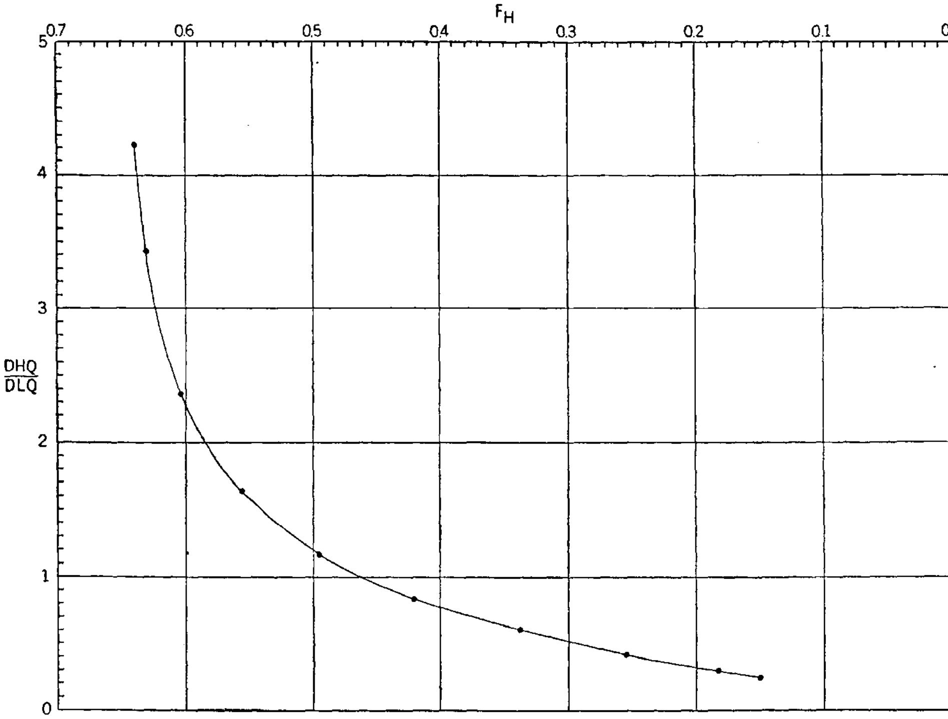
Since $(K_1 + O_1) \times F_H = DHQ$
and $(K_1 + O_1) \times F_L = DLQ$

$$\frac{F_H}{F_L} = \frac{DHQ}{DLQ}.$$

Using the theoretical ratio $O_1/K_1 = 0.7$ and entering Table 13 with this as an argument, corresponding values of F_H and F_L for each value of P are obtained

P (for DHQ)	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
F_H	.639	.631	.604	.557	.495	.422	.339	.255	.183	.151
P (for DLQ)	90°	100°	110°	120°	130°	140°	150°	160°	170°	180°
F_L	.151	.183	.255	.339	.422	.495	.557	.604	.631	.639
$\dfrac{F_H}{F_L}$ (or $\dfrac{DHQ}{DLQ}$)	4.23	3.45	2.37	1.64	1.17	0.85	0.61	0.42	0.29	0.24

Since $\dfrac{F_H}{F_L} = \dfrac{DHQ}{DLQ}$, corresponding values of F_H and $\dfrac{DHQ}{DLQ}$ can be plotted on Figure 1. These points are connected by a smooth curve, making it possible to obtain a value of F_H for any ratio $\frac{DHQ}{DLQ}$. This solves the problem as $K_1 + O_1 = \frac{DHQ}{F_H}$.



In actual observations, $\frac{DHQ}{DLQ}$ will sometimes be greater than 4.23 or less than 0.24. It is presumed this will be due to a ratio $O_1/K_1 > 0.7$. From the curve in Figure 1, F_H is approaching a maximum value at $\frac{DHQ}{DLQ} = 4.23$ and will not increase for higher values of the ratio.

If we limit the portion of Figure 1 to $\frac{DHQ}{DLQ} \geqq 1$, and use the same portion of the curve for the solution for F_L for any ratio $\frac{DLQ}{DHQ} > 1$, $K_1 + O_1 = \frac{DLQ}{F_L}$. In this manner we can avoid the portion of the curve in which F_H

varies rapidly with a small change in $\frac{DHQ}{DLQ}$ and also eliminate the problem of how to handle a ratio $\frac{DHQ}{DLQ} < 0.24$.

Two solutions are therefore used depending on whether DHQ is greater or less than DLQ.

(a) If DHQ $\geqq$ DLQ, determine $\frac{DHQ}{DLQ}$ and then F_H from Table 1.

$$K_1 + O_1 = \frac{DHQ}{F_H}$$

(b) If DHQ $<$ DLQ, determine $\frac{DLQ}{DHQ}$ and then F_L from Table 1.

$$K_1 + O_1 = \frac{DLQ}{F_L}$$

Table 1 gives values for F_H and F_L but calls both F_1. This requires that the limiting character of the choice of solutions be observed.

To obtain M_2 :

The formula for mean range (No. 43 on page 10 of Special Publication 260) is

$$\begin{aligned} Mn = {} & 1.02\,(\text{Cos } v + \text{Cos } w + \text{Table 4} + \text{Table 5})\, M_2 \\ & + 1.02\, M_4\, [\text{Cos } (P_4 - 2v) - \text{Cos } (P_4 - 2w)] \\ & + 1.02\, M_6\, [\text{Cos } (P_6 - 3v) + \text{Cos } (P_6 - 3w)]. \end{aligned}$$

If M_4 and M_6 are taken as zero, the equation simplifies to

$$Mn = 1.02\,(2.00 + \text{Table 4} + \text{Table 5})\, M_2.$$

Table 4 is the theoretical increment in mean range due to the lunar semi-diurnal constituents with speeds incommensurable with that of M_2 and to the solar semidiurnal constituents. Its value is determined by the ratio S_2/M_2.

Table 5 is the theoretical increment in mean range due to the diurnal constituents. Its value is determined by the ratio $\frac{K_1+O_1}{M_2}$.

The factor 1.02 is an empirical value which takes account of non-predictable inequalities.

Then $$M_2 = \frac{Mn}{1.02\,(2.00 + \text{Table 4} + \text{Table 5})}$$

Using the theoretical relationship of $\frac{S_2}{M_2} = 0.47$, the corresponding value in Table 4 is 0.15.

The denominator becomes

$1.02\,(2.00 + 0.15 + \text{Table 5}) = 2.19 + 1.02 \text{ Table 5}$

Since the factor from Table 5 is ordinarily small, we let

$1.02 \text{ Table 5} = \text{Table 5}$

Then $M_2 = \dfrac{Mn}{2.19 + \text{Table 5}}$

Table 2 in this paper is the same as Table 5 (Special Publication No. 260). The argument for determining its value is $\dfrac{K_1 + O_1}{M_2}$. If we approximate $M_2 = \dfrac{Mn}{2.2}$ and enter Table 2 with the argument $\dfrac{2.2\,(K_1 + O_1)}{Mn}$, we can solve for a more accurate value of M_2. The factor F_2 is therefore obtained by entering Table 2 with $\dfrac{2.2\ (K_1 + O_1)}{Mn}$ as an argument.

$$M_2 = \frac{Mn}{2.19 + F_2}$$

In the development of the formulas and tables, theoretical ratio ($O_1/K_1 = 0.7$ and $S_2/M_2 = 0.47$) are used and it is assumed that M_4 and M_6 (overtides of M_2) are zero. While variations from the above assumptions are normally to be expected, many such variations are represented by the stations given in Table 3 and the results shown in that table indicate that the procedures for approximating $K_1 + O_1$ and M_2 are ordinarily satisfactory. The inferences of $K_1 + O_1$ for Buenaventura and Talara are poor in terms of percentages but the numerical error is less than 0.2 foot. The inaccuracy is due primarily to a ratio O_1/K_1 considerably less than the theoretical value. A survey of harmonic constants for places throughout the world disclosed a limited number of other places for which a low value of the ratio O_1/K_1 would give inaccurate results. In every case but one (a 15 day series at Obbia, Africa), the amplitudes of K_1 and O_1 were less than a foot so that the numerical error in inferring by this method would be small.

TABLE I

$\frac{DHQ}{DLQ}$ or $\frac{DLQ}{DHQ}$	F_1
1.0	.46
1.1	.48
1.2	.50
1.3	.52
1.4	.53
1.5	.54
1.6	56
1.7	.56
1.8	.57
1.9	.58
2.0	.58
2.1	.59
2.2	.60
2.3	.60
2.4	.61
2.5	.61
2.6	.61
2.7	.62
2.8	.62
2.9	.62
3.0	.62
3.1	.63
3.2	.63
3.3	.63
3.4	.63
3.5	.63
3.6	.63
3.6	.64

TABLE 2

$\frac{2.2\,(K_1 + O_1)}{Mn}$	F_2
0.0	.00
0.1	.00
0.2	.00
0.3	.01
0.4	.01
0.5	.02
0.6	.03
0.7	.04
0.8	.05
0.9	.06
1.0	.07
1.1	.09
1.2	.10
1.3	.12
1.4	.14
1.5	.16
1.6	.18
1.7	.21
1.8	.23
1.9	.26
2.0	.29
2.1	.32
2.2	.35
2.3	.38
2.4	.42
2.5	.45
2.6	.49
2.7	.52
2.8	.56
2.9	.61
3.0	.65

Table 3 COMPARISON OF ANALYZED AND INFERRED AMPLITUDES
WEST COAST, NORTH AND SOUTH AMERICA

Place	Series	Observed Values* Mn ft.	Observed Values* DHQ ft.	Observed Values* DLQ ft.	Harmonic Analysis K_1+O_1 ft.	Harmonic Analysis M_2 ft.	Inferred Values K_1+O_1 ft.	Inferred Values M_2 ft.	Error of Inference K_1+O_1 %	Error of Inference M_2 %
Kodiak	1 yr., 1935-1936	6.61	0.90	1.03	2.12	3.11	2.15	2.96	1	5
Seward	1 yr., 1931	8.29	0.88	1.33	2.32	3.88	2.46	3.72	6	4
Cordova	1 yr., 1929-1930	9.78	0.84	1.47	2.62	4.67	2.58	4.41	2	6
Yakutat	1 yr., 1941	7.82	0.88	1.39	2.37	3.65	2.48	3.51	5	4
Sitka	1 yr., 1938-1939	7.68	0.79	1.39	2.40	3.59	2.44	3.44	2	4
Juneau	1 yr., 1939	13.90	1.01	1.59	2.70	6.51	2.84	6.32	5	3
Skagway	1 yr., 1908-1909	13.96	0.97	1.52	2.73	6.65	2.71	6.35	1	5
Craig	1 yr., 1917	7.96	0.88	1.35	2.35	3.72	2.50	3.57	6	4
Ketchikan	1 yr., 1939	13.00	0.93	1.54	2.69	6.15	2.75	5.88	2	4
Neah Bay	1 yr., 1939	5.76	0.37	1.59	2.58	2.67	2.79	2.53	8	5
Port Angeles	1 yr., 1934-1935	4.14	0.59	2.22	3.41	1.70	3.47	1.71	2	1
Friday Harbor	1 yr., 1940	4.46	0.79	2.58	3.89	1.83	4.10	1.80	5	2
Blaine	1 yr., 1934-1935	5.85	0.79	2.65	4.17	2.53	4.21	2.47	1	2
Port Townsend	1 yr., 1934-1935	5.05	0.61	2.45	3.90	2.14	3.83	2.10	2	2
Seattle	1 yr., 1939	7.70	0.87	2.91	4.23	3.53	4.62	3.33	9	6
Olympia	1 yr., 1934-1935	10.46	0.92	2.97	4.44	4.77	4.71	4.63	6	3
Raymond	1 yr., 1939	7.80	0.75	1.42	2.23	3.50	2.45	3.50	10	0
Astoria	1 yr., 1939	6.56	0.73	1.06	2.00	3.01	1.96	2.94	2	2
Crescent City	1 yr., 1939	5.10	0.68	1.24	2.06	2.33	2.18	2.27	6	3
San Francisco	1 yr., 1940	3.97	0.63	1.10	1.95	1.80	1.96	1.74	1	3
Alameda	1 yr., 1940	4.67	0.64	1.10	1.97	2.14	1.96	2.08	1	3
Benecia	1 yr., 1936-1937	4.16	0.52	0.95	1.73	1.87	1.67	1.85	3	1
Antioch	1 yr., 1936-1937	3.14	0.47	0.59	1.25	1.34	1.13	1.40	10	4
Stockton	1 yr., 1938	2.80	0.41	0.43	0.98	1.16	0.93	1.26	5	9

Table 3 (Cont'd)

COMPARISON OF ANALYZED AND INFERRED AMPLITUDES
WEST COAST, NORTH AND SOUTH AMERICA

Place	Series	Observed Values*			Harmonic Analysis		Inferred Values		Error of Inference	
		Mn ft.	DHQ ft.	DLQ ft.	K_1+O_1 ft.	M_2 ft.	K_1+O_1 ft.	M_2 ft.	K_1+O_1 %	M_2 %
Collinsville	1 yr., 1936-1937	3.20	0.48	0.61	1.28	1.37	1.17	1.43	9	4
Dunbarton Bridge	1 yr., 1936-1937	6.50	0.58	1.20	2.11	2.98	2.03	2.91	4	2
Avila	1 yr., 1949	3.58	0.67	1.02	1.89	1.60	1.89	1.56	0	2
Port Hueneme	1 yr., 1941	3.65	0.78	0.98	1.86	1.62	1.88	1.60	1	1
Santa Monica	1 yr., 1940	3.72	0.80	0.96	1.83	1.66	1.92	1.63	5	2
Los Angeles	1 yr., 1940	3.78	0.79	0.96	1.81	1.70	1.92	1.66	6	2
La Jolla	1 yr., 1940	3.66	0.76	0.94	1.77	1.63	1.88	1.61	6	1
San Diego	1 yr., 1936	4.12	0.76	0.92	1.79	1.79	1.84	1.82	3	2
Salina Cruz	1 yr., 1941-1942	3.40	0.36	0.13	0.56	1.60	0.58	1.55	4	3
La Union	1 yr., 1947-1948	8.01	0.32	0.20	0.55	3.90	0.57	3.66	4	6
Puntarenas	1 yr., 1941-1942	7.48	0.26	0.21	0.47	3.62	0.52	3.42	11	6
Buenaventura	1 yr., 1941-1942	10.40	0.24	0.32	0.46	4.89	0.62	4.75	35	3
Talara	1 yr., 1942-1943	4.02	0.21	0.33	0.51	1.90	0.59	1.83	16	4
Callao	1 yr., 1941-1942	1.79	0.42	0.28	0.71	0.76	0.78	0.79	10	4
Matarani	1 yr., 1941-1942	2.14	0.51	0.17	0.79	1.00	0.82	0.96	4	4
Valparaiso	1 yr., 1941-1942	2.99	0.54	0.19	0.84	1.40	0.87	1.35	4	4
Puerto Montt	1 yr., 1942-1943	13.22	0.73	0.43	1.26	6.15	1.30	6.04	3	2
Punta Arenas	1 yr., 1942-1943	3.77	0.88	0.71	1.69	1.65	1.76	1.67	4	1
					Mean error of inference =				5 %	3 %

* Observed values corrected in accordance with Tables 6 and 7 in U. S. Coast and Geodetic Survey Special Publication No. 135, TIDAL DATUM PLANES, by H.A. Marmer, 1927.