DETERMINATION OF AZIMUTH AND LATITUDE FROM OBSERVATIONS OF A SINGLE UNKNOWN STAR BY A NEW METHOD

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The purpose of this paper is to describe a new and easy method of determining the (astronomical) latitude and azimuth at any place and to explain the line of approach and the formulae. It will be seen that the method should be useful to a wide circle of land surveyors. One of its principal advantages is that identification of the star is not necessary and it can be used when no star chart or star catalogue is available.

UNDERLYING PRINCIPLE

By knowing three positions in space of a single star, the plane of the apparent movement of the star in space can be found. The normal to this plane is the axis of the celestial sphere, and this axis points to the pole. The altitude of this axis will give the latitude and its direction will give the azimuth.

Before going into the details of the method a list of the symbols used is given for easy reference.

SYMBOLS

θ = The horizontal angle between the R.O. and the star, *i.e.*, the angle between the two vertical circles, through the R.O. and through the star, on the horizontal plane.

θ_1, θ_2, θ_3 = θ for the first, the second, and third positions of the same star, as obtained from the means of the pair of observations on the two faces of the theodolite.

φ = The altitude of the star above the horizontal plane, corrected for refraction, etc.

φ_1, φ_2, φ_3, = φ for the first, the second, the third positions of the same star, as obtained from the means of the pair of observations on the two faces of the theodolite.

ψ = Zenith distance of a star.

x_1, y_1, z_1 = The rectangular co-ordinates in three dimensions for the first mean position of the star, considering the direction of the R.O as the x-co-ordinate, and the zenith-direction as the z-co-ordinate.

x_2, y_2, z_2 = The rectangular co-ordinates in three dimensions for the second mean position of the star.

x_3, y_3, z_3 = The rectangular co-ordinates in three dimensions for the third mean position of the star.

l, m, n = The three unknowns in the general equation of a plane. They represent the direction ratios of the normal to that plane.

$\alpha\ \beta\ \gamma$ = The angles the axis of the celestial sphere makes with the x, y and z co-ordinates respectively.

b = The true bearing of the R.O. from the point of observation.

λ = Latitude of the place of observation, before the application of the corrections.

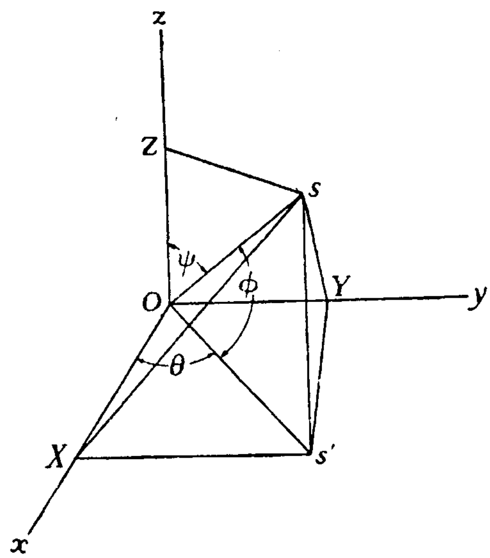

Fig. 1.

THEORY

For a particular position of an unknown star, as in Fig. 1, let the point of observation be O and let Ox be the direction of the R.O. (Reference Object), x being the reference mark. Let s be the position of the star so that Os is the direction of the star from the point of observation.

Let θ be the horizontal angle between the R.O. and the star (this is the mean of the two angles recorded by the horizontal circle of the theodolite with which the observations are made).

Let ϕ be the altitude of the star (this is the mean of the two angles recorded by the vertical circle of the theodolite after the application of proper corrections).

Let ψ be the zenith distance of the star (*i.e.* the angle between Os and Oz, the zenith-direction at the point of observation).

Apparently, Ox and Oz are at right-angles to each other. Let Oy be another straight line at right angles to both Ox and Oz ; so that Ox, Oy and Oz represent the co-ordinates in space in the usual manner.

From s perpendiculars are drawn on Ox, Oy and Oz ; meeting them at X, Y and Z, respectively. Also ss' is the normal from s to the plane xOy meeting the plane at s'. Join Xs' and Ys' and Os'.

Now, from the figure :

$Os' = Os. \cos \angle sOs' = Os. \cos \phi$,

$\therefore OX = Os' \cos \angle s'OX = Os. \cos \phi. \cos \theta$,

$OY = Xs' = Os' \sin \angle s'Ox = Os. \cos \phi \sin \theta$,

and $OZ = Os, \cos \psi = Os. \sin \phi$.

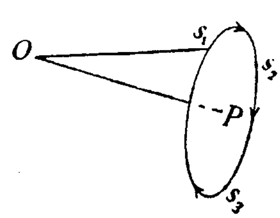

Fig. 2.

Os is evidently the radius of the celestial sphere. For our convenience and ease of computations Os is considered to be *unity*.

Hence, the position of s in three dimensions with respect to the direction of the R.O. is given by :

$$(\cos \phi \cos \theta, \cos \phi \sin \theta, \sin \phi) \qquad \text{(A)}$$

For a single position of the star these three are known quantities because both ϕ and θ are known from observations.

For a star three such observations are made and the three apparent positions of the same star, viz., s_1, s_2 and s_3 with respect to the R.O. according to (A) are determined. These will determine the plane $s_1 s_2 s_3$ of the star's apparent movement (Fig. 2).

The co-ordinates of the three positions, s_1, s_2 and s_3, can be written :

$$\cos \phi_1 \cos \theta_1, \cos \phi_1 \sin \theta_1, \sin \phi_1 \ldots \text{ for } s_1$$
$$\cos \phi_2 \cos \theta_2, \cos \phi_2 \sin \theta_2, \sin \phi_2 \ldots \text{ for } s_2$$
$$\cos \phi_3 \cos \theta_3, \cos \phi_3 \sin \theta_3, \sin \phi_3 \ldots \text{ for } s_3$$

The general equation of a plane is $lx + my + nz = 1$ where l, m, and n are three unknows and x, y and z represent the three-dimensional co-ordinates along the x, y and z axes respectively.

Now, substituting the proper notation in the equation $lx + my + nz = 1$ for the three positions of the star, we obtain three equations :

$$lx_1 + my_1 + nz_1 = 1 \quad \text{or} \quad l \cos \phi_1 \cos \theta_1 + m \cos \phi_1 \sin \theta_1 + n \sin \phi_1 = 1$$
$$lx_2 + my_2 + nz_2 = 1 \quad \text{or} \quad l \cos \phi_2 \cos \theta_2 + m \cos \phi_2 \cos \theta_2 + n \sin \phi_2 = 1 \quad \text{(B)}$$
$$lx_3 + my_3 + nz_3 = 1 \quad \text{or} \quad l \cos \phi_3 \cos \theta_3 + m \cos \phi_3 \sin \theta_3 + n \sin \phi_3 = 1$$

in the three unknowns l, m and n, which represent the direction ratios of the normal to the plane $s_1 s_2 s_3$, *i.e.* the direction ratios of the axis OP with respect to the three co-ordinates. These equations (B) are solved for l, m and n.

Now, if α, β and γ be the three angles OP makes with the x, y and z axes respectively (see Fig. 3), the direction cosines of OP (*i.e.* the normal to the plane of the star which is again the polar axis or the axis of the celestial sphere) are given by :

$$\cos \alpha = \frac{1}{\sqrt{l^2 + m^2 + n^2}}$$
$$\cos \beta = \frac{m}{\sqrt{l^2 + m^2 + n^2}} \qquad \text{(C)}$$
$$\cos \gamma = \frac{n}{\sqrt{l^2 + m^2 + n^2}}$$

For our purposes only the values of $\cos \alpha$ and $\cos \gamma$ will be required.

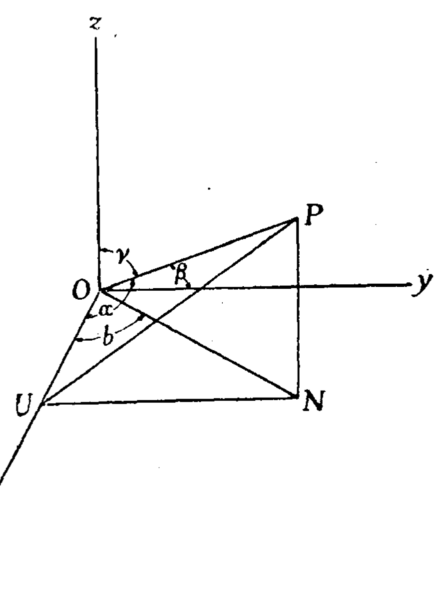

Fig. 3.

LATITUDE DETERMINATION

It is now evident that γ is the zenith distance of the pole and hence the co-latitude. Thus the latitude of the place of observation is given by :

$$\lambda = 90^\circ - \gamma \qquad \text{(D)}$$

AZIMUT DETERMINATION

In Fig. 3 *ON* is the projection of *OP* on the horizontal plane and points to true north. *PU* is drawn perpendicular to *OX* from *P*. Join *NU*.

$$\therefore \angle xON = b = \text{Bearing of the R.O.}$$

$$\therefore ON = OP \sin\gamma \; [\because \angle PNO = 90^\circ]$$

$$\text{and} \quad OU = OP \cos\alpha \; [\because \angle PUO = 90^\circ]$$

$$\therefore \cos b = \frac{OU}{ON} = \frac{\cos\alpha}{\sin\gamma} \qquad \text{(E)}$$

From equation (C) $\cos\gamma$ is known and hence $\sin\gamma$, as also $\cos\alpha$. Thus $\cos b$ gives us the angle b and hence the true bearing of the line Ox, whence we get the azimuth,

$$A = b \pm 180^\circ \qquad \text{(F)}$$

INSTRUMENTS

The following instruments are required for the combined determination of latitude and azimuth by this method :

(i) A theodolite of precision with the following accessories :

(a) Means for illuminating the field of view at night to enable the cross-hairs to be seen ;

(b) A diagonal eye piece for easy and safe observations ;

(c) Sight vanes fitted on the upper and lower sides of the telescope tube to facilitate pointing to a star.

(ii) A computing machine, if available.

OBSERVATIONS

Before proceeding further it must be stated that for such astronomical work great care should be taken while observing with the theodolite.

This method deals with co-ordinate geometry for which the angles should be reckoned anti-clockwise So one important job will be to convert horizontal angles into angles reckoned anti-clockwise from the R.O. This is required for observations with an ordinary theodolite where the angles are marked in a clockwise sense.

Observations should be made as accurately as possible. Collimation errors, etc., may be eliminated by changing face and taking the mean of two readings with opposite faces. After taking down the readings of the theodolite, the following corrections have to be applied to the mean of the two observed altitudes for each star position :

Correction for mean refraction = +

Barometric Correction .... = +

Temperature Correction ...... = —

After applying these corrections (the sum of which is subtracted from the observed altitude to give the correct value of the vertical angle) the corrected value of the angle of altitude, ϕ, is used for the computations.

COMPUTATIONS

The equations (B) are best solved by the « Sweep-Out » method. They can also be solved by the method of determinants, *i.e.* the ordinary method of solving simlutaneous equations of three unknowns. While the « Sweep-Out » method provides automatic checks and is quicker, particularly with an ordinary computing machine, the method of determinants is slightly more laborious but standard forms can be made out for this process and thus the computations can be done by less expert computers.

RESULTS

After the computations are finished, we arrive at the astronomical latitude of the place of observation. The values of λ can be accepted for all topographical work (for determining the geographical latitude) subject to the application of any correction that may be necessary on account of local deviation of the vertical, etc.

Accuracy and good results depend on the following :

(i) The nearer the star is to the equator the better will be the results obtained, because in that case the plane of the apparent movement of the star will be more accurately defined. Slight errors or mistakes in observations will not throw the plane much out of the actual and thus the normal to that plane will be more accurately determined. Hence, the star chosen for such observations should not be too close to the Pole.

(ii) The three positions of the star should be well balanced so that the triangle thus formed is wide. The wider is the triangle the more accurate is the result. In this connection, best results could be obtained if the observations could be made at intervals of eight hours (because a complete apparent circular movement of a star takes 24 hours). But daylight and other difficulties will not permit one to make such observations.

It is, therefore, suggested that observations should be made at the maximum possible interval of time so that the triangle is as wide as possible.

(iii) Again, for places in very high northern latitudes, observations of a southern star, and vice versa, may create some computational troubles for less expert surveyors. Care should be exercised in this respect.

(iv) If the altitude of the star at the time of any particular observation is too low, errors of atmospheric refraction may be large and hence observations at low altitudes should be avoided as far as possible.

(v) Corrections for refraction, barometric pressure and temperature should be applied correctly.

(vi) Personal errors in observations have to be taken into account.

(vii) The more the number of sets observed for a star the more accurate will be the results.

The observations for latitude and azimuth determinations by this method should be planned keeping in mind these precautions. As an example of the accuracy of the results obtained by this method, it can be mentioned, for the information of those interested, that the author obtained an accuracy of 0.5 second in the values of latitude and azimuth from three sets of observations of a single star with a theodolite which reads to a second of arc.

ADVANTAGES OF METHOD

The method has the following advantages :

(a) The use of a chronometer, a star catalogue, a star chart or a prismatic astrolabe is not necessary for determining latitude or azimuth by this method.

(b) The identification of the star is not necessary. Only for high precision work is it desirable to identify the star.

(c) Pre-observation computations are unnecessary.

(d) While observing a star, there will obviously be a great amount of time left at the disposal of the surveyor between two sets of observations. This time the surveyor can conveniently utilize by observing some other stars so that the mean of the final results obtained from several stars will contribute towards a more accurate result.

(e) From a single set of three pairs of observations the surveyor gets both the latitude and azimuth.

It is claimed that, because of these advantages, the method is at least as good as, if not better than, most of the existing methods of determining combined latitude and azimuth, and the author believes that it will thus be of interest and use to practising surveyors. The basic theory might possibly be extended to research in other similar astronomical determinations.

Finally, the author wishes to acknowledge the valuable suggestions and encouragement received from Sri G.B. Das, M.A., A.R.I.C.S., etc., of the Survey of India, who is investigating further possibilities.

EXAMPLE

Observations to a northern star were made with a Tavistock theodolite at intervals of about 3 hours.

Place of Observation : Rajpur, Dehra Dun, U.P., India.
Date of Observation : February, 1948.
The observed angles after correction for refraction, etc., were :

$$\theta_1 = 44^\circ 27' 56'' \qquad \theta_2 = 63^\circ 17' 17'' \qquad \theta_3 = 88^\circ 01' 39''$$

$$\phi_1 = 32^\circ 36' 06'' \qquad \phi_2 = 44^\circ 24' 30'' \qquad \phi_3 = 50^\circ 47' 29''$$

Now,

$$\cos\phi_1 = 0{\cdot}842437 \quad \cos\theta_1 = 0{\cdot}713672 \quad \sin\theta_1 = 0{\cdot}700480 \quad \sin\phi_1 = 0{\cdot}538795$$

$$\cos\phi_2 = 0{\cdot}714371 \quad \cos\theta_2 = 0{\cdot}449505 \quad \sin\theta_2 = 0{\cdot}893278 \quad \sin\phi_2 = 0{\cdot}699767$$

$$\cos\phi_3 = 0{\cdot}632146 \quad \cos\theta_3 = 0{\cdot}034420 \quad \sin\theta_3 = 0{\cdot}999408 \quad \sin\phi_3 = 0{\cdot}774859$$

Hence,

$$x_1 = \cos\phi_1 \cos\theta_1 = 0{\cdot}601224 \quad y_1 = \cos\phi_1 \sin\theta_1 = 0{\cdot}590110 \quad z_1 = \sin\phi_1 = 0{\cdot}538795$$

$$x_2 = \cos\phi_2 \cos\theta_2 = 0{\cdot}321113 \quad y_2 = \cos\phi_2 \sin\theta_2 = 0{\cdot}638132 \quad z_2 = \sin\phi_2 = 0{\cdot}699767$$

$$x_3 = \cos\phi_3 \cos\theta_3 = 0{\cdot}021758 \quad y_3 = \cos\phi_3 \sin\theta_3 = 0{\cdot}631772 \quad z_3 = \sin\phi_3 = 0{\cdot}774859$$

METHOD I. *By the method of Determinants* :—

1	$y_1 z_2$ =	0·412939	$y_1 z_3$ =	0·457252
2	$y_2 z_3$ =	0·494462	$y_2 z_1$ =	0·343822
3	$y_3 z_1$ =	0·340396	$y_3 z_2$ =	0·442093
4	Algebraic sum of rows 1, 2 and 3	1·247797	—	1·243167
5	Algebraic difference of the second sum from the first sum of row 4 = l/k =			+0·004630
6	$x_1 z_3$ =	0·465864	$x_1 z_2$ =	0·420717
7	$x_2 z_1$ =	0·173014	$x_2 z_3$ =	0·248817
8	$x_3 z_2$ =	0·015226	$x_3 z_1$ =	0·011723
9	Algebraic sum of rows 6, 7 and 8	0·654104	—	0·681257
10	Algebraic difference of the second sum from the first sum of row 9 = m/k =			−0·027153
11	$x_1 y_2$ =	0·383660	$x_1 y_3$ =	0·379836
12	$x_2 y_3$ =	0·202870	$x_2 y_1$ =	0·189492
13	$x_3 y_1$ =	0·012840	$x_3 y_2$ =	0·013885
14	Algebraic sum of rows 11, 12 and 13	0·599370	—	0·583213
15	Algebraic difference of the second sum from the first sum of row 14 = n/k =			+0·016157
16	Sum of the squares of rows 5, 10 and 15 $= \dfrac{l^2+m^2+n^2}{k^2} =$			0·001019771236
17	Square root of the sum in row 16 $= \dfrac{\sqrt{l^2+m^2+n^2}}{k} =$			0·031934
18	$\cos\gamma = \dfrac{n}{\sqrt{l^2+m^2+n^2}}$ [Divide row 15 by row 17] =			0·505950
19	$\therefore\ \gamma =$			59° 36′ 20″
20	LATITUDE of the place of observation $= \lambda = 90° - \gamma =$			30° 23′ 40″
21	$\cos\alpha = \dfrac{l}{\sqrt{l^2+m^2+n^2}}$ [Divide row 5 by row 17] =			0·144987
22	$\sin\gamma$ [From row 19] =			0·862563
23	$\cos b = \dfrac{\cos\alpha}{\sin\gamma}$ [Divide row 21 by row 22] =			0·168089
24	$\therefore\ b =$			80° 19′ 24″
25	AZIMUTH of the R.O. $= A = b \pm 180°$ =			260° 19′ 24″

METHOD II. *By the "Sweep-Out" method*:—

The equations

$$lx_1+my_1+nz_1=1$$

$$lx_2+my_2+nz_2=1$$

$$lx_3+my_3+nz_3=1$$

are solved from the matrix

$$\begin{Vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{Vmatrix}$$

Hence, substituting the relevant values:

$$\begin{vmatrix} 0{\cdot}601224 & 0{\cdot}590110 & 0{\cdot}538795 & 1 \\ 0{\cdot}321113 & 0{\cdot}638132 & 0{\cdot}699767 & 1 \\ 0{\cdot}021758 & 0{\cdot}631772 & 0{\cdot}774859 & 1 \end{vmatrix}$$

$$\begin{vmatrix} -1 & -0{\cdot}981514 & -0{\cdot}896163 & -1{\cdot}663274 \\ 0 & +0{\cdot}322955 & +0{\cdot}411997 & +0{\cdot}465901 \\ 0 & +0{\cdot}610416 & +0{\cdot}755360 & +0{\cdot}963810 \end{vmatrix}$$

$$\begin{vmatrix} -1 & 0 & +0{\cdot}355964 & -0{\cdot}247323 \\ 0 & -1 & -1{\cdot}275710 & -1{\cdot}442619 \\ 0 & 0 & -0{\cdot}023354 & +0{\cdot}083212 \end{vmatrix}$$

$$\begin{vmatrix} -1 & 0 & 0 & +1{\cdot}021003 \\ 0 & -1 & 0 & -5{\cdot}988067 \\ 0 & 0 & -1 & +3{\cdot}563073 \end{vmatrix}$$

Whence

$$l=+1{\cdot}021003$$

$$m=-5{\cdot}988067$$

$$n=+3{\cdot}563073$$

$$\therefore\quad l^2+m^2+n^2=49{\cdot}594882725827$$

$$\therefore\quad \sqrt{(l^2+m^2+n^2)}=7{\cdot}042363$$

$$\therefore\quad \cos\gamma=\frac{n}{\sqrt{l^2+m^2+n^2}}=0{\cdot}505949$$

$$\therefore\quad \gamma=59^\circ\,36'\,20''$$

$$\therefore\quad \text{LATITUDE}=\lambda=90^\circ-\gamma=30^\circ\,23'\,40''. \qquad (1)$$

Also,

$$\cos\alpha = \frac{l}{\sqrt{l^2+m^2+n^2}} = 0{\cdot}144986$$

$$\sin\gamma = 0{\cdot}862563$$

$$\therefore \quad \cos b = \frac{\cos\alpha}{\sin\gamma} = 0{\cdot}168088$$

$$\therefore \quad b = 80^\circ\ 19'{\cdot}24''.$$

$$\therefore \quad \text{Azimuth} = A = b \pm 180^\circ = 260^\circ\ 19'\ 24''. \qquad (2)$$

Observed and computed by S. K. Ghosh, Survey of India.