A SLIDE-RULE FOR VECTORIAL COMBINATIONS

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The task of determining R and θ from R cos θ and R sin θ, is frequently met in several fields of computation and although its solution involves only a simple calculation in two stages, $\sqrt{R^2 \cos^2 \theta + R^2 \sin^2 \theta}$ and $\tan^{-1} \dfrac{R \sin \theta}{R \cos \theta}$, the process can be exceedingly tedious if a large number of such computations are required. This is particularly true of the harmonic analysis of tides in which the amplitude and phase angle of some sixty separate sinusoidal oscillations are extracted to define the tide of a single port. With the object of economising in time and labour in this process, Dr. A.T. Doodson of the Liverpool Tidal Institute has investigated the possibility of introducing a mechanical aid and has been successful in designing a slide-rule capable of producing the solution by a single setting of the sliding scale. After a thorough testing over a period of years, such slide-rules, manufactured on the premises, have not only justified the energy put into their construction, but have proved to be invaluable items of the computing equipment of the Institute. With the assurance that the rule would be greatly appreciated in other fields of computational practice, the details of its design are here outlined.

The most convenient introduction to the rule is by the demonstration of a specific example. Consider the case where R cos θ = 472 and R sin θ = 268, then figure No. 2 shows the relevant section of the slide-rule required for its solution. The centre sliding rule is set so that 472 appears against the 45° graduation on the lower rule. If the cursor is set to read 268 on the centre scale, then θ can be read off the lower rule as 29°6. At the same time the factor 1.150 is noted on the adjacent scale and if the cursor is now set to this value on the scale to the right of the 45° graduation then R can be read off on the centre scale as 543.

If on the other hand the values had been given as R cos θ = 472 and R sin θ = 26.8 then the upper rule would have been brought into use to read off the angle and its factor, 3°3 and 1.002 respectively. The value of R, in this case, would be 473.

It is, of course, possible to compute the results on the conventional slide-rule, but it must be remembered that this procedure, in its simplest form, applies only to a limited number of cases. With fundamental changes in method as the ratio $\dfrac{R \sin \theta}{R \cos \theta}$ becomes unduly large, or alternatively unduly small, and particularly where R must be found by evaluating such an expression as $R \cos \theta \sqrt{1 + \dfrac{R^2 \sin^2 \theta}{R^2 \cos^2 \theta}}$,

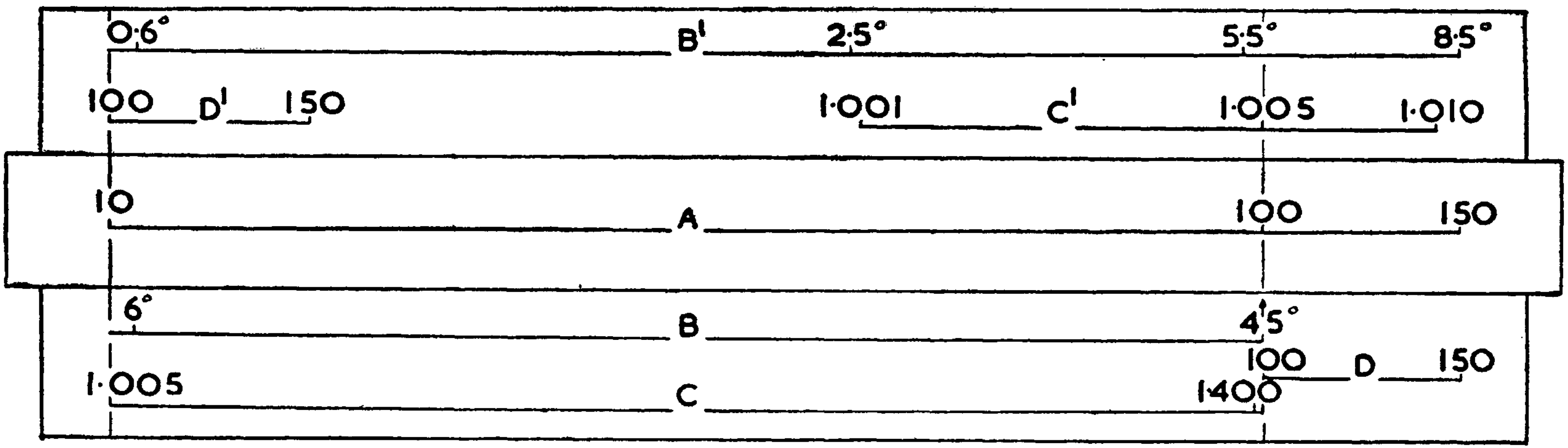


Fig. 1.

Details :

A = ordinary logarithmic scale.
B and B' = tangent logarithm scale.
C and C' = natural secant scale.
D and D' = ordinary logarithmic scale.

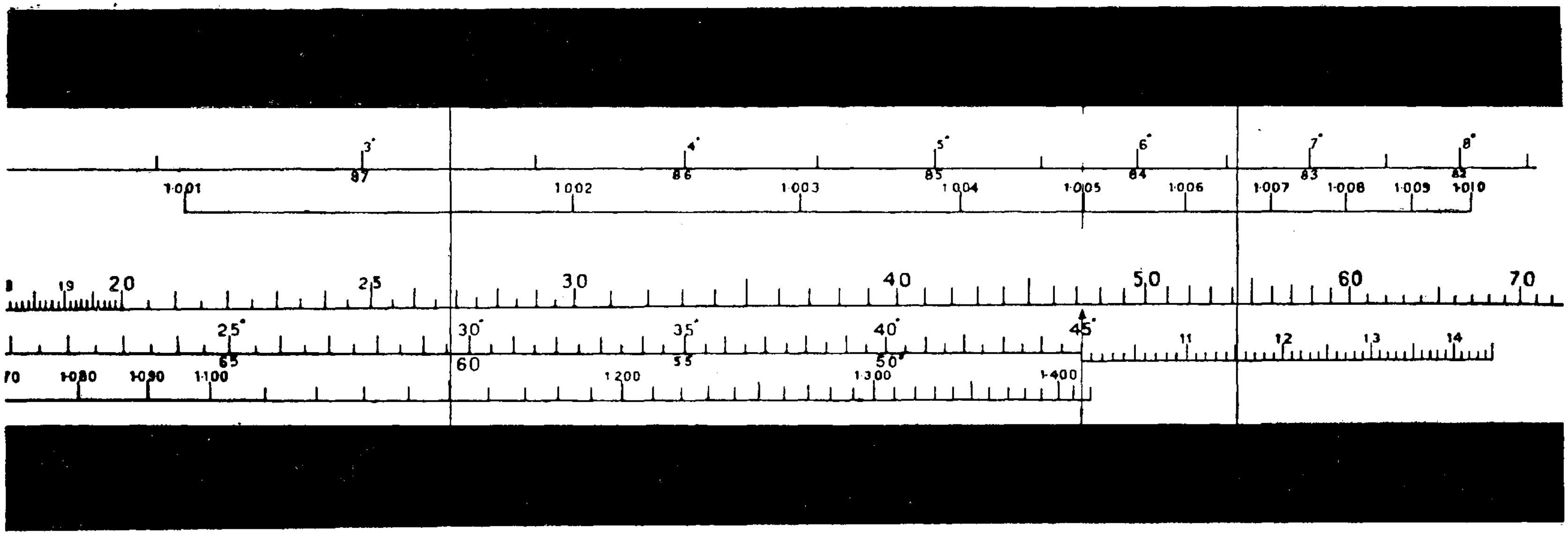
then the procedure becomes tedious. It fails to relieve the computer of considerable mental calculation and consequently is difficult to reduce to a routine. In comparison, a rule designed to perform this particular operation has very considerable advantages over that constructed for as wide an application as possible. No one is more conscious of this argument than the computer who, while consistently requiring his results in terms of degrees and decimals, is faced with the inconvenience of using a rule graduated in degrees and minutes.

Examining the construction of the rule in greater detail it can be seen that, as in direct computation, θ is derived from its tangent which is known. On a simple log scale, this ratio can be expressed as a difference, $\log R \sin \theta - \log R \cos \theta$, which, when applied to a second scale graduated in log tangents will give θ. In practice the log scale (A) is computed for the range 10 to 100 and affixed to the sliding rule. The log tangent scale (B), of the same length and attached to the lower fixed rule, is graduated between $\tan^{-1} 0.1$ (approx. 5°7) and $\tan^{-1} 1.0$ (45°). It is of assistance, if these angles, which, for the purpose of this explanation, shall be named the « natural » angles, are accompanied by their complements, preferably inserted below their respective graduations.

Now where θ lies in the range $5°7 < \theta < 45°0$, $R \cos \theta$ and $R \sin \theta$ are both positive and $R \cos \theta > R \sin \theta$. If the value of $R \cos \theta$ on the log scale (A) is set against 45° on the log tan scale (B), then θ can be read off on (B) against $R \sin \theta$ on (A). Where $R \sin \theta$ and $R \cos \theta$ are positive but $R \sin \theta > R \cos \theta$, then clearly θ lies in the range $45° < \theta < 84°3$. Setting the larger of the values on (A) against 45° on (B), then θ can be found against the smaller value on (A), but this time making use of the complement scale. The extension of this procedure into the other quadrants can be effected by an examination of the sign $R \cos \theta$ and $R \sin \theta$ in the usual way.

In the conventional computation, R is defined as $\sqrt{R^2 \cos^2 \theta + R^2 \sin^2 \theta}$ which can be more conveniently expressed as $R \cos \theta \sqrt{1 + \tan^2 \theta}$ or $R \cos \theta \sec \theta$. For this purpose, a subsidiary scale (C) is drawn up on the lower fixed rule to give the natural secants of the angles shown on the adjacent (B) scale. The simple log scale (A) is now extended in range up to 150 and this extension is repeated in a scale (D), on the same fixed rule, commencing at the 45° graduation on the (B) scale and continuing to the right. Now, when reading off θ against the smaller value, the same position of the cursor can be used to determine $\sec \theta$ on the (C) scale. If the cursor is set to this value on (D), then R can be read off directly on (A).

All that now remains is to cover those « natural » angles in the range $O < \theta < 5°7$, and this is accomplished by drawing up a second log tangent scale (B') for the range $\tan^{-1} 0.01$ (approx. 0°57) to $\tan^{-1} 0.1$ (approx. 5°7), these angles being accompanied by their complements as before. This scale is attached to the upper fixed rule together with a second secant scale (C') to correspond with the new angular range. In practice this secant scale need only be drawn for values of $\sec \theta > 1.001$ i.e. commencing at approximately 2°6. A further refinement, the advantage of which will appear later, is achieved by extending both these scales to 8°5 and 1.010 respectively. Finally if the extension log scale (D) is again repeated as (D'), this time on the upper fixed rule commencing at 0°57 and proceeding to the right, then all cases can be covered. Figure No. 3 indicates the layout of the completed scales ready for application, in three strips, to the slide-rule itself.



3°
4°
5°
6°
7°
8°
87
86
85
84
83
82
1·001
1·002
1·003
1·004
1·005
1·006
1·007
1·008
1·009
1·010
19
20
25
30
40
50
60
70
25°
30°
35°
40°
45°
50°
11
12
13
14
65
60
55
1·080
1·090
1·100
1·200
1·300
1·400

Fig. 2.

The detail of the constructional work tends to give a false impression of the actual speed and ease with which the computation is performed in practice. Such an instrument, designed especially for this single operation, enables confident manipulation even by computers with only an elementary mathematical knowledge, guided by a few simple rules which may be summarized as follows:

First examine the signs of R cos θ and R sin θ to determine the quadrant in which θ lies.

THEN (i) *Where R cos θ and R sin θ are of the same order.*

Set the larger of the two values on (A) against 45° on (B).

If R cos θ>R sin θ

Where θ is in 1st or 3rd quadrant, read off the « natural » angle on (B) against the smaller value on (A), adding 0° or 180°.

Where θ is in 2nd or 4th quadrant, read off the complement angle on B against the smaller value on (A) adding 90° or 270°.

If R sin θ>R cos θ

Proceed as above but read « complement » angle for « natural » angle and vice versa.

To find R

Note the factor on (C) shown under the cursor, then read off R on (A) against the value of this factor on (D).

(ii) *Where one value is greater than* 10 *times the other.*

Proceed as before but read off the angle on (B') against 10 times the smaller value on (A). In this case the secant factor is, of course, taken from scale (C').

(iii) *Where one value is greater than* 100 *times the other.*

Ten times the angle is read off the (B') scale against 100 times the smaller value on (A). Here, within the limits of accuracy, R equals the magnitude of the larger value.

(iv) Some exceptional cases occur in which, having set the larger value on (A) against 45°, the smaller value is found to be off the scale to the left. A large number of such cases can be solved as in procedure (ii) or (iii) by utilizing the (B') and (C') scales which were extended from 5°7 to 8°5 and from 1.005 to 1.010 for the purpose. The remaining cases can be solved by means of an expedient common to all slide rule practice. The equivalent of setting the larger value against 45° is effected by setting 1/10 of its value against the other end of the scale, (5°7). To determine R in this event, the (D') scale is brought into use by means of which 1/10 R is read off on (A).

At this stage it is interesting to note a secondary virtue of the rule in that, although not ideal for the purpose, it can be used in reverse to find R cos θ and R sin θ given R and θ. In this case one must first read off the secant on (C) or (C') against the adjusted θ on (B) or (B'), then by setting R on (A) against this factor on (D), R cos θ and R sin θ can be read off against 45° and θ on (B).

Referring again to figure No. 2, if given R = 542 and θ = 29°6 and required to find R cos θ and R sin θ :

Using scales (B) and (C) read off sec θ = 1.150

Against 1.150 on (D) set 542 on scale (A).

Then R cos θ, being the larger, can be read off against 45° = 472 and R sin θ can be read off against θ = 268.

The computation and drawing of the scales is, of course, the most laborious part of the work and once completed, reproduction is best effected photographically by contact copying techniques. The accuracy of the rule will depend to a great extent upon the care exercised in the actual application of the scales to their respective rules. It is of course necessary to ensure that the graduations of 0°57 on (B'), 10 on (A) and 5°7 on (B) lie in a straight line on the left hand side when 5°7 on (B), 100 on (A) and 45° on B, similarly lie on a straight line on the right hand side, both lines being parallel to the line of the cursor. It is convenient, too, that the scale (A) should be affixed along the lower edge of the centre rule so that the cursor need not be called into use in the setting of the sliding scale.

Little need be said of the construction of the rule itself, in fact three lengths of one-inch lath, two of them secured to a base board and equipped with an improvised cursor, would suffice. However, figure No. 3 will perhaps suggest a simple form of construction designed to give ease of manipulation.

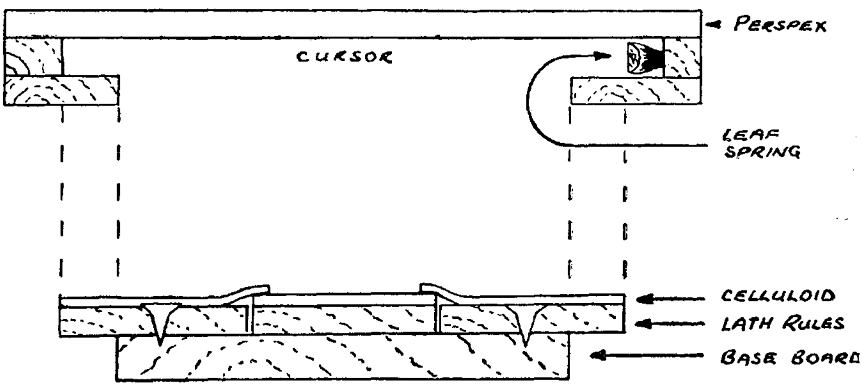


Fig. 3.

It is well to realize the imperfections of the hand-made rule, in particular the probable errors of draughtsmanship in hand-divided scales, and the distortions inherent in paper strips especially during their application to the rule itself. It would therefore be futile to attempt to construct, say, a 10 inch rule to give an accuracy comparable with the average manufactured article of the same size. At the Tidal Institute it has been found convenient to design scales of the order of 18 inches or so in length, then with average draughtsmanship and with care, one can expect at least a linear accuracy of three significant figures. No slide rule can claim a place in precise computation, but in those mathematical processes which normally call for the use of the four-figure log tables the rule will certainly prove to be a great asset.